CORRECTIONS TO
-
AXIOM A POLYNOMIAL SKEW PRODUCTS OF $\mathbb{C}^2$ AND THEIR POSTCRITICAL SETS

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1. Introduction

A polynomial skew product of $\mathbb{C}^2$ is a map of the form

$$f(z,w) = (p(z), q(z,w)) = (p(z), q_z(w)),$$

where p and q are polynomials. We consider skew products which are Axiom A and extend holomorphically to endomorphisms of $\mathbb{P}^2$ of degree $d \geq 2$. In the article [DH], we studied orbits of critical points in a distinguished subset of $\mathbb{C}^2$, and we constructed new examples of Axiom A maps.

We made an erroneous assumption about polynomial skew products, which holds for our main examples but fails in general. In this Correction, we describe the mistake and fix the proofs of our main results. We also indicate which statements in the original article do not hold without the extra assumption.

We would like to thank Hiroki Sumi for bringing the mistake to our attention and carefully describing an example for which the assumption fails (see [Su, Remark 4.13]). We also thank Shizuo Nakane for his careful reading of the original article.

1.1. **An extra assumption.** Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be an Axiom A polynomial skew product. Let $J_p \subset \mathbb{C}$ denote the Julia set of the base polynomial p. We define $\Lambda \subset \mathbb{C}^2$ to be the subset of the nonwandering set of f contained in $J_p \times \mathbb{C}$ and of saddle type. In particular, $f|\Lambda$ is expanding in the base direction (where it acts as the hyperbolic polynomial p), and it is contracting in the vertical direction (acting by the polynomials q_z on fibers). The unstable manifold $W^u(\Lambda)$ consists of all points $x \in \mathbb{C}^2$ for which there exists a backward orbit x_{-k} of x converging to Λ. The stable manifold $W^s(\Lambda)$ consists of all points $x \in \mathbb{C}^2$ which converge to Λ under iteration.

The saddle set Λ decomposes into a disjoint union of *saddle basic sets* $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_n$ on which f acts transitively and satisfies $f(\Lambda_i) = \Lambda_i$. For each saddle basic set, we have

$$W^s(\Lambda_i) \cap W^u(\Lambda_i) = \Lambda_i.$$

See [Jo2, Appendix A]. We incorrectly assumed

$$\text{Assumption } (*) \qquad W^s(\Lambda) \cap W^u(\Lambda) = \Lambda$$

in many of our arguments.

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Assumption $(*)$ holds when Λ is itself a saddle basic set. It also holds for products and for the main examples we give, as we explain below. It does not hold for Example 5.10 of the original article.

1.2. **Summary of the incorrect statements in the original article.** Lemma 3.5, Theorem 5.2, and Corollary 5.3 are false without Assumption $(*)$ and should be replaced with:

Lemma 3.5. *The unstable manifold of Λ satisfies*

$$W^u(\Lambda) \cap K_{J_p} = W^u(\Lambda) \cap W^s(\Lambda).$$

Theorem 5.2. *The following two conditions are equivalent:*

(a) $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$, *and*
(b) $z \mapsto \Lambda_z$ *is continuous for all* $z \in J_p$.

These conditions imply Assumption $(*)$ *and also*

(c) $z \mapsto K_z$ *is continuous for all* $z \in J_p$.

Under Assumption $(*)$, *condition* (c) *is equivalent to* (a) *and* (b).

Corollary 5.3 *If Assumption* $(*)$ *holds and* J_z *is connected for all* $z \in J_p$, *then* $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$.

Claim (2) of Example 5.10 (which we added to the original example of Sumi) is false. In fact, Example 5.10 does not satisfy Assumption $(*)$ and provides a counterexample to the original versions of Lemma 3.5, Theorem 5.2, and Corollary 5.3.

Proposition 6.3 and Lemma 6.4 are proved under Assumption $(*)$; we do not have a proof of these statements in general and do not know if they are true.

2. Corrected proofs of the main Theorems

In this section, we fix the proofs of Theorems 1.1, 5.1, 6.1, and 7.1, which used Assumption $(*)$ in the original article, and we give the proof of the corrected Theorem 5.2.

2.1. **Proof of Theorem 1.1.** The proof of Theorem 1.1 relies on the lemmas of Section 3.

Proof of the corrected Lemma 3.5. The original proof of Lemma 3.5 provides a proof of the corrected version stated above in §1.2.

Proof of Lemma 3.6. The inclusion $W^s(\Lambda) \subset K_{J_p} \setminus J_2$ follows from the definitions, since $J_p \times \mathbb{C}$ is totally invariant, and p is expanding on J_p. For the reverse inclusion, suppose $x \in K_{J_p} \setminus J_2$. Its accumulation set $A(x)$ lies in the nonwandering set of f within $J_p \times \mathbb{C}$. On the other hand, it is contained in $W^u(\Lambda)$ by Proposition 3.3, so it is disjoint from J_2. Therefore $A(x) \subset \Lambda$, so $x \in W^s(\Lambda)$. $\square$

Proof of Lemma 3.7. Let c be an element of the critical set C_{J_p}. Then the accumulation set $A(c)$ is either empty or contained in the nonwandering set within K_{J_p}. From Proposition 3.3, $A(c) \subset W^u(\Lambda)$, so it is disjoint from J_2. Consequently $A(c) \subset \Lambda$, so $A_{pt}(C_{J_p}) \subset \Lambda$. The proof of the reverse inclusion is correct in the original article. □

2.2. **Proof of the corrected Theorem 5.2.** The corrected Theorem 5.2 is stated above in §1.2. The proofs that (a) $\implies$ (b) and that (b) $\implies$ (c) from the original article are correct. The proof that (c) $\implies$ (a) is correct under Assumption ($*$).

We first prove that (b) $\implies$ (a). Suppose $z \mapsto \Lambda_z$ is continuous over J_p, and fix $x \in W^u(\Lambda)$. Then there exists a prehistory x_{-k}, with $f^k(x_{-k}) = x$, converging to Λ. Continuity of Λ_z implies that a vertical neighborhood of Λ is in fact a neighborhood of Λ in the ambient space $J_p \times \mathbb{C}$. Consequently the points x_{-k} lie in a vertical trapping neighborhood of Λ for all large enough k. But then all forward iterates of these x_{-k} lie in smaller vertical trapping neighborhood, by Proposition 3.2, which implies that x is in Λ.

An easy argument shows that (a) implies Asssumption ($*$). Indeed, if $x \in W^s(\Lambda) \cap W^u(\Lambda)$, then Lemma 3.6 (with corrected proof given above in §2.1) implies that x lies in $J_p \times \mathbb{C}$. Then (a) implies that $x \in \Lambda$. The inclusion $\Lambda \subset W^s(\Lambda) \cap W^u(\Lambda)$ is clearly true. □

2.3. **Proof of Theorem 5.1.** The proofs of (1) and (2) are correct as stated, so it remains to establish the validity of part (3). We apply the corrected version of Theorem 5.2, as stated above in §1.2 and proved in §2.2.

Assume that F_a is Axiom A, so that g_a is hyperbolic. Let $P_a \subset \mathbb{C}$ denote the unique attracting cycle of g_a. Recall the notation for each fixed $x \in \mathbb{C}$,

$$S_x = \{(e^{2it}, xe^{it}) \in \mathbb{C}^2 : t \in [0, 2\pi]\},$$

so that $F_a(S_x) = S_{g_a(x)}$. Define

$$\Lambda_a = \bigcup_{x \in P_a} S_x.$$

We will show that $\Lambda = \Lambda_a$. Observe first that $F_a(\Lambda_a) = \Lambda_a$. By computing derivatives of F_a along S_x, it is easy to see that saddle periodic points are dense in S_x for each $x \in P_a$. Therefore $\Lambda_a \subset \Lambda$. On the other hand, all points in $K_{J_p} \setminus J_2$ converge to Λ_a, and since $W^s(\Lambda) = K_{J_p} \setminus J_2$ by Lemma 3.6, we may conclude that $\Lambda \subset \Lambda_a$.

It follows immediately that $z \mapsto \Lambda_z$ is continuous over J_p for these examples, and therefore by Theorem 5.2, we have $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$.

2.4. **Proof of Theorem 6.1.** It remains to establish the validity of Theorem 6.1 parts (5) and (6).

At the end of the proof of Theorem 6.1, we aplied Lemma 6.4 to conclude that $A_{pt}(C_{J_p}) \neq A_{cc}(C_{J_p})$. The proof of Lemma 6.4 relies on Assumption ($*$). Assumption ($*$) holds for these examples because they have a unique saddle basic set.

Alternatively, we can simply observe that $A_{cc}(C_{J_p})$ must contain a unbounded connected set which intersects K_{J_p}, while $A_{pt}(C_{J_p})$ is compact. This proves Theorem 6.1 (5).

To prove Theorem 6.1 (6), we used Proposition 6.3, which again relies on Assumption $(*)$. Alternatively, we can simply observe that the fiber Julia sets of these f_n are not all homeomorphic; so applying Corollary 4.4 we can conclude that f_n does not lie in the same hyperbolic component as a product.

2.5. **Proof of Theorem 7.1.** It remains to justify the final statement of part (7), that f does not lie in the same hyperbolic component of a product. We will only give the proof under an additional assumption: suppose that either (a) s_1 and s_2 are not topologically conjugate on their Julia sets, or (b) at least one of the s_i has connected Julia set. In case (a), statement (7) follows from the fact that the motion of J_2 must preserve the skew structure (Theorem 4.2) so the fiberwise conjugacy induces a conjugacy of s_1 and s_2 on their Julia sets. In case (b), statement (7) follows immediately from Corollary 4.4. This is enough to provide an infinite family of examples.

3. $\hat{\Lambda}$-stability and hyperbolic components

In this final section, we complete the proofs of Propositions 1.2 and 5.4 of the original article. We show that the hypotheses imply Assumption $(*)$ and that the assumption holds throughout the hyperbolic component. We use the stability of the *natural extension* of the non-wandering set for Axiom A endomorphisms, as described in [Jo2, Appendix A]. Specifically, the natural extension of Λ moves holomorphically, in the sense of [Jo1, Theorem B], and the motion respects the skew-product structure.

3.1. **Skew motion of $\hat{\Lambda}$.** Jonsson showed that there is a motion of the natural extension of Λ on a neighborhood of f in the space of polynomial skew products. We begin by showing that this motion preserves the skew structure, exactly as in our Theorem 1.4 of the original article for the expanding part of the non-wandering set.

Let $\{f_a = (p_a, q_a) : a \in \mathbb{D}\}$ be a holomorphic family of Axiom A skew products, and let Λ be the saddle set for f_0 inside $J_p \times \mathbb{C}$. Combining [Jo1, Theorem B] and [Jo2, Corollary 8.14], there exist a radius $r > 0$ and a continuous map

$$h : \mathbb{D}_r \times \hat{\Lambda} \to \mathbb{C}^2$$

satisfying

(1) for each $a \in \mathbb{D}_r$, $\Lambda_a := h_a(\hat{\Lambda})$ is the saddle set of f_a in $J_{p_a} \times \mathbb{C}$;
(2) for each $a \in \mathbb{D}_r$, h_a lifts to a conjugating homeomorphism $\hat{h}_a : \hat{\Lambda} \to \hat{\Lambda}_a$; and
(3) $h(\cdot, \hat{x})$ is holomorphic for each fixed $\hat{x}$ in $\hat{\Lambda}$.

Let $\phi_a : J_p \to J_{p_a}$ be the conjugating homeomorphism for the holomorphic motion of the base Julia set J_p. We use density of periodic points in Λ to show that the diagram

$$\begin{array}{ccc} \hat{\Lambda} & \xrightarrow{\hat{h}_a} & \hat{\Lambda}_a \\ \hat{\pi}\downarrow & & \downarrow\hat{\pi} \\ \hat{J}_p & \xrightarrow[\hat{\phi}_a]{} & \hat{J}_{p_a} \end{array}$$

commutes. Here $\pi : \mathbb{C}^2 \to \mathbb{C}$ is the projection to the first coordinate, so $\hat{\pi}$ is the extension of this projection to the natural extension of the skew product.

Indeed, if $\hat{x} \in \hat{\Lambda}$ is a periodic itinerary for $f = f_0$, then the first coordinate $\hat{z} = \hat{\pi}(\hat{x})$ is periodic in $\hat{J}_p$. Under the extended motion $\hat{h}_a$, the itinerary $\hat{x}_a$ remains periodic, so the projection $\hat{z}_a = \hat{\pi}(\hat{x}_a)$ does too. By uniqueness of local solutions to $p_a^n(z) = z$ near repelling periodic points, we conclude that $\hat{z}_a = \hat{\phi}_a(\hat{z})$, the image of $\hat{z}$ under the extended motion of J_p. Consequently, the diagram commutes on periodic cycles. Periodic cycles are dense and the maps in the diagram are continuous; therefore the diagram commutes everywhere. In other words, the motion preserves the skew structure.

3.2. **The motion of $\hat{\Lambda}$ induces a fiberwise continuity.** Fix a point $z \in J_p$, and let $\hat{\Lambda}_z$ be the set of itineraries (x_{-k}) in $\hat{\Lambda}$ with initial point x_0 in the fiber $\{z\} \times \mathbb{C}$. Then if f_a is a family of skew products as in §3.1, the skew-structure of the motion h implies that for each fixed $a \in \mathbb{D}_r$, the image $h_a(\hat{\Lambda}_z)$ lies in the fiber $\{z_a\} \times \mathbb{C}$, with $z_a = \phi_a(z)$, and it coincides with the intersection $(\Lambda_a)_{z_a} := \Lambda_a \cap (\{z_a\} \times \mathbb{C})$. The continuity of h implies, and here is the key point, that the function

$$a \mapsto (\Lambda_a)_{z_a}$$

is continuous in the Hausdorff topology for each fixed $z \in J_p$.

3.3. **Discontinuity of $z \mapsto \Lambda_z$ is an open condition.** Suppose $z \mapsto \Lambda_z$ is discontinuous for f, and let f_a be a holomorphic family with $f_0 = f$ and for which there exists a holomorphic motion as in §3.1. Because Λ is closed, it must be that $z \mapsto \Lambda_z$ fails to be *lower* semicontinuous at a point $z_0 \in J_p$. Thus, there is a point $x = (z_0, w_0) \in \Lambda_{z_0}$, a neighborhood U of w_0 in $\mathbb{C}$, and a sequence of points z_k converging to z_0 in J_p so that Λ_{z_k} does not intersect U. The continuity of $(\Lambda_a)_{z_a}$ from §3.2 implies that under perturbation, the point x_a persists in Λ_a in the fiber over $(z_0)_a$ while the neighborhood U remains empty in the fibers over moved points $(z_k)_a$. In other words, $z \mapsto (\Lambda_a)_z$ is discontinuous also for nearby maps f_a.

While the discussion of holomorphic motions is done for 1-parameter families, defined over a disk $\mathbb{D}$, it should be remarked that the same arguments go through for higher-dimensional parameter spaces. Thus, the discontinuity of $z \mapsto \Lambda_z$ is an open condition in the space of Axiom A skew products.

3.4. **Assumption** $(*)$ **holds on an open set.** The meaning of Assumption $(*)$ is that there are no relations among the basic sets of Λ in the preording by intersection of stable and unstable manifolds. If Assumption $(*)$ is satisfied for a map f, then the $\hat{\Lambda}$-stability of an Axiom A polynomial skew product guarantees that Assumption $(*)$ is satisfied on an open set containing f: no new relations can appear under perturbation [Jo2, Corollary 8.14].

3.5. **Proof of Proposition 5.4.** We aim to show that the condition $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$ holds throughout a hyperbolic component if it holds for a single map. From the corrected Theorem 5.2, this condition implies Assumption $(*)$ and the continuity of $z \mapsto K_z$. The original proof of Proposition 5.4 shows that the continuity of $z \mapsto K_z$ must hold throughout the entire hyperbolic component. Now, from §3.4, we know that Assumption $(*)$ holds on an open set. On the other hand, if there is a map f_1 in the component for which $z \mapsto \Lambda_z$ is discontinuous, then continuity of $z \mapsto K_z$ implies that Assumption $(*)$ fails for f_1. As the discontinuity of $z \mapsto \Lambda_z$ is an open condition, we must have that Assumption $(*)$ fails on an open set. This is a contradiction, so f_1 cannot exist. In other words, $z \mapsto \Lambda_z$ is continuous for all maps in the hyperbolic component, and by Theorem 5.2, this is equivalent to $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$. □

3.6. **Proof of Proposition 1.2.** We need to show that $\Lambda = W^u(\Lambda) \cap (J_p \times \mathbb{C})$ for all maps in the hyperbolic component of a product. This is an immediate consequence of Proposition 5.4, because the condition holds for products. Indeed, it is immediate to see that $z \mapsto \Lambda_z$ is continuous for products. □

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