Math 535
Homework 8
Due Friday, April 10

Read Chapter 4 §5 and Chapter 5 §1. You are encouraged to work on *all* of the exercises in the text, but you only need to turn in the following problems.

1. Compute the following residues:

$$(a)\ \operatorname{Res}_1\left(\frac{1}{z^5-1}\right) \qquad (b)\ \operatorname{Res}_0\left(\frac{\sin z}{z^2}\right)$$

2. Use the Residue Theorem to prove that

$$\int_0^\infty \frac{x^2}{x^4+1}\,dx = \frac{\pi}{2\sqrt{2}}.$$

3. Suppose f is meromorphic in $\mathbb{C}$ and doubly periodic, with

$$f(z) = f(z+1) = f(z+i)$$

for all $z \in \mathbb{C}$. Let S be the square region with vertices $0, 1, i, 1+i$, and let γ be its boundary, oriented positively (counterclockwise). Assume that f has no poles on γ.

(a) Prove that the sum of the residues at poles in S is zero.

(b) Deduce that f cannot have only one pole of order 1 and no other poles.

Recall that Liouville's theorem implies that if f has no poles, then it must be constant. Do you remember how the proof goes? Such f do exist with more poles. Do you know of one? Take a look at Chapter 7 in Ahlfors.

4. If f has an isolated singularity at ∞, its *residue* at infinity is defined by

$$\operatorname{Res}_\infty(f) = \frac{-1}{2\pi i}\int_C f(z)\,dz$$

where C is a circle of radius R, oriented counterclockwise, for R large enough that f is analytic on $\{|z| \geq R\}$. For example, the residue of z at ∞ is 0, while the residue of $1/z$ at ∞ is -1. Prove that for any meromorphic function on the extended complex plane $\hat{\mathbb{C}}$, the sum of its residues is 0.

Note that statement (a) of the previous exercise implies the same is true for the torus: the sum of the residues of a meromorphic function on a torus is always zero. The residue $\operatorname{Res}_\infty$ may strike you as an artificial definition, just to make this exercise work, but exercise 5 explains that this choice is natural.

5. The residue is a value naturally associated to 1-forms, not to functions. Show that the differential $f(z)\,dz$ becomes $-f(1/w)\,dw/w^2$ under the change of coordinates $w = 1/z$. Conclude that the residue of f at infinity coincides with that of $-f(1/w)/w^2$ at $w = 0$.

6. Find the radius of convergence for the power series expansion of

$$f(z) = \frac{z^2 - 1}{z^3 - 1}$$

at $z_0 = 2$.

7. Suppose f is analytic in a neighborhood of 0 and satisfies

$$f(z) = z + f(z)^2.$$

What is the radius of convergence of its power series expansion at $z_0 = 0$?

8. Show that the famous series

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

converges for $\operatorname{Re} s > 1$, and represent its derivative in series form.

9. Suppose f_k is a sequence of analytic functions on a disk D, converging uniformly on compact subsets to a function f. If the f_k are all one-to-one, prove that f is either constant or one-to-one.

10. Suppose f_k is a sequence of analytic functions on a disk D, converging uniformly on compact subsets to a function f. If the f_k have at most m zeroes in D, prove that f is either identically zero or has at most m zeroes in D.