

Math 535
Homework 1
Due Friday, January 23

Read Chapter 1. You are encouraged to work on *all* of the exercises in the text, but you only need to turn in the following problems.

1. Sketch the image of the line $\{z = x + iy : y = 2\}$ under the functions:

- (a) $1/z$
- (b) z^2
- (c) $iz - 3$

2. Fix $r > 0$, $r \neq 1$, and points $a, b \in \mathbb{C}$. Show that the set of $z \in \mathbb{C}$ with

$$|z - a| = r|z - b|$$

is a circle. Sketch it for $r = 1/2$ and $r = 2$, with $a = 0$ and $b = 1$. What happens when $r = 1$?

3. Consider the equation $az + b\bar{z} + c = 0$, where a, b, c are complex numbers. When does this equation determine a line in \mathbb{C} ? When is there exactly 1 solution, and what is that solution?

4. Prove that $a, b, c \in \mathbb{C}$ are the vertices of an equilateral triangle if and only if

$$a^2 + b^2 + c^2 = ab + ac + bc.$$

Hint: consider the cubed roots of unity, $1, \zeta, \zeta^2$, where $\zeta = e^{2\pi i/3}$, and show that a triple of points satisfies the equation if and only if any translation or dilation of the three points satisfies the equation.

5. Consider the 2×2 matrices of the form

$$\begin{pmatrix} s & t \\ -t & s \end{pmatrix}$$

with $s, t \in \mathbb{R}$, not both 0. Show that any such matrix defines a linear transformation $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ which is a composition of a rotation and a dilation. What is the angle of rotation; what is the dilation factor in terms of s and t ? Express L as a complex-valued function of a complex variable z .

6. Show that $a, b \in \mathbb{C}$ are antipodal points on the Riemann sphere if and only if $a\bar{b} = -1$.