PREPERIODIC POINTS AND UNLIKELY INTERSECTIONS

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Abstract. In this article, we combine complex-analytic and arithmetic tools to study the preperiodic points of one-dimensional complex dynamical systems. We show that for any fixed $a, b \in \mathbb{C}$, and any integer $d \geq 2$, the set of $c \in \mathbb{C}$ for which both a and b are preperiodic for $z^d + c$ is infinite if and only if $a^d = b^d$. This provides an affirmative answer to a question of Zannier, which itself arose from questions of Masser concerning simultaneous torsion sections on families of elliptic curves. Using similar techniques, we prove that if rational functions $f, g \in \mathbb{C}(z)$ have infinitely many preperiodic points in common, then they must have the same Julia set. This generalizes a theorem of Mimar, who established the same result assuming that f and g are defined over $\bar{\mathbb{Q}}$. The main arithmetic ingredient in the proofs is an adelic equidistribution theorem for preperiodic points over number fields and function fields, with non-archimedean Berkovich spaces playing an essential role.

1. Introduction

1.1. **Statement of main results.** A complex number a is *preperiodic* for a polynomial map $f \in \mathbb{C}[z]$ if the forward orbit of a under iteration by f is finite. In this article, we examine preperiodic points for the unicritical polynomials, those of the form $z^d + c$ for a complex parameter c. The main result of this article is the following.

Theorem 1.1. *Let $d \geq 2$ be an integer, and fix $a, b \in \mathbb{C}$. The set of parameters $c \in \mathbb{C}$ such that both a and b are preperiodic for $z^d + c$ is infinite if and only if $a^d = b^d$.*

One direction of Theorem 1.1 follows easily from Montel's theorem: if $a^d = b^d$ then a is preperiodic for $z^d + c$ if and only if b is preperiodic, and the set of complex numbers $c \in \mathbb{C}$ such that a is preperiodic for $z^d + c$ is always infinite (see §3). The reverse implication combines ideas from number theory and complex analysis, the main arithmetic ingredient being an equidistribution theorem for points of *small*

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height with respect to an *adelic height function* (see §2.3 below for details). When a and b are algebraic, the equidistribution in question takes place over $\mathbb{C}$, but in the transcendental case, we require an equidistribution theorem which takes place on the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},K}$ over some complete and algebraically closed non-archimedean field K. (The Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},K}$ is a canonical compact, Hausdorff, path-connected space containing $\mathbb{P}^1(K)$ as a dense subspace. It is for many applications the "correct" setting for non-archimedean potential theory and dynamics, see e.g. [Bak08, BR09].) The idea is to think of the field $k = \bar{\mathbb{Q}}(a) \subset \mathbb{C}$ as the function field of $\mathbb{P}^1_{\bar{\mathbb{Q}}}$, and to consider the distribution of the preperiodic parameters $c \in \bar{k}$ inside a collection of non-archimedean Berkovich analytic spaces, one for each completion of k.

Our method of proof also provides an analog of Theorem 1.1 in the dynamical plane, which we state in the more general context of rational functions.

Theorem 1.2. *Let $f, g \in \mathbb{C}(z)$ be rational functions of degrees at least 2 with Julia sets $J(f) \neq J(g)$. The set of points which are preperiodic for both f and g is finite.*

From the proof of Theorem 1.2, we obtain:

Corollary 1.3. *Let $\varphi, \psi \in \mathbb{C}(z)$ be rational functions of degree at least 2. Then* $\mathrm{Preper}(\varphi) \cap \mathrm{Preper}(\psi)$ *is infinite if and only if* $\mathrm{Preper}(\varphi) = \mathrm{Preper}(\psi)$.

When f and g are defined over $\bar{\mathbb{Q}}$, Theorem 1.2 is a special case of a result of Mimar. The transcendental case (which is again handled using Berkovich spaces) appears to be new.[1]

The statements of Theorems 1.1 and 1.2 are false if "are preperiodic for" is replaced with "are in the Julia set of". For example, for any two points $a, b \in (-2, 2)$, there is an infinite family of polynomials $z^2 + c$, with $c \in \mathbb{R}$ descending to -2, with Julia sets containing both a and b (in fact, containing a closed interval in $\mathbb{R}$ increasing to $[-2, 2]$ as $c \to -2$). These same examples show that distinct Julia sets can have infinite intersections. Because of such examples, it seems hard to prove results such as Theorems 1.1 and 1.2 using complex analysis alone, without making use of some arithmetic information about preperiodic points.

It would be interesting to obtain a generalization of Theorem 1.1 in which a, b are allowed to depend algebraically on c. (This problem was suggested by Joe Silverman.) It would also be interesting to study analogs of Theorem 1.1 for some other 1-parameter families of rational maps on $\mathbb{P}^1(\mathbb{C})$, or for a two-parameter family of rational maps in which three points a, a', a'' are required to be simultaneously preperiodic.

[1]During the final stages of preparing this article, the authors learned that Shouwu Zhang and Xinyi Yuan have recently (and independently) used similar techniques to prove a generalization of Corollary 1.3 to polarized algebraic dynamical systems of any dimension [YZ09].

1.2. **Motivation and historical background.** The motivation for Theorem 1.1 came from a topic of discussion at the AIM workshop "The uniform boundedness conjecture in arithmetic dynamics" in Palo Alto in January 2008. By analogy with questions due to David Masser concerning simultaneous torsion sections on families of elliptic curves, Umberto Zannier asked:

Question 1.4. *Is the set of complex numbers $c \in \mathbb{C}$ such that 0 and 1 are both preperiodic for $z^2 + c$ finite?*

Theorem 1.1 provides an affirmative answer to Question 1.4, in analogy with the following recent theorem of Masser and Zannier:

Theorem 1.5. [MZ08, MZ09] *The set of complex numbers $\lambda \neq 0, 1$ such that both $P_\lambda = (2, \sqrt{2(2-\lambda)})$ and $Q_\lambda = (3, \sqrt{6(3-\lambda)})$ have finite order on the Legendre elliptic curve E_λ defined by $Y^2 = X(X-1)(X-\lambda)$ is finite.*

An application of Siegel's theorem shows that there are infinitely many λ such that either P_λ or Q_λ alone has finite order; however, in each case the set of such λ is rather sparse (for example, it is countable), and imposing both torsion conditions at once makes the set of λ finite. There is nothing special about the numbers 2 and 3; Masser and Zannier have announced that they can extend the main result of [MZ08, MZ09] to arbitrary sections $P_\lambda, Q_\lambda \in E_\lambda(\bar{\mathbb{Q}})$ (under a suitable independence hypothesis), and even to $P_\lambda, Q_\lambda \in E_\lambda(\mathbb{C})$.

Both Question 1.4 and Theorem 1.5 arose from the earlier work of Bombieri, Masser, and Zannier [BMZ99] (see also [BMZ03, BMZ08]) on "unlikely intersections" between a curve embedded in an algebraic torus $G = \mathbf{G}_m^n$ and the union of all algebraic subgroups of G of codimension at least 2. The main theorem of [BMZ99] is itself a special case of general conjectures of Pink and Zilber concerning subschemes of varying semiabelian schemes.

Like Theorem 1.1, Theorem 1.2 also fits into a conceptual framework related to a larger body of literature, in this case Zhang's "dynamical Manin-Mumford conjecture"; see [GT09, Mim97, Zha06] for further discussion. Theorem 1.2 is also reminiscent of the following recent result of Ghioca, Tucker, and Zieve:

Theorem 1.6. [GTZ08] *Let $\varphi, \psi \in \mathbb{C}[z]$ be polynomials of degree at least 2, and let $a \in \mathbb{C}$. If the forward orbits of a under φ and ψ have infinite intersection, then φ and ψ have a common iterate.*

Philosophically, Theorem 1.6 is related to dynamical analogs of the Mordell conjecture, while Theorem 1.2 is closer in spirit to the Manin-Mumford conjecture. Like Ghioca, Tucker, and Zieve in [GTZ08], we use number-theoretic methods to prove theorems in complex dynamics, and we treat the algebraic and transcendental cases separately. However, our methods in the algebraic case are completely different from those of Ghioca-Tucker-Zieve, and their reduction of the transcendental case to the algebraic case is based on a more "traditional" specialization argument. It is interesting to note, though, that in both their reduction step and ours, Benedetto's theorem (Theorem 3.11 below) plays a key role. It seems difficult to prove Theorems 1.1 and 1.2 using standard specialization techniques; our use of Berkovich spaces circumvents this difficulty, and provides a new conceptual method for reducing certain questions about complex numbers to the algebraic case.

Observe that the hypothesis $J(f) \neq J(g)$ in Theorem 1.2 cannot be replaced with the weaker hypothesis that f and g do not have a common iterate, as one sees by taking polynomials $f(z) = z^2$ and $g(z) = z^3$; all points on the unit circle of the form $e^{2\pi i\theta}$ with θ rational are preperiodic for both f and g.

1.3. **Overview of the proof of Theorem 1.1.** A key role in the proof of Theorem 1.1 is played by certain generalizations of the famous *Mandelbrot set.* For $a \in \mathbb{C}$, let M_a denote the set of all $c \in \mathbb{C}$ such that a stays bounded under iteration of $z^d + c$. (When $d = 2$ and $a = 0$, M_a is just the usual Mandelbrot set.) We let μ_a denote the *equilibrium measure* on M_a relative to ∞, in the sense of complex potential theory; by classical results, μ_a is a probability measure whose support is equal to ∂M_a.

With this terminology in mind, an overview of the proof of Theorem 1.1 is as follows. Assume for the sake of contradiction that there is an infinite sequence $c_1, c_2, \ldots$ of complex numbers such that a and b are both preperiodic for $z^d + c_n$ for all n.

Case 1: a, b are *algebraic* numbers. In this case, all c_n's must also be algebraic. Let δ_n be the discrete probability measure on $\mathbb{C}$ supported equally on the Galois conjugates of c_n. Using the fact that a is preperiodic for each c_n, an arithmetic equidistribution theorem based on the product formula for number fields shows that the measures δ_n converge weakly to the equilibrium measure μ_a for M_a on $\mathbb{P}^1(\mathbb{C})$. By symmetry, the measures δ_n also converge weakly to μ_b. Thus $\mu_a = \mu_b$, which implies that $M_a = M_b$. A complex-analytic argument using Green's functions and univalent function theory then shows that $a^d = b^d$. (In the special case $a = 0$ and $b = 1$ corresponding to Question 1.4, one can show directly that $M_0 \neq M_1$; for example, $i \in M_0$ but $i \notin M_1$.)

Case 2: a is transcendental. In this case, one can show that b is also transcendental, and that a, b, and all the c_n's are defined over the algebraic closure $\bar{k}$ of $k = \bar{\mathbb{Q}}(a)$ in $\mathbb{C}$. The field k is isomorphic to the field $\bar{\mathbb{Q}}(T)$ of rational functions over the constant field $\bar{\mathbb{Q}}$, and in particular k has a (non-archimedean) product formula structure on it. An arithmetic equidistribution theorem based on the product formula for function fields, together with the assumption that a is preperiodic for each c_n, shows that for every place v of k, the v-adic analogue of the measures δ_n above converge weakly on the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},v}$ over $\mathbb{C}_v$ to a probability measure whose support is the v-adic analogue $M_{a,v} \subseteq \mathbb{P}^1_{\mathrm{Berk},v}$ of M_a. (Here $\mathbb{C}_v$ denotes the completion of an algebraic closure of the v-adic completion k_v.) By symmetry, it follows that $M_{a,v} = M_{b,v}$ for all places v of K. A theorem of Benedetto implies that for every $c \in \overline{k}$, and hence for every complex number c, a is preperiodic for $z^d + c$ if and only if b is. We conclude that $M_a = M_b$, and finish the argument as in Case 1.

2. Potential theory background

In this section we discuss some results from potential theory which are used in the rest of the paper.

2.1. **Complex potential theory.** Let E be a compact subset of $\mathbb{C}$. The *logarithmic capacity* $\gamma(E)$ of E relative to ∞ is $e^{-V(E)}$, where

$$-\log \gamma(E) = V(E) = \inf_{\nu} \iint_{E\times E} -\log|x-y|\,d\nu(x)\,d\nu(y). \tag{2.1}$$

The infimum in (2.1) is over all probability measures ν supported on E. If $\gamma(E) > 0$ (equivalently, $V(E) < \infty$), then there is a unique probability measure μ_E which achieves the infimum in (2.1), called the *equilibrium measure* for E. The support of μ_E is contained in the "outer boundary" of E, i.e., in the boundary of the unbounded component U_E of $\mathbb{C}\backslash E$.

If $\gamma(E) > 0$, the *Green's function* G_E is defined by

$$G_E(z) = V(E) + \int_E \log|z-w|\,d\mu_E(w);$$

it is a nonnegative real-valued subharmonic function on $\mathbb{C}$. The following facts are well known; we include some proofs for lack of a convenient reference.

Lemma 2.2. *Let E be a compact subset of $\mathbb{C}$ for which $\gamma(E) = e^{-V(E)} > 0$, and let U be the unbounded component of $\mathbb{C}\backslash E$. Then:*

1. *$G_E(z) = V(E) + \log|z| + o(1)$ for $|z|$ sufficiently large.*
2. *If $G : \mathbb{C} \to \mathbb{R}$ is a continuous subharmonic function which is harmonic on U, identically zero on E, and such that $G(z) - \log^+|z|$ is bounded, then $G = G_E$.*
3. *If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on $\mathbb{C}$, $\operatorname{Supp}\mu_E = \partial U$, and $G_E(z) > 0$ if and only if $z \in U$.*

Proof. Assertion (1) is [Ran95, Theorem 5.2.1].

For (2), first note that G_E is continuous at every point $q \in E$ where $G_E(q) = 0$. Indeed, G_E is upper semicontinuous and bounded below by zero, so

$$0 \le \liminf_{z\to q} G_E(z) \le \limsup_{z\to q} G_E(z) \le G_E(q) = 0. \tag{2.3}$$

By Frostman's Theorem ([Ran95, Theorem 3.3.4]), G_E is identically zero on $\mathbb{C}\backslash U$ outside a set $e \subset \partial U$ of capacity 0, and hence the same is true for $f := G_E - G$. Since G_E is continuous on $\mathbb{C}\backslash e$ and G is continuous everywhere, f is continuous outside e.

And by assumption, f is harmonic and bounded on U. By the Extended Maximum Principle [Ran95, Proposition 3.6.9], we conclude that $f \equiv 0$ on U, and hence on $\mathbb{C}\backslash e$. Thus $G_E(z) = G(z)$ for all $z \in \mathbb{C}\backslash e$. Since e has measure zero by [Ran95, Corollary 3.2.4], the generalized Laplacians $\Delta(G_E)$ and $\Delta(G)$ coincide. Since G_E and G are both subharmonic on $\mathbb{C}$, it follows from Weyl's Lemma [Ran95, Lemma 3.7.10] that f is harmonic on all of $\mathbb{C}$. Since f is also bounded, Liouville's Theorem [Ran95, Corollary 2.3.4] implies that f is identically zero. This proves (2).

The continuity assertion in (3) follows from (2.3), and the rest of (3) follows easily from the Maximum Principle. □

2.2. **Non-archimedean potential theory.** In [BR09] (see also [FRL06, Thu05]), one finds non-archimedean Berkovich space analogs of various classical results from complex potential theory, including a theory of Laplacians, harmonic functions, subharmonic functions, Green's functions, and capacities. These results closely parallel the classical theory over $\mathbb{C}$. For the reader's convenience, we give a quick summary in this section of the results from [BR09] which are used in the present paper.[2] Although this theory is used heavily in the proofs of Lemma 2.5 and Theorem 2.7, the reader who wishes to accept these results as "black boxes" does not need a detailed understanding of non-archimedean potential theory in order to understand the proof of Theorem 1.1 below.

Let K be an algebraically closed field which is complete with respect to some absolute value $|\cdot|$. The *Berkovich affine line* $\mathbb{A}^1_{\text{Berk}} = \mathbb{A}^1_{\text{Berk},K}$ over K is a locally compact, Hausdorff, path-connected space containing K (with the given metric topology) as a dense subspace. As a topological space, $\mathbb{A}^1_{\text{Berk},K}$ is the set of all multiplicative seminorms $[\cdot]_x : K[T] \to \mathbb{R}$ on the polynomial ring $K[T]$ which extend the given absolute value on K, endowed with the weakest topology for which $x \mapsto [f]_x$ is continuous for all $f \in K[T]$. The *Berkovich projective line* $\mathbb{P}^1_{\text{Berk},K}$ can be identified with the one-point compactification of $\mathbb{A}^1_{\text{Berk},K}$, with the extra point denoted ∞. It is a consequence of the Gelfand-Mazur theorem that if $K = \mathbb{C}$, then $\mathbb{A}^1_{\text{Berk},\mathbb{C}}$ is homeomorphic to $\mathbb{C}$ (and $\mathbb{P}^1_{\text{Berk}}$ is homeomorphic to the Riemann sphere $\mathbb{P}^1(\mathbb{C})$). When K is non-archimedean, however, there are lots of multiplicative seminorms $x \in \mathbb{A}^1_{\text{Berk},K}$ which do not come from evaluation at a point of K; for example, the *Gauss point* $\zeta_{\text{Gauss}} \in \mathbb{A}^1_{\text{Berk},K}$ corresponds to the seminorm $[f]_{\zeta_{\text{Gauss}}} := \sup_{z \in K, |z| \le 1} |f(z)|$.

[2] Amaury Thuillier has independently developed non-archimedean potential theory on $\mathbb{P}^1_{\text{Berk},K}$ [Thu05], and in fact his results are formulated in the context of arbitrary Berkovich curves, and without assuming that the field K is algebraically closed. Also, Charles Favre and Juan Rivera-Letelier [FRL04, FRL06] have independently developed most of the non-archimedean potential theory needed for the present applications to complex dynamics; their work relies heavily on potential theory for $\mathbb{R}$-trees as developed in the book by Favre and Jonsson [FJ04].

For the rest of this section, we assume that the absolute value on K is *non-archimedean* and non-trivial. If $z \in \mathbb{A}^1_{\text{Berk}}$, we will sometimes write $|z|$ instead of the more cumbersome $[T]_z$; the function $z \mapsto |z|$ is a natural extension of the absolute value on K to $\mathbb{A}^1_{\text{Berk}}$.

There is a canonical extension of the fundamental potential kernel $-\log|x-y|$ to $\mathbb{A}^1_{\text{Berk}}$. It can be defined as $-\log\delta(x,y)$, where $\delta(x,y)$ (called the *Hsia kernel* in [BR09]) is defined as

$$\delta(x,y) := \limsup_{\substack{z,w\in K \\ z\to x, w\to y}} |z-w|.$$

Let E be a compact subset of $\mathbb{A}^1_{\text{Berk}}$. The *logarithmic capacity* $\gamma(E)$ of E relative to ∞ is $e^{-V(E)}$, where

$$-\log\gamma(E) = \inf_{\nu} \iint_{E\times E} -\log\delta(x,y)d\nu(x)\delta\nu(y). \tag{2.4}$$

The infimum in (2.4) is over all probability measures ν supported on E. If $\gamma(E) > 0$ (equivalently, $V(E) < \infty$), there is again a unique probability measure μ_E which achieves the infimum in (2.4), called the *equilibrium measure* for E relative to ∞. The support of μ_E is contained in the outer boundary of E (the boundary of the unbounded component of $\mathbb{A}^1_{\text{Berk}}\backslash E$).

If $\gamma(E) > 0$, the *Green's function of E relative to infinity* is defined by

$$G_E(z) = V(E) + \int_E \log\delta(z,w)\, d\mu_E(w);$$

it is a nonnegative real-valued subharmonic (in the sense of [BR09, Chapter 8]) function on $\mathbb{A}^1_{\text{Berk}}$. For example, if $E = \mathcal{D}(0,1)$ is the closed unit disc in $\mathbb{A}^1_{\text{Berk}}$, defined as

$$\mathcal{D}(0,1) = \{x \in \mathbb{A}^1_{\text{Berk}} \;:\; |x| \le 1\},$$

then

$$G_E(z) = \log\max\{|z|, 1\}.$$

The following is the non-archimedean counterpart of Lemma 2.2:

Lemma 2.5. *Let E be a compact subset of $\mathbb{A}^1_{\text{Berk}}$ for which $\gamma(E) = e^{-V(E)} > 0$, and let U be the unbounded component of $\mathbb{A}^1_{\text{Berk}}\backslash E$. Then:*

1. *$G_E(z) = V(E) + \log|z|$ for all $z \in \mathbb{A}^1_{\text{Berk}}$ with $|z|$ sufficiently large.*
2. *If $G : \mathbb{A}^1_{\text{Berk}} \to \mathbb{R}$ is a continuous subharmonic function which is harmonic on U, identically zero on E, and such that $G(z) - \log^+|z|$ is bounded, then $G = G_E$.*
3. *If $G_E(z) = 0$ for all $z \in E$, then G_E is continuous on $\mathbb{A}^1_{\text{Berk}}$,* $\operatorname{Supp}\mu_E = \partial U$, *and $G_E(z) > 0$ if and only if $z \in U$.*

Proof. Assertion (1) follows from [BR09, Proposition 7.37(A7)], and (3) is [BR09, Corollary 7.39].

For (2), note that by [BR09, Proposition 7.37(A4)], G_E is identically zero on $\mathbb{A}^1_{\mathrm{Berk}} \backslash U$ outside a set $e \subset \partial U$ of capacity 0, and hence the same is true for $f := G_E - G$. Since G_E is continuous on $\mathbb{A}^1_{\mathrm{Berk}} \backslash e$ by [BR09, Proposition 7.37(A5)] and G is continuous everywhere, f is continuous outside e. And by assumption, f is harmonic and bounded on U. By the Strong Maximum Principle [BR09, Proposition 7.17], we conclude that $f \equiv 0$ on U. Thus $G_E(z) = G(z)$ for all $z \in \mathbb{A}^1_{\mathrm{Berk}} \backslash e$.

Note that G_E is subharmonic on $\mathbb{A}^1_{\mathrm{Berk}}$ by [BR09, Example 8.9] and G is subharmonic on $\mathbb{A}^1_{\mathrm{Berk}}$ by assumption. Since $e \subset \mathbb{P}^1(K)$ by [BR09, Example 6.3], and the Laplacian of a function on $\mathbb{P}^1_{\mathrm{Berk}}$ depends only on its restriction to $\mathbb{P}^1_{\mathrm{Berk}} \backslash \mathbb{P}^1(K)$ (see [BR09, Remark 5.12]), we have $\Delta_{\mathbb{A}^1_{\mathrm{Berk}}}(G_E) = \Delta_{\mathbb{A}^1_{\mathrm{Berk}}}(G)$. Since G_E and G are both subharmonic on $\mathbb{A}^1_{\mathrm{Berk}}$, have the same Laplacian, and agree on $\mathbb{A}^1_{\mathrm{Berk}} \backslash K$, it follows from [BR09, Corollary 8.37] that $G = G_E$ on $\mathbb{A}^1_{\mathrm{Berk}}$. □

2.3. **Adelic equidistribution of small points.** In this section, we state the arithmetic equidistribution result needed for our proof of Theorem 1.1. In order to state the result (Theorem 2.7 below), we first need some definitions.

Definition. A *product formula field* is a field k, together with the following extra data:

(1) a set $\mathcal{M}_k$ of non-trivial absolute values on k (which we may assume to be pairwise inequivalent), and
(2) for each $v \in \mathcal{M}_k$, an integer $N_v \geq 1$

such that

(3) for each $\alpha \in k^\times$, we have $|\alpha|_v = 1$ for all but finitely many $v \in \mathcal{M}_k$, and
(4) every $\alpha \in k^\times$ satisfies the *product formula*

$$\prod_{v \in \mathcal{M}_k} |\alpha|_v^{N_v} \;=\; 1.$$

The most important examples of product formula fields are number fields and function fields of normal projective varieties (see [Lan83, §2.3] or [BG06, §1.4.6]). It is known (see [Art06, Chapter 12, Theorem 3]) that a product formula field for which at least one $v \in \mathcal{M}_k$ is archimedean must be a number field. If all $v \in \mathcal{M}_k$ are non-archimedean, then we define the *constant field* k_0 of k to be the set of all $\alpha \in k$ such that $|\alpha|_v \leq 1$ for all $v \in \mathcal{M}_k$. By the product formula, if $\alpha \in k_0$ is nonzero then in fact $|\alpha|_v = 1$ for all $v \in \mathcal{M}_k$. Any finitely generated extension k of an algebraically closed field k_0 can be endowed with a product formula structure in such a way that the field of constants of k is k_0 (cf. [BG06, Lemma 1.4.10]).

For simplicity, we will assume throughout this paper that our product formula fields have characteristic zero (since this is the only case needed for our applications). However, the equidistribution theorems 2.7 and 6.4 below are proved in [BR09] without this assumption.

Let k be a product formula field of characteristic zero, and let $\bar{k}$ denote a fixed algebraic closure of k. For $v \in \mathcal{M}_k$, let k_v be the completion of k at v, let $\bar{k}_v$ be an algebraic closure of k_v, and let $\mathbb{C}_v$ denote the completion of $\bar{k}_v$. For each $v \in \mathcal{M}_k$, we fix an embedding of $\bar{k}$ in $\mathbb{C}_v$ extending the canonical embedding of k in k_v, and view this embedding as an identification. By the discussion above, if v is archimedean then $\mathbb{C}_v \cong \mathbb{C}$. For each $v \in \mathcal{M}_k$, we let $\mathbb{P}^1_{\mathrm{Berk},v}$ denote the Berkovich projective line over $\mathbb{C}_v$, which we take to mean $\mathbb{P}^1(\mathbb{C})$ if v is archimedean.

A *compact Berkovich adelic set* (relative to ∞) is a set of the form

$$\mathbb{E} \;=\; \prod_v E_v$$

where E_v is a nonempty compact subset of $\mathbb{A}^1_{\mathrm{Berk},v} = \mathbb{P}^1_{\mathrm{Berk},v} \backslash \{\infty\}$ for each $v \in \mathcal{M}_k$, and where E_v is the closed unit disc $\mathcal{D}(0,1)$ in $\mathbb{A}^1_{\mathrm{Berk},v}$ for all but finitely many nonarchimedean $v \in \mathcal{M}_k$.

For each $v \in \mathcal{M}_k$, let $\gamma(E_v)$ be the logarithmic capacity of E_v relative to ∞; see (2.1) and (2.4). The *logarithmic capacity* (relative to ∞) of a compact Berkovich adelic set $\mathbb{E}$, denoted $\gamma(\mathbb{E})$, is

$$\gamma(\mathbb{E}) \;=\; \prod_v \gamma(E_v)^{N_v} \;.$$

We will assume throughout the rest of this section that $\gamma(\mathbb{E}) \neq 0$, i.e., that $\gamma(E_v) > 0$ for all $v \in \mathcal{M}_k$.

For each $v \in \mathcal{M}_k$, let $G_v : \mathbb{A}^1_{\mathrm{Berk},v} \to \mathbb{R}$ be the Green's function for E_v relative to ∞, i.e., $G_v(z) = G_{E_v}(z)$. If $S \subset \bar{k}$ is any finite set invariant under $\mathrm{Gal}(\bar{k}/k)$, we define the *height of S relative to* $\mathbb{E}$, denoted $h_{\mathbb{E}}(S)$, by

$$h_{\mathbb{E}}(S) \;=\; \sum_{v \in \mathcal{M}_k} N_v \left(\frac{1}{|S|} \sum_{z \in S} G_v(z) \right). \tag{2.6}$$

By Galois-invariance, the sum $\sum_{z \in S} G_v(z)$ does not depend on our choice of an embedding of $\bar{k}$ into $\mathbb{C}_v$.

If $z \in \bar{k}$, let $S_k(z) = \{z_1, \ldots, z_n\}$ denote the set of $\mathrm{Gal}(\bar{k}/k)$-conjugates of z over k, where $n = [k(z) : k]$. We define a function $h_{\mathbb{E}} : \bar{k} \to \mathbb{R}_{\geq 0}$ by setting $h_{\mathbb{E}}(z) = h_{\mathbb{E}}(S_k(z))$. If $E_v = \mathcal{D}(0,1)$ for all $v \in \mathcal{M}_k$, then $G_v(z) = \log_v^+ |z|_v$ for all $v \in \mathcal{M}_k$ and all $z \in \bar{k}$, and $h_{\mathbb{E}}$ coincides with the *standard logarithmic Weil height* h on $\bar{k}$.

Finally, we let μ_v denote the equilibrium measure for E_v relative to ∞. We can now state the needed equidistribution result from [BR09]:

Theorem 2.7. *Let k be a product formula field of characteristic zero, and let $\mathbb{E}$ be a compact Berkovich adelic set with $\gamma(\mathbb{E}) = 1$. Suppose S_n is a sequence of $\mathrm{Gal}(\bar{k}/k)$-invariant finite subsets of $\bar{k}$ with $|S_n| \to \infty$ and $h_{\mathbb{E}}(S_n) \to 0$. Fix $v \in \mathcal{M}_k$, and for each n let δ_n be the discrete probability measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ supported equally on the elements of S_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v on $\mathbb{P}^1_{\mathrm{Berk},v}$.*

Remark 2.8. A slightly more general version of Theorem 2.7 is proved in [BR09, Theorem 7.52] without the assumption that k has characteristic zero. When k is a number field, a slightly weaker version of Theorem 2.7 is proved in [BR06] and a stronger version (which also generalizes Theorem 6.4 below) is proved in [FRL06].

For concreteness, we explicitly state the special case of Theorem 2.7 in which k is a number field and S_n is the $\mathrm{Gal}(\bar{k}/k)$-orbit of a point $z_n \in \bar{k}$ (note in this case that $h_{\mathbb{E}}(z_n) \to 0$ implies $|S_n| \to \infty$ by Northcott's theorem):

Corollary 2.9. *Let k be a number field, and let $\mathbb{E}$ be a compact Berkovich adelic set with $\gamma(\mathbb{E}) = 1$. Suppose $\{z_n\}$ is a sequence of distinct points of $\bar{k}$ with $h_{\mathbb{E}}(z_n) \to 0$. Fix a place v of k, and for each n let δ_n be the discrete probability measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ supported equally on the $\mathrm{Gal}(\bar{k}/k)$-conjugates of z_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to μ_v on $\mathbb{P}^1_{\mathrm{Berk},v}$.*

Remark 2.10. When $k = \mathbb{Q}$ and $\mathbb{E}$ is the *trivial* Berkovich adelic set (i.e., E_v is the v-adic unit disc for all v), Corollary 2.9 is Bilu's equidistribution theorem [Bil97] for v archimedean, and it is Chambert-Loir's generalization of Bilu's theorem [CL06] for v non-archimedean.

Remark 2.11. If k is a number field and $\gamma(\mathbb{E}) < 1$, there are only finitely many $z \in \bar{k}$ with $h_{\mathbb{E}}(z) = 0$; this follows from the adelic version of the Fekete-Szegö theorem proved in [BR09, Theorem 6.28]. This observation helps explain the role played by the condition $\gamma(\mathbb{E}) = 1$ in Theorem 2.7 and Corollary 2.9.

3. Generalized Mandelbrot sets

Let K be an algebraically closed field which is complete with respect to a nontrivial (archimedean or nonarchimedean) absolute value. Fix an integer $d \geq 2$, and for $c \in K$ let $f_c(z) = z^d + c$. We denote by $f_c^{(n)}(z)$ the n^{th} iterate of $f_c(z)$. In this section, we introduce a family of generalized Mandelbrot sets, defined as the set of parameters c for which a given point $z = a$ remains bounded under iteration.

3.1. **The archimedean case.** If $K = \mathbb{C}$, define the *generalized Mandelbrot set* M_a for $a \in \mathbb{C}$ by

$$M_a = \left\{ c \in \mathbb{C} \ : \ \sup_n |f_c^{(n)}(a)| < \infty \right\}. \tag{3.1}$$

When $d = 2$ and $a = 0$, M_a is the usual Mandelbrot set. It is clear that every parameter $c \in K$ for which a is preperiodic for $z^d + c$ is contained in M_a.

We need some basic potential-theoretic properties of M_a. The proofs follow the same reasoning as for the Mandelbrot set, but we provide some details for the reader's convenience. Recall that for each fixed f_c, the Green's function for the filled Julia set

$$K_c = \left\{ z : \sup_n |f_c^{(n)}(z)| < \infty \right\}$$

of f_c is given by the *escape rate*

$$G_c(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f_c^{(n)}(z)|.$$

These escape-rate functions are continuous in both c and z, and $G_c(z) = 0$ if and only if $z \in K_c$. Furthermore, there is an analytic homeomorphism ϕ_c defined in a neighborhood of ∞ which satisfies $\phi_c(f_c(z)) = (\phi_c(z))^d$ and $G_c(z) = \log |\phi_c(z)|$. The conjugating map ϕ_c is uniquely determined if we require that it has derivative 1 at infinity. See for example [DH84] or [CG93].

Lemma 3.2. *For each fixed $a \in \mathbb{C}$, we have $G_c(a^d + c) > G_c(0)$ for all c sufficiently large. Consequently, the value $f_c(a)$ lies in the domain of the conjugating isomorphism ϕ_c.*

Proof. The proof relies on a standard distortion theorem for univalent functions; see [BH88, Corollary 3.3] or [Rud87, Theorem 14.14]. The main observation is that $\phi_c(z) = z + O(1/z)$ for z near infinity; see [BH88, §3]. Let U_R denote the domain $\{|z| > R\}$ in the complex plane. Setting $R_c = e^{G_c(0)}$, we find that $\phi_c^{-1}(U_{R_c}) \supset U_{2R_c}$ because ϕ_c^{-1} is univalent with derivative 1 at infinity and constant term 0. In particular, the set U_{2R_c} is in the domain of ϕ_c, and therefore the critical point $z = 0$ and all of its preimages $(-c)^{1/d}$ lie in the closed disk of radius $2R_c$. Thus, $|c| \leq 2^d R_c^d$. This implies that $R_c \to \infty$ as $c \to \infty$.

Note that $|\phi_c(c)| = R_c^d$. When c is large enough so that $R_c^d/2 > 2R_c$, we apply the same distortion estimate to conclude that $\phi_c(U_{R_c^d/2}) \supset U_{R_c^d}$, so $|c| \geq R_c^d/2$. It follows that for any fixed a, since $R_c \to \infty$ with c, we have $|a^d + c| \geq R_c^d/2 - |a|^d > 2R_c$ for all sufficiently large c. That is, the value $f_c(a) = a^d + c$ lies in the domain of ϕ_c and has escape rate $G_c(a^d + c) > G_c(0)$. □

Proposition 3.3. *For each $a \in \mathbb{C}$, the generalized Mandelbrot set M_a satisfies:*

(1) *M_a is a compact and full subset of $\mathbb{C}$;*

(2) *the function* $G_a(c) := G_c(a^d + c)$ *defines the Green's function for* M_a *and satisfies* $G_a(c) = 0$ *for all* $c \in M_a$*;*
(3) *the function* $\Phi_a(c) := \phi_c(a^d + c)$ *defines a conformal isomorphism in a neighborhood of infinity, and it is uniquely determined by the conditions* $G_a(c) = \log|\Phi_a(c)|$ *and* $\Phi_a'(\infty) = 1$*;*
(4) *the logarithmic capacity is* $\gamma(M_a) = 1$*; and*
(5) *the support of the equilibrium measure* μ_a *on* M_a *is equal to the boundary* ∂M_a*.*

Proof. The set M_a is closed because $M_a = \{c : G_c(a) = 0\}$ and $(c, z) \mapsto G_c(z)$ is continuous. It is bounded by Lemma 3.2: for all sufficiently large c, the escape rate of a is positive and therefore $f_c^{(n)}(a) \to \infty$. The maximum modulus principle implies that M_a is full (meaning that its complement is connected), completing the proof of statement (1).

The conjugating isomorphisms ϕ_c satisfy

$$\phi_c(z) = z \prod_{n=0}^{\infty} \left(1 + \frac{c}{(f_c^{(n)}(z))^d}\right)^{1/d^{n+1}}$$

on their domains $\{z : G_c(z) > G_c(0)\}$. By Lemma 3.2, the function $\Phi_a(c)/c = \phi_c(a^d + c)/c$ can be expressed by this infinite product for c near infinity. The terms in the infinite product each tend to 1 as $c \to \infty$, so (setting $\Phi_a(\infty) = \infty$) we conclude $\Phi_a'(\infty) = 1$. In particular, Φ_a defines a conformal isomorphism in a neighborhood of infinity.

The Green's function G_c for the filled Julia set K_c satisfies $G_c(z) = \log|\phi_c(z)|$ where defined. The function $G_a(c) = G_c(a^d + c)$ therefore satisfies $G_a(c) = \log|\Phi_a(c)| = \log|c + O(1)| = \log|c| + o(1)$ for all c large. Furthermore, G_a is harmonic on $\mathbb{C} \setminus M_a$, as a locally uniform limit of the harmonic functions $c \mapsto G_n(c) = d^{-n} \log|f_c^{(n)}(a^d + c)|$, and $G_a(c) = 0$ if and only if $c \in M_a$; we conclude that G_a is the Green's function for M_a. The conditions stated in (3) clearly determine Φ_a uniquely near ∞. Statement (4) follows because $G_a(c) = \log|c| + o(1)$ near infinity.

Finally, statement (5) follows from Lemma 2.2, because M_a is full. □

Fix a degree $d \geq 2$. For each $a \in \mathbb{C}$, define

$$\text{Preper}(a) := \{c \in \mathbb{C} \;:\; a \text{ is preperiodic for } z^d + c\}.$$

Combining Proposition 3.3 with Montel's theorem, we obtain:

Theorem 3.4. *For each degree* $d \geq 2$ *and any* $a, b \in \mathbb{C}$*, the following are equivalent:*

(1) $M_a = M_b$
(2) $a^d = b^d$
(3) $\text{Preper}(a) = \text{Preper}(b)$.

Proof. First suppose that $a^d = b^d$. Then for every c, we have $f_c(a) = a^d + c = b^d + c = f_c(b)$, so a is preperiodic for f_c if and only if b is preperiodic for f_c. Thus (2) implies (3).

Now assume (3) and consider the sequence of functions $g_n(c) := f_c^{(n)}(a)$. This sequence forms a normal family except on the boundary ∂M_a. Consider the set $\{a, a^d + c\}$. First note that $a^d + c = a$ implies that a is a fixed point for f_c, so $c \in \text{Preper}(a) \subset M_a$. Now fix an open set U intersecting ∂M_a which does not contain the parameter $c = a - a^d$. Then by Montel's theorem, the union of images $g_n(U)$ must intersect the set $\{a, a^d + c\}$. In particular, there is an iterate n so that either $f_c^{(n)}(a) = a$ or $f_c^{(n)}(a) = f_c(a)$; in either case, the set U must intersect Preper(a). Consequently, the boundary ∂M_a is contained in the closure of Preper(a). As M_a is a full set by Proposition 3.3 (1), it is determined by its boundary. Therefore, Preper(a) = Preper(b) implies that $M_a = M_b$.

Finally, assume that $M_a = M_b$. Then by Proposition 3.3 (3) the uniformizing maps $\Phi_a(c)$ and $\Phi_b(c)$ coincide on a neighborhood of infinity. In other words, for all large c, we have $\phi_c(a^d + c) = \phi_c(b^d + c)$. The conjugating isomorphisms ϕ_c are injective, so we conclude that $a^d + c = b^d + c$. □

The following simple statement is used for one implication of Theorem 1.1.

Lemma 3.5. *For each $a \in \mathbb{C}$, the set* Preper(a) *is infinite.*

Proof. From Proposition 3.3, the set M_a has capacity 1, so its boundary cannot be a finite set. From the proof of Theorem 3.4, the boundary ∂M_a is contained in the closure of Preper(a), so the set Preper(a) must contain infinitely many points. □

Note that if $a \in \bar{\mathbb{Q}}$, then the set Preper(a) is a subset of $\bar{\mathbb{Q}}$ with bounded Weil height (since $h_{\mathbf{M}_a}(c) = 0$ for all $c \in \text{Preper}(a)$ and the difference $h - h_{\mathbf{M}_a}$ is bounded). It is thus a rather "sparse" set (compare with the discussion following Theorem 1.5 above).

3.2. **The non-archimedean case.** If K is a non-archimedean field, one can define $M_a \subset \mathbb{A}^1_{\text{Berk},K}$ similarly and prove basic potential-theoretic statements about M_a. As the arguments are very similar to the archimedean case, we omit some of the details.

Let $g_n(T) = f_T^{(n)}(a)$; this is a monic polynomial in T of degree d^{n-1} which depends on $a \in K$. Define

$$M_a := \left\{ c \in \mathbb{A}^1_{\text{Berk},K} \ : \ \sup_n \, [g_n(T)]_c < \infty \right\}, \tag{3.6}$$

where $[\cdot]_c$ is the multiplicative seminorm on $K[T]$ corresponding to $c \in \mathbb{A}^1_{\text{Berk},K}$. (Note that for $c \in K$, we have $[g_n(T)]_c = |g_n(c)| = |f_c^{(n)}(a)|$.)

Proposition 3.7. *For each $a \in K$,*

(1) *the boundary of M_a coincides with the outer boundary in $\mathbb{A}^1_{\mathrm{Berk},K}$, and it is equal to the support of μ_{M_a};*
(2) *the logarithmic capacity $\gamma(M_a)$ is equal to 1; and*
(3) *the Green's function for M_a relative to ∞ is 0 at all points of M_a.*

Proof. Fix $a \in K$, and for $c \in \mathbb{A}^1_{\mathrm{Berk}}$ define

$$G_a(c) := \lim_{n\to\infty} \frac{1}{d^n} \log^+ [g_{(n+1)}(T)]_c. \tag{3.8}$$

Note that the limit in (3.8) exists for all $c \in \mathbb{A}^1_{\mathrm{Berk}}$: if $c \in M_a$, then the limit is zero, while if $c \notin M_a$, then the sequence $\frac{1}{d^n} \log^+[g_{(n+1)}(T)]_c$ is eventually constant (since the ultrametric inequality implies that if $z \in K$ and $|z|^d > |c|$ then $[T + z^d]_c = |z|^d$). Note also that for $c \in K$, we have

$$G_a(c) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f_c^{(n+1)}(a)| = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f_c^{(n)}(a^d + c)|,$$

which is the same formula we used to define G_a over $\mathbb{C}$.

Claim 1: $\frac{1}{d^n} \log^+[g_{(n+1)}(T)]_c$ converges to $G_a(c)$ uniformly on compact subsets of $\mathbb{A}^1_{\mathrm{Berk}}$ (as functions of c).

To see this, one can employ essentially the same argument as in the archimedean case ([BH88, Proposition 1.2]; compare with [BR09, §10.1]). Briefly, fix a compact set $E \subset \mathbb{A}^1_{\mathrm{Berk}}$. Then there is a constant $C > 0$, depending only on E, such that $[T + z^d]_c = |z|^d$ for $c \in E$ and $|z| \geq C$. Thus there are constants $C_1, C_2 > 0$ (depending only on E) such that for $z \in K$ and $c \in E$,

$$C_1 \max(1, |z|^d) \leq \max(1, [T + z^d]_c) \leq C_2 \max(1, |z|^d).$$

Taking logarithms and iterating shows that there is a constant $C' > 0$ (depending only on E) such that for each fixed $a \in K$,

$$\left| \frac{1}{d^n} \log^+[g_n(T)]_c - \frac{1}{d^{n-1}} \log^+[g_{n-1}(T)]_c \right| \leq \frac{C'}{d^n}.$$

A telescoping series argument now gives the desired uniform convergence on E, proving Claim 1.

Claim 2: G_a is the Green's function for M_a relative to ∞.

Indeed, G_a is harmonic on $U_a := \mathbb{A}^1_{\mathrm{Berk}} \backslash M_a$ by [BR09, Example 7.5] and [BR09, Proposition 7.31], since on U_a the function G_a is the limit of the harmonic functions $\frac{1}{d^n} \log[g_{(n+1)}(T)]_c$. Moreover, since the sequence of continuous subharmonic functions $\frac{1}{d^n} \log^+[g_{(n+1)}(T)]_c$ converges uniformly to G_a on compact subsets of $\mathbb{A}^1_{\mathrm{Berk}}$, it follows from [BR09, Proposition 8.26(C)] that G_a is continuous and subharmonic on $\mathbb{A}^1_{\mathrm{Berk}}$. In addition, G_a is zero on M_a, and for $|c| > \max\{1, |a|^d\}$ we have $G_a(c) = \log^+ |c|$. Claim 2 therefore follows from part (2) of Lemma 2.5.

Assertion (3) is now immediate, and assertions (1) and (2) follow from parts (3) and (1) of Lemma 2.5, respectively. □

3.3. **Global generalized Mandelbrot sets.** Let k be a product formula field, and fix $a \in k$. For each $v \in \mathcal{M}_k$, define $M_{a,v} \subseteq \mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v}$ following the local recipes above. Recall that $\mathbb{A}^1_{\mathrm{Berk},\mathbb{C}_v} = \mathbb{C}$ if v is archimedean. Define a compact Berkovich adelic set $\mathbf{M}_a$ by

$$\mathbf{M}_a := \{M_{a,v}\},$$

observing that $M_{a,v} = \mathcal{D}(0,1)$ whenever $|a|_v \le 1$. Propositions 3.3 (4) and 3.7 (2) imply that the global capacity $\gamma(\mathbf{M}_a)$ is equal to 1. Moreover, for each $v \in \mathcal{M}_k$ the local Green's function $G_{M_{a,v}} : \mathbb{A}^1_{\mathrm{Berk},v} \to \mathbb{R}_{\ge 0}$ is continuous, with $G_{M_{a,v}}(z) = 0$ if and only if $z \in M_{a,v}$.

If $S \subset \bar{k}$ is any finite set invariant under $\mathrm{Gal}(\bar{k}/k)$, then following (2.6) the height of S relative to $\mathbf{M}_a$ is given by

$$h_{\mathbf{M}_a}(S) \;=\; \sum_{v \in \mathcal{M}_k} N_v \left(\frac{1}{|S|} \sum_{z \in S} G_{M_{a,v}}(z) \right) . \tag{3.9}$$

Remark 3.10. The adelic height function attached to the usual Mandelbrot set appeared previously in [BH05] and [FRL06].

3.4. **The function field setting.** For later use, we recall a result of Benedetto and note its relevant consequences. By an *abstract function field*, we mean a product formula field k for which all $v \in \mathcal{M}_k$ are non-archimedean. A polynomial $\varphi \in k[T]$ is called *isotrivial over* k if it is conjugate (by an invertible linear map $T \mapsto \alpha T + \beta$ defined over k) to a polynomial defined over the constant field of k.

Theorem 3.11. [Ben05] *Let k be an abstract function field. If $\varphi \in k[T]$ is not isotrivial over k, then $a \in \bar{k}$ is preperiodic for φ if and only if a belongs to the v-adic filled Julia set of φ (i.e., a stays v-adically bounded under iteration of φ) for all $v \in \mathcal{M}_k$.*

Remark 3.12. If k is a number field, then it is well known and follows easy from Northcott's theorem that $a \in \bar{k}$ is preperiodic for φ if and only if a belongs to the v-adic filled Julia set of φ for all $v \in \mathcal{M}_k$. But if k is an abstract function field and $\varphi \in k[T]$ is isotrivial over k, then it is easy to see that the conclusion of Theorem 3.11 fails, since every element of the constant field k_0 of k stays v-adically bounded for all v but not every element of k_0 is preperiodic.

Corollary 3.13. *Let k be an abstract function field such that every field k_v for $v \in \mathcal{M}_k$ has residue characteristic zero, and fix $a, c \in k$ with c not in the constant field k_0 of k. Then the following are equivalent:*

(1) *a is preperiodic for the iteration of $f_c(z) = z^d + c$.*
(2) *c is contained in $M_{a,v}$ for all $v \in \mathcal{M}_k$.*
(3) *$h_{\mathbf{M}_a}(c) = 0$.*

Proof. By definition, we have $h_{\mathbf{M}_a}(c) = 0$ if and only if c is contained in $M_{a,v}$ for all $v \in \mathcal{M}_k$. We claim that $z^d + c$ is isotrivial if and only if $c \in k_0$. Thus $f_c(z) = z^d + c$ is not isotrivial by assumption, and Benedetto's theorem implies that a is preperiodic for f_c if and only if a belongs to the v-adic filled Julia set of f_c for all $v \in \mathcal{M}_k$. The desired result follows, since by definition, a belongs to the v-adic filled Julia set of f_c if and only if $c \in M_{a,v}$.

To prove the claim, suppose for the sake of contradiction that $c \notin k_0$ and $\alpha z + \beta$ conjugates $z^d + c$ into a polynomial defined over k_0. Then

$$\frac{1}{\alpha}(\alpha z + \beta)^d + \frac{1}{\alpha}(c - \beta) = \alpha^{d-1} z^d + \cdots + d\beta^{d-1} z + \frac{c + \beta^d - \beta}{\alpha} \in k_0[z]. \tag{3.14}$$

Since $c \notin k_0$, there exists $v \in \mathcal{M}_k$ such that $|c|_v > 1$. By (3.14), $|\alpha|_v = 1$ and (since k_v has residue characteristic zero) $|d\beta^{d-1}|_v = |\beta|_v^{d-1} \leq 1$, hence $|\beta|_v \leq 1$. Thus $|c + \beta^d - \beta|_v > 1$ by the ultrametric inequality, contradicting (3.14). ☐

4. Proof of Theorem 1.1

Proof of Theorem 1.1. First suppose that $a^d = b^d$. Then a is preperiodic for f_c if and only if b is preperiodic for f_c. From Lemma 3.5, the set of parameters c for which a is preperiodic is infinite.

Now fix a and b in $\mathbb{C}$, and assume that there is an infinite sequence $c_1, c_2, \ldots$ of distinct complex numbers such that a and b are both preperiodic for $z^d + c_n$ for all n.

Case 1: $a, b \in \bar{\mathbb{Q}}$.

In this case, c_n must be algebraic for all n. Indeed, let $g_m(c) = f_c^{(m)}(a)$; this is a monic polynomial in c of degree d^{m-1} with coefficients in the number field $k := \mathbb{Q}(a) \subset \bar{\mathbb{Q}}$. Since a is preperiodic for $f_{c_n}(z)$ $(n = 1, 2, \ldots)$, there exist integers $\ell > m \geq 1$ (depending on n) such that $g_\ell(c_n) = g_m(c_n)$. Thus c_n is a root of the nonzero polynomial $g_\ell(z) - g_m(z) \in \bar{\mathbb{Q}}[z]$, and hence $c_n \in \bar{k}$ for all n. Further, we see that a is also preperiodic for all $\mathrm{Gal}(\bar{k}/k)$-conjugates of c_n, and we deduce that $h_{\mathbf{M}_a}(c_n) = 0$ for all n.

Let δ_n be the discrete probability measure on $\mathbb{C}$ supported equally on the $\mathrm{Gal}(\bar{k}/k)$-conjugates of c_n. By Corollary 2.9 the measures δ_n converge weakly to the probability measure μ_{M_a} (the equilibrium measure relative to ∞ for the set M_a) on $\mathbb{P}^1(\mathbb{C})$. By symmetry, the measures δ_n also converge weakly to μ_{M_b}. Thus $\mu_{M_a} = \mu_{M_b}$. By Proposition 3.3, the support of μ_{M_a} (respectively μ_{M_b}) is precisely ∂M_a (resp. ∂M_b), and therefore $M_a = M_b$. By Theorem 3.4, we conclude that $a^d = b^d$.

Case 2: a is transcendental.

In this case, b is also transcendental, as otherwise each c_n would be algebraic, contradicting the transcendence of a. In fact, the values a, b, and c_n for all n are defined over the algebraic closure $\bar{k}$ of $k = \bar{\mathbb{Q}}(a)$ in $\mathbb{C}$. Indeed, for each n there exist integers $\ell > m \geq 1$ such that c_n is a root of the nonzero polynomial $g_\ell(z) - g_m(z) \in k[z]$, and hence $c_n \in \bar{k}$ for all n. Moreover (setting $c = c_n$ for any n), there exist $\ell > m \geq 1$ such that b is a root of the nonzero polynomial $f_c^{(\ell)}(z) - f_c^{(m)}(z) \in \bar{k}[z]$, and hence $b \in \bar{k}$ as well.

Since a is transcendental, the field $k = \bar{\mathbb{Q}}(a)$ is isomorphic to the field $\bar{\mathbb{Q}}(T)$ of rational functions over $\bar{\mathbb{Q}}$, and in particular k can be viewed as a product formula field with $\bar{\mathbb{Q}}$ as its field of constants. Since a is preperiodic for $f_{c_n}(z)$, we have $h_{\mathbf{M}_a}(c_n) = 0$ for all n. Fix a place $v \in \mathcal{M}_k$, let $\mathbb{C}_v$ be the completion of an algebraic closure of the v-adic completion k_v, and identify $\bar{k}$ with a subfield of $\mathbb{C}_v$. Let T_m be the set of $\mathrm{Gal}(\bar{k}/k)$-conjugates of $c_m \in \bar{k}$, and define

$$S_n = \bigcup_{m=1}^{n} T_m.$$

Then S_n is a $\mathrm{Gal}(\bar{k}/k)$-stable subset of $\bar{k}$, $h_{\mathbf{M}_a}(c) = 0$ for every $c \in S_n$, and $|S_n| \to \infty$ as $n \to \infty$. Let δ_n be the discrete probability measure on the Berkovich projective line $\mathbb{P}^1_{\mathrm{Berk},v}$ over $\mathbb{C}_v$ supported equally on the elements of S_n. Let $M_{a,v} \subset \mathbb{P}^1_{\mathrm{Berk},v}$ be the v-adic generalized Mandelbrot set corresponding to a (cf. (3.6)). By Theorem 2.7, the sequence δ_n converges weakly on $\mathbb{P}^1_{\mathrm{Berk},v}$ to the equilibrium measure $\mu_{M_{a,v}}$ for $M_{a,v}$ relative to ∞. Moreover, by Proposition 3.7, the support of $\mu_{M_{a,v}}$ is equal to $\partial M_{a,v}$.

Applying the same reasoning to b, it follows by symmetry that $M_{a,v} = M_{b,v}$ for all places v of k. Hence, by Corollary 3.13, for each fixed $c \in \bar{k}$, a is preperiodic for $z^d + c$ if and only if b is preperiodic. Recall from the discussion at the beginning of Case 2 that if $c \in \mathbb{C}$ and a is preperiodic for $z^d + c$, then $c \in \bar{k}$. It follows that for every complex number c, a is preperiodic for $z^d + c$ if and only if b is. Theorem 3.4 then implies that $a^d = b^d$, completing the proof of the theorem. □

In the case where $a, b \in \bar{\mathbb{Q}}$, the proof of Theorem 1.1 actually yields the following stronger result:

Theorem 4.1. *Let $a, b \in \bar{\mathbb{Q}}$ with $a^d \neq b^d$. Then there is a real number $\varepsilon > 0$ such that $h_{\mathbf{M}_a}(c) + h_{\mathbf{M}_b}(c) \geq \varepsilon$ for all but finitely many $c \in \bar{\mathbb{Q}}$.*

5. Effective bounds

As it stands, the proof of Theorem 1.1 is not effective. In particular, recall the original question of Zannier which motivated the present paper:

Question 5.1. *Is the set $S_{0,1}$ of complex numbers $c \in \mathbb{C}$ such that 0 and 1 are both preperiodic for $z^2 + c$ finite?*

By Theorem 1.1, the answer to this question is yes; however, we do not know which parameters the set $S_{0,1}$ contains. It is easily checked that 0 and 1 are both preperiodic for $z^2 + c$ when $c \in \{0, -1, -2\}$. A straightforward computation with Mathematica shows that there are no other values of c for which $f_c^{(\ell_0)}(0) = f_c^{(m_0)}(0)$ and $f_c^{(\ell_1)}(1) = f_c^{(m_1)}(1)$ with $0 \leq m_i < \ell_i < 15$ for $i = 0, 1$ and $\ell_0 + \ell_1 < 20$. Figure 5.1 (generated by Daniel Connelly, an undergraduate student of the first author) illustrates the shape of M_1 and how it compares with the Mandelbrot set M_0. Note that the the sets M_0 and M_1, and even their boundaries, have considerable overlap. Nevertheless, we believe:

Conjecture 5.2. $S_{0,1} = \{0, -1, -2\}$.

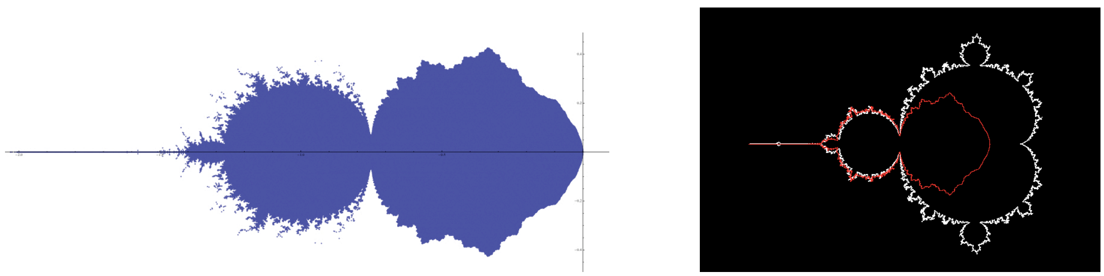

Figure 5.1. At left, a sketch of the set M_1. The full extent of M_1 is not visible, due to the crude algorithm used to draw the picture; for example, $a = 1$ is periodic for $f_{-3}(z) = z^2 - 3$ which does not appear in this figure. At right, M_1 is superimposed over the Mandelbrot set.

It would be interesting to study the corresponding sets $S_{a,b}$ for arbitrary $a, b \in \mathbb{C}$ with $a \neq \pm b$. An examination of the sets M_a and M_b and their Green's functions may give clues to the structure of $S_{a,b}$.

One could also ask for an explicit value of $\varepsilon > 0$ such that $h_{\mathbf{M}_0}(c) + h_{\mathbf{M}_1}(c) \geq \varepsilon$ for all $c \in \bar{\mathbb{Q}}$ outside an explicit finite set (as in Theorem 4.1). In the case $a = 0$ and $b = 1$, Theorem 4.1 is reminiscent of the following consequence of the Bogomolov conjecture for curves in algebraic tori (proved by Shouwu Zhang in [Zha92]): There is a real number $\varepsilon > 0$ such that $h(\alpha) + h(1-\alpha) \geq \varepsilon$ for all but finitely many $\alpha \in \bar{\mathbb{Q}}^*$ (where h denotes the usual logarithmic Weil height). Using elementary but clever observations from complex potential theory, Don Zagier proved the following quantitative version of Zhang's result in [Zag93]: For all $\alpha \in \bar{\mathbb{Q}}^*$ not equal to a primitive sixth root of unity, $h(\alpha) + h(1 - \alpha) \geq \frac{1}{2} \log \frac{-1+\sqrt{5}}{2}$, with equality if and only if α is a primitive tenth root of unity.

The existence of such an $\varepsilon > 0$ (though not the optimal value of ε computed by Zagier) can be deduced directly from Bilu's equidistribution theorem. Indeed, if there were an infinite sequence of $\alpha_n \in \bar{\mathbb{Q}}^*$ with $h(\alpha_n) \to 0$ and $h(1 - \alpha_n) \to 0$, then by Bilu's theorem the sequence δ_n of probability measures supported equally on the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$-conjugates of α_n would have to be equidistributed with respect to both the uniform measure on the unit circle $|z| = 1$ and the uniform measure on the circle $|z-1| = 1$, a contradiction. (This argument is a simplified model of our proof of the algebraic case of Theorem 1.1.) Note that the α's for which α and $1-\alpha$ both have height zero (i.e., are roots of unity) are precisely the primitive sixth roots of unity, which are the two points where the complex circles $|z| = 1$ and $|z-1| = 1$ intersect.

6. A variant of Theorem 1.1

Our goal in this section is to prove Theorem 1.2, whose statement we recall:

Theorem. *Let $\varphi, \psi \in \mathbb{C}(T)$ be rational functions of degrees at least 2, and assume that the complex Julia sets of φ and ψ are distinct. Then there are only finitely many $a \in \mathbb{C}$ which are preperiodic for both φ and ψ.*

Remark 6.1. For example, if $\varphi(z) = z^2 + c$ and $\psi(z) = z^2 + c'$ with $c \neq c'$, then φ and ψ have distinct Julia sets; see [Bea90, §4].

In the special case where φ, ψ are defined over $\bar{\mathbb{Q}}$, Theorem 1.2 is a special case of a theorem of Mimar [Mim97]. Thus the novelty of the present result consists in extending Mimar's result to the transcendental case by making use of (non-archimedean) Berkovich analytic spaces and equidistribution over function fields.

6.1. **Equidistribution.** The main references for this section are [BR06] and [BR09, Chapter 10]; see also [FRL06]. Let k be a product formula field, and let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. Associated to φ is the *Call-Silverman canonical height function* $\hat{h}_\varphi : \mathbb{P}^1(\bar{k}) \to \mathbb{R}_{\geq 0}$. If k is a number field, then a point $P \in \mathbb{P}^1(\bar{k})$ is preperiodic for φ if and only if $\hat{h}_\varphi(P) = 0$. In general, things are a little more subtle; see Theorem 6.6 below.

For each $v \in \mathcal{M}_k$, the v-adic *Arakelov-Green function* of φ is a function

$$g_{\varphi,v} : \mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v} \to \mathbb{R} \cup \{+\infty\}$$

that takes the value $+\infty$ on the intersection of the diagonal with $\mathbb{P}^1(\mathbb{C}_v)$ and is finite-valued elsewhere. There is also a canonical probability measure $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\mathrm{Berk},v}$; when v is archimedean $\mu_{\varphi,v}$ is the measure of maximal entropy for φ on $\mathbb{P}^1(\mathbb{C})$ studied in [Lyu83], [FLM83]. Each of $\mu_{\varphi,v}$ and $g_{\varphi,v}(x,y)$ determines the other by the equation

$$\Delta_x\, g_{\varphi,v}(x,y) = \mu_{\varphi,v} - \delta_y \tag{6.2}$$

for every fixed $y \in \mathbb{P}^1_{\mathrm{Berk},v}$, where g is normalized so that

$$\iint_{\mathbb{P}^1_{\mathrm{Berk},v} \times \mathbb{P}^1_{\mathrm{Berk},v}} g_{\varphi,v}(x,y)\, d\mu_{\varphi,v}(x)\, d\mu_{\varphi,v}(y) = 0.$$

(The Laplacian in (6.2) is the negative of the one studied in [BR09].) Moreover, for all $x, y \in \mathbb{P}^1(\bar{k})$ with $x \neq y$ we have

$$\hat{h}_\varphi(x) + \hat{h}_\varphi(y) = \sum_{v \in \mathcal{M}_k} N_v \, g_{\varphi,v}(x,y). \tag{6.3}$$

If v is archimedean, it is well known that $\mathrm{Supp}(\mu_{\varphi,v})$ is equal to the complex Julia set of φ. For v non-archimedean, the Berkovich Julia set of φ is *defined* in [BR09, Chapter 10] to be the support of $\mu_{\varphi,v}$. This turns out to be equivalent to several other, more topological, characterizations of the Berkovich Julia set.

If S is a finite subset of $\mathbb{P}^1(\bar{k})$ which is stable under $\mathrm{Gal}(\bar{k}/k)$, we define

$$\hat{h}_\varphi(S) = \frac{1}{|S|} \sum_{P \in S} \hat{h}_\varphi(P).$$

The following equidistribution theorem was proved independently by Baker–Rumely [BR06], Chambert-Loir [CL06], and Favre–Rivera-Letelier [FRL06] in the number field case. For the present formulation in terms of an arbitrary product formula field, see [BR09, Theorem 10.38].

Theorem 6.4. *Let k be a product formula field of characteristic zero, and let $\varphi \in k(T)$ be a rational function of degree $d \geq 2$. Suppose S_n is a sequence of $\mathrm{Gal}(\bar{k}/k)$-invariant finite subsets of $\bar{k}$ with $|S_n| \to \infty$ and $\hat{h}_\varphi(S_n) \to 0$. Fix $v \in \mathcal{M}_k$, and for each n let δ_n be the discrete probability measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ supported equally on the elements of S_n. Then the sequence of measures $\{\delta_n\}$ converges weakly to $\mu_{\varphi,v}$ on $\mathbb{P}^1_{\mathrm{Berk},v}$.*

6.2. **Isotriviality.** Let k be an abstract function field; see §3.4 for the definition. A rational map $\varphi \in k(T)$ is called *isotrivial* if it is conjugate (by a linear fractional transformation defined over $\bar{k}$) to a rational map defined over the constant field of k. A map $\varphi \in k(T)$ of degree d is said to have *good reduction* at $v \in \mathcal{M}_k$ if $\varphi = f/g$ for some $f, g \in \mathcal{O}_v[T]$ whose reductions $\overline{f}, \overline{g} \in \tilde{k}_v[T]$ are degree d polynomials with no common roots over $\bar{k}_v$. (Here $\mathcal{O}_v$ denotes the valuation ring of k_v, and $\tilde{k}_v$ its residue field.)

The following two results are proved in [Bak09]:

Proposition 6.5. *Let k be an abstract function field, and let $\varphi \in k(T)$ be a rational map of degree at least 2. Then the following are equivalent:*

1. *φ is defined over the constant field of k*
2. *φ has good reduction at every $v \in \mathcal{M}_k$*
3. *The canonical measure $\mu_{\varphi,v}$ is a point mass supported at the Gauss point of $\mathbb{P}^1_{\mathrm{Berk},v}$ for all $v \in \mathcal{M}_k$.*

Theorem 6.6. *Let k be an abstract function field with an algebraically closed field of constants. If $\varphi \in k(T)$ is a rational map of degree $d \geq 2$ which is not isotrivial, then a point $P \in \mathbb{P}^1(\bar{k})$ satisfies $\hat{h}_\varphi(P) = 0$ if and only if P is preperiodic for φ.*

Modulo the assumption that the constant field of k is algebraically closed, Theorem 6.6 is a generalization of Theorem 3.11.

6.3. **Preperiodic points for rational functions.** Let $\mathrm{Preper}(\varphi)$ denote the set of preperiodic points for φ. For the proof of Theorem 1.2, we will need the following well-known result from complex dynamics:

Lemma 6.7. *The Julia set of a rational map $\varphi \in \mathbb{C}(T)$ of degree at least 2 is equal to the set of accumulation points of* $\mathrm{Preper}(\varphi)$*.*

Proof. Periodic points are dense in $J(\varphi)$, so clearly $J(\varphi) \subset \overline{\mathrm{Preper}(\varphi)}$. Furthermore, the Julia set has no isolated points, so every point of $J(\varphi)$ is an accumulation point of $\mathrm{Preper}(\varphi)$. Conversely, the preperiodic points form a discrete subset of the Fatou set: there are only finitely many periodic cycles in the Fatou set, and backwards orbits will accumulate on the Julia set. See e.g. [Mil99] for details. □

We can now give the proof of Theorem 1.2.

Proof of Theorem 1.2. Suppose for the sake of contradiction that there is an infinite sequence a_n of complex numbers which are preperiodic for both φ and ψ.

Case 1: $\varphi, \psi \in \bar{\mathbb{Q}}(T)$.

In this case, a_n is algebraic for all n. Let k be a number field over which both φ and ψ are defined, and let δ_n be the discrete probability measure on $\mathbb{P}^1(\mathbb{C})$ supported equally on the $\mathrm{Gal}(\bar{k}/k)$-conjugates of a_n. By Theorem 6.4, the sequence δ_n converges weakly on $\mathbb{P}^1(\mathbb{C})$ to both μ_φ and μ_ψ, hence $\mu_\varphi = \mu_\psi$. Thus

$$J_\varphi = \mathrm{Supp}(\mu_\varphi) = \mathrm{Supp}(\mu_\psi) = J_\psi,$$

a contradiction.

Case 2: Neither φ nor ψ is conjugate to a rational map defined over $\bar{\mathbb{Q}}(T)$.

In this case, all a_n's are defined over $\bar{k}$, where k is the finitely generated field extension of $\bar{\mathbb{Q}}$ generated by the coefficients of φ and ψ. The field k can be endowed with a product formula structure by thinking of it as the function field of some normal projective variety X over $\bar{\mathbb{Q}}$. All places $v \in \mathcal{M}_k$ are non-archimedean, and the constant field of k is $\bar{\mathbb{Q}}$. For each $v \in \mathcal{M}_k$, let δ_n be the discrete probability measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ supported equally on the $\mathrm{Gal}(\bar{k}/k)$-conjugates of a_n. By Theorem 6.4, the sequence δ_n converges weakly on $\mathbb{P}^1_{\mathrm{Berk},v}$ to both $\mu_{\varphi,v}$ and $\mu_{\psi,v}$, hence $\mu_{\varphi,v} = \mu_{\psi,v}$ for all $v \in \mathcal{M}_k$. By (6.2), we therefore have $g_{\mu_\varphi,v}(x,y) = g_{\mu_\psi,v}(x,y)$ for all $v \in \mathcal{M}_k$ and all $x, y \in \mathbb{P}^1_{\mathrm{Berk}}$. From the identity (6.3), it follows (letting $y = a_n$ for some n, so that $\hat{h}_\varphi(y) = \hat{h}_\psi(y) = 0$) that $\hat{h}_\varphi(x) = \hat{h}_\psi(x)$ for all $x \in \mathbb{P}^1(\bar{k})$.

By assumption, neither φ nor ψ is conjugate to a map defined over $\bar{\mathbb{Q}}$. By Theorem 6.6, a point $x \in \mathbb{P}^1(\bar{k})$ is preperiodic for φ (resp. ψ) if and only if $\hat{h}_\varphi(x) = 0$ (resp. $\hat{h}_\psi(x) = 0$). We conclude that φ and ψ have the same set of preperiodic points in $\bar{k}$, and therefore the same set of preperiodic points in $\mathbb{C}$. By Lemma 6.7, we conclude that $J_\varphi = J_\psi$, a contradiction.

Case 3: φ is conjugate to a rational map defined over $\bar{\mathbb{Q}}(T)$.

Replacing φ by a conjugate, we may assume without loss of generality that φ is defined over $\bar{\mathbb{Q}}(T)$. We claim that (in these new coordinates) ψ is defined over $\bar{\mathbb{Q}}(T)$ as well, so that we are back in Case 1.

As in Case 2, all a_n's are defined over $\bar{k}$, where k is the finitely generated field extension of $\bar{\mathbb{Q}}$ generated by the coefficients of φ and ψ. We may assume that $k/\mathbb{Q}$ is transcendental, since otherwise we're done. So as in Case 2, k can be endowed with a product formula structure with respect to which all places $v \in \mathcal{M}_k$ are non-archimedean and the constant field of k is $\bar{\mathbb{Q}}$. For each $v \in \mathcal{M}_k$, let δ_n be the discrete probability measure on $\mathbb{P}^1_{\mathrm{Berk},v}$ supported equally on the $\mathrm{Gal}(\bar{k}/k)$-conjugates of a_n. By Theorem 6.4, the sequence δ_n converges weakly on $\mathbb{P}^1_{\mathrm{Berk},v}$ to both $\mu_{\varphi,v}$ and $\mu_{\psi,v}$, hence $\mu_{\varphi,v} = \mu_{\psi,v}$ for all $v \in \mathcal{M}_k$.

Since φ is defined over the constant field $\bar{\mathbb{Q}}$ of k, φ has good reduction at every $v \in \mathcal{M}_k$. Equivalently, $\mu_{\varphi,v}$ is a point mass supported at the Gauss point of $\mathbb{P}^1_{\mathrm{Berk},v}$ for all $v \in \mathcal{M}_k$. As $\mu_{\varphi,v} = \mu_{\psi,v}$ for all $v \in \mathcal{M}_k$, we deduce from Proposition 6.5 that ψ has good reduction at every $v \in \mathcal{M}_k$ and hence is defined over the constant field $\bar{\mathbb{Q}}$ of k, as claimed. □

Note that by applying the adelic argument from Case 2, one can conclude in Case 1 as well that φ and ψ have the same set of preperiodic points. (The argument is actually simpler in the algebraic case, since one can use Northcott's theorem instead of Proposition 6.5.) So from the proof of Theorem 1.2, we obtain Corollary 1.3 from the Introduction.

Note also that the argument in Case 1 of Theorem 1.2 actually proves the following stronger result when φ, ψ are defined over $\bar{\mathbb{Q}}$:

Theorem 6.8. *Let $\varphi, \psi \in \bar{\mathbb{Q}}(T)$ be rational functions of degree at least 2, and assume that the canonical probability measures μ_φ and μ_ψ on $\mathbb{P}^1(\mathbb{C})$ are distinct. (This will be true, in particular, if the complex Julia sets of φ and ψ are distinct.) Then there is a real number $\varepsilon > 0$ such that $\hat{h}_\varphi(a) + \hat{h}_\psi(a) \geq \varepsilon$ for all but finitely many $a \in \bar{\mathbb{Q}}$.*

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