Math 535
Final Exam
Due Wednesday, May 6

Guidelines: You are on your honor to work independently on this exam. You may use the textbook or other resources from the library; if you consult a reference, please indicate where you looked (title, author, theorem or page number). You may also come ask me questions. Please write your solutions carefully. Your final score depends on the clarity of the presentation as much as your ability to solve the problem.

1. Specify a branch of the logarithm which is well-defined on the union of the two lines $\{z = x + iy : x = e\}$ and $\{z = x + iy : y = e\}$. Determine (and draw) the images of these two lines under your branch.

2. Evaluate the following integrals, using any method you wish. Justify your steps.

(a) $\int_\gamma \cos z \, dz$, where γ is a path from $-\pi i$ to πi

(b) $\int_\gamma \frac{z^3}{z+1} \, dz$, where γ is the circle of radius 2, centered at 0, oriented positively

(c) $\int_\gamma \frac{e^z}{z^2(z+5)} \, dz$, where γ is the circle of radius 2, centered at 0, oriented positively

3. Construct a conformal isomorphism from the half-infinite strip

$$\{z = x + iy : x < 0 \text{ and } -\pi < y < \pi\}$$

to the upper half-plane $\{z = x + iy : y > 0\}$. Use compositions of exponentials, logs, powers, and/or linear fractional transformations, and draw pictures for each step. If you use $\sin z$ or $\cos z$, can you find a simpler formula?

4. Let $f(z) = \tan z = \frac{\sin z}{\cos z}$, and let

$$\sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent expansion of f in the annulus $\{3 < |z| < 4\}$. Define

$$f_+(z) = \sum_{n=0}^{\infty} a_n z^n \qquad f_-(z) = \sum_{n=-\infty}^{-1} a_n z^n$$

(a) Prove that f is meromorphic on $\mathbb{C}$. What are the orders of its poles?

(b) Compute a_{-1}, and obtain an explicit expression for f_-.

(c) What is the radius of convergence of the power series f_+, and why?

5. Suppose f is a bounded analytic function on the upper half plane, $\mathrm{H} = \{z = x+iy : y > 0\}$, and $M > 0$ is a constant such that

$$\limsup_{z \to X} |f(z)| \leq M$$

for each point $X \in \mathbb{R}$. Show that $|f(z)| \leq M$ for all $z \in \mathrm{H}$.

Hint: For small $\varepsilon > 0$, consider the function $(z+i)^{-\varepsilon} f(z)$ on a large half-disk.

6. Let $U(\varepsilon) \subset \mathbb{C}$ be the ε-neighborhood of the real interval $[0,1]$. That is, $U(\varepsilon)$ is the set of all points $z \in \mathbb{C}$ such that $|z - x| < \varepsilon$ for some $0 \leq x \leq 1$. Fix point $p = 1/2$, and let f_ε be the Riemann map from $U(\varepsilon)$ to the disk $\mathbb{D}$, with $f_\varepsilon(p) = 0$ and $f_\varepsilon'(p) > 0$. Prove that

$$f_\varepsilon'(p) \to \infty$$

as $\varepsilon \to 0$.

Hint: Consider the family of inverse functions f_ε^{-1}.

7. Suppose Ω is a simply connected region in $\mathbb{C}$ which is symmetric about the real axis. That is, $z \in \Omega$ if and only if $\bar{z} \in \Omega$. Fix a point $p \in \Omega \cap \mathbb{R}$. Prove that the Riemann map $f : \Omega \to \mathbb{D}$, with $f(p) = 0$ and $f'(p) > 0$, satisfies

$$f(\bar{z}) = \overline{f(z)}.$$

Conclude, in particular, that $f(x) \in \mathbb{R}$ for all $x \in \Omega \cap \mathbb{R}$.

8. (a) Give an example of a region $\Omega \subset \mathbb{C}$ with a trivial group of conformal automorphisms. That is, find an Ω so that if $f : \Omega \to \Omega$ is analytic and invertible, then f is the identity. Justify your answer.

(b) What are all of the conformal automorphisms of the annulus

$$A = \{z \in \mathbb{C} : 1/R < |z| < R\}$$

where $R > 1$? **Hint:** Use Schwarz Reflection to extend the automorphism to all of $\mathbb{C} \setminus \{0\}$.