EDUCATION AND EXAMINATION COMMITTEE

OF THE

SOCIETY OF ACTUARIES

FINANCIAL MATHEMATICS STUDY NOTE

USING DURATION AND CONVEXITY TO APPROXIMATE CHANGE IN PRESENT VALUE

by

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FM-24-17

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October 28, 2016

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1 Introduction

The study of interest theory includes the concept of duration and how it may be used to approximate the change in the present value of a cash flow series resulting from a small change in interest rate. The purpose of this study note is to demonstrate a non-linear approximation using Macaulay duration that is more accurate than the linear approximation using modified duration, and that a corresponding second-order approximation using Macaulay duration and convexity is more accurate than the usual second-order approximation using modified duration and convexity. These Macaulay approximations are found in formulas (4.2) and (6.2) below.

Most textbooks give the following formula using modified duration to approximate the change in the present value of a cash flow series due to a change in interest rate:

$$P(i) \approx P(i_0) \cdot \left(1 - (i - i_0) \cdot D_{\text{mod}}(i_0)\right).$$

This approximation uses only the difference in interest rates and two facts about the cash flow series based on the initial interest rate, i_0, to provide an approximation of the present value at a new interest rate, i. These two facts are (1) the present value of the cash flow series and (2) the modified duration of the cash flow series. Furthermore, the approximation of the change in present value is directly proportional to the change in interest rate, facilitating mental computations. We will refer to this approximation as the **first-order modified approximation**.

The following approximation, using Macaulay duration, is, under very general conditions, at least as accurate as the first-order modified approximation and has other pleasant attributes:

$$P(i) \approx P(i_0) \cdot \left(\frac{1 + i_0}{1 + i}\right)^{D_{\text{mac}}(i_0)},$$

We will refer to this approximation as the **first-order Macaulay approximation**.

The methods discussed in this note are based on the assumption that the timings and amounts of the cash flow series are unaffected by a small change in interest rate. This assumption is not always valid. On one hand, in the case of a callable bond, a change in interest rates may trigger the calling of the bond, thus stopping the flow of future coupons. On the other hand, non-callable bonds, or payments to retirees in a pension plan are situations where the assumption is usually valid.

The developments in this note are also predicated on a flat yield curve, that is to say that cash flows at all future times are discounted to the present using the same interest rate.

This note is not intended to be a complete discussion of duration. In fact, we assume the reader already is acquainted with the concept of duration, although it is not absolutely required.

2 Cash Flow Series and Present Value

A **cash flow** is a pair, (a,t), where a is a real number, and t is a non-negative real number. Given a cash flow (a,t), the **amount** of the cash flow is a and the **time** of the cash flow is t. Notice that we have allowed the amount to be negative, although the time is non-negative. A **cash flow series** is a sequence (finite or infinite) of cash flows (a_k, t_k) defined for $k \in N$, where N is a subset of the set of non-negative integers.

For the purpose of calculating present values and durations, we introduce a periodic effective interest rate, i, where the period of time is the same time unit used to measure the times of the cash flows. For example, if the times are measured in months, then the interest rate, i, is a monthly effective interest rate. We define P to represent the present value of the cash flow series as a function of the interest rate as follows.

$$P(i) = \sum_{k \in N} \left(a_k \cdot (1+i)^{-t_k} \right) \tag{2.1}$$

If the cash flow series is infinite, the sum in (2.1) may not converge or be finite. In what follows, we implicitly make the assumption that any sums so represented converge. In the case that N is a finite set of the form $\{1,\ldots,n\}$, we may choose to write the sum as

$$\sum_{k=1}^{n} \left(a_k \cdot (1+i)^{-t_k} \right).$$

The following examples show the present value of a 10-year annuity immediate calculated at an annual effective interest rate of 7.0% and at an annual effective interest rate of interest of 6.5%. We will use this same cash flow series as an example throughout this note.

Suppose $(a_k, t_k) = (1000, k)$ and $N = \{1,\ldots,10\}$. Then,

$$P(0.07) = 1000 \cdot a_{\overline{10}|0.07} = 7023.5815 \tag{2.2}$$

and

$$P(0.065) = 1000 \cdot a_{\overline{10}|0.065} = 7188.8302. \tag{2.3}$$

We would like to approximate the change in the present value of a cash flow series resulting from a small change in the interest rate. This is a valuable technique for several reasons. First, much of actuarial science involves the use of mathematical models of various levels of complexity and sophistication. To be able to use a model effectively, one needs to understand the dynamics of the model, i.e., how one variable changes based on a change to a different variable. The present value formula is such a mathematical model. An actuary should understand how present value changes when the amounts change, when the times change, and when the interest rate changes.

A second reason is that as a practical matter, actuaries are required sometimes to approximate changes in present value without being able to use the computer power needed for a complete calculation. For example, consider an investment actuary meeting with the president of a large insurance company with a substantial bond portfolio. The president is concerned that interest rates will increase, which will decrease the value of the bond portfolio. The investment actuary has recently calculated the value of the bond portfolio using an interest rate of 6.5%. The president wants to know the value of the bond portfolio if interest rates increase to 6.75% or even 7.0%. Since the value of the bond portfolio is merely the present value of future cash flows, using the concepts of duration defined below, such approximations can be done quickly using nothing more than a handheld calculator.

Even when full computing power is available, approximations like the ones in this note are essential. For example, when doing multi-year projections using Monte Carlo techniques for interest rate scenarios, thousands of present value calculations may be needed. It is not feasible to do full calculations and approximations make it possible for such projections to be done.

3 Macaulay and Modified Duration

The definition of **Macaulay duration** is

$$D_{\text{mac}}(i) = \frac{\sum_{k\in N}\left(t_k \cdot a_k \cdot (1+i)^{-t_k}\right)}{\sum_{k\in N}\left(a_k \cdot (1+i)^{-t_k}\right)} = \frac{\sum_{k\in N}\left(t_k \cdot a_k \cdot (1+i)^{-t_k}\right)}{P(i)}. \tag{3.1}$$

The definition of **modified duration** is

$$D_{\text{mod}}(i) = \frac{-P'(i)}{P(i)} = \frac{\sum_{k\in N}\left(t_k \cdot a_k \cdot (1+i)^{-t_k-1}\right)}{P(i)}. \tag{3.2}$$

Macaulay duration is the weighted average of the times of the cash flows, where the weights are the present values of the cash flows. Modified duration is the negative derivative of the present-value function with respect to the effective interest rate, and expressed as a fraction of the present value. Therefore it is expected that modified duration gives us information about the rate of change of the present-value function as the interest rate changes. We note the following relation between the two notions of duration:

$$D_{\text{mod}}(i) = \frac{D_{\text{mac}}(i)}{1+i}. \tag{3.3}$$

Because both definitions of duration involve division by $P(i)$, we will assume for the remainder of this note that

$$P(i) \neq 0. \tag{3.4}$$

As an example of Macaulay and modified duration, we first consider a cash flow series that consists of a single flow, (a_1, t_1). For this situation, we have

$$D_{\text{mac}}(i) = t_1 \text{ and } D_{\text{mod}}(i) = \frac{t_1}{1+i}. \tag{3.5}$$

Next, using the 10-year immediate annuity and setting $i = 0.07$ we have

$$D_{\text{mac}}(0.07) = \frac{\sum_{k=1}^{10}(k \cdot 1000 \cdot 1.07^{-k})}{\sum_{k=1}^{10}(1000 \cdot 1.07^{-k})} = \frac{34739.1332}{7023.5815} = 4.9460710. \tag{3.6}$$

Alternatively, for this example, we can see that

$$D_{\text{mac}}(0.07) = \frac{1000 \cdot (Ia)_{\overline{10}|0.07}}{1000 \cdot a_{\overline{10}|0.07}} = \frac{(Ia)_{\overline{10}|0.07}}{a_{\overline{10}|0.07}} = \frac{34.7391332}{7.0235815} = 4.9460710. \tag{3.7}$$

Also, for this example, we have

$$D_{\text{mod}}(0.07) = \frac{D_{\text{mac}}(0.07)}{1.07} = \frac{4.9460710}{1.07} = 4.6224963. \tag{3.8}$$

4 First-Order Approximations of Present Value

The **first-order modified approximation** of the present-value function is

$$P(i) \approx P(i_0) \cdot (1 - (i - i_0) \cdot D_{\text{mod}}(i_0)). \tag{4.1}$$

This approximation is presented on Page 369 in [1], on Page 396 in [2], on Page 455 in [3], and on Page 216 in [4]. It is derived using the first-order Taylor approximation for $P(i)$ about i_0.

The **first-order Macaulay approximation** of the present-value function is

$$P(i) \approx P(i_0) \cdot \left(\frac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)}. \tag{4.2}$$

The derivation of this approximation is given in Appendix A.

Using the 10-year annuity immediate, we calculate the first-order modified approximation for $P(0.065)$ and compare it to the true present value. The result is

$$\begin{aligned} P(0.065) &\approx P(0.07) \cdot \left(1 - (0.065 - 0.07) \cdot D_{\text{mod}}(0.07)\right) \\ &= 7023.5815 \cdot (1 + 0.005 \cdot 4.6224963) = 7185.9139. \end{aligned} \tag{4.3}$$

Because $P(0.065) = 7188.8302$, the percent error is –0.0406%.

Next we calculate the corresponding values for the first-order Macaulay approximation:

$$P(0.065) \approx 7023.5815 \cdot \left(\frac{1.07}{1.065}\right)^{4.9460710} = 7188.1938. \tag{4.4}$$

The percent error is –0.0089%.

Thus, the error from Macaulay approximation is about 22% of the error from the modified approximation.

It is worthwhile noting that in the case where the cash flow series consists of a single cash flow, the first-order Macaulay approximation gives the exact present value, while the first-order modified approximation does not.

In Appendix B, we have compared the two approximations over 180 scenarios. At worst, the error from the first-order Macaulay approximation is 39% of the error from the first-order modified approximation. At best, the error from the first-order Macaulay approximation is 14% of the error from the first-order modified approximation.

In Appendix C, it is shown that the first-order Macaulay approximation is more accurate than the first-order modified approximation whenever the cash flow amounts are positive. When this condition is not met, it is possible for the first-order modified approximation to be more accurate than the first-order Macaulay approximation.

5 Modified and Macaulay Convexity

The definition of **modified convexity** is:

$$C_{\text{mod}}(i) = P''(i) / P(i) = \frac{\sum_{k \in N} \left(t_k \cdot (t_k + 1) \cdot a_k \cdot (1+i)^{-t_k - 2}\right)}{P(i)}. \tag{5.1}$$

The definition of **Macaulay convexity** is:

$$C_{\text{mac}}(i) = \frac{\sum_{k \in N} \left({t_k}^2 \cdot a_k \cdot (1+i)^{-t_k}\right)}{\sum_{k \in N} \left(a_k \cdot (1+i)^{-t_k}\right)} = \frac{\sum_{k \in N} \left({t_k}^2 \cdot a_k \cdot (1+i)^{-t_k}\right)}{P(i)}. \tag{5.2}$$

Thus, Macaulay convexity is the weighted average of the squares of the times of the cash flows, where the weights are the present values of the cash flows. The following relationship is easily derived:

$$C_{\text{mod}}(i) = \frac{C_{\text{mac}}(i) + D_{\text{mac}}(i)}{(1+i)^2}. \tag{5.3}$$

As an example of Macaulay and modified convexity, we first consider a cash flow series that consists of a single cash flow, (a_1, t_1). For this situation, we have

$$C_{mac}(i) = t_1^2 \text{ and } C_{mod}(i) = \frac{t_1 \cdot (t_1 + 1)}{(1+i)^2}. \tag{5.4}$$

Using the 10-year annuity example from Sections 2 and 3, we can see that

$$C_{mac}(0.07) = \frac{\sum_{k=1}^{10} (1000 \cdot k^2 \cdot 1.07^{-k})}{\sum_{k=1}^{10} (1000 \cdot 1.07^{-k})} = \frac{228,451.20}{7,023.5815} = 32.526311 \tag{5.5}$$

and

$$C_{mod}(0.07) = \frac{C_{mac}(.07) + D_{mac}(0.07)}{1.07^2} = 32.729830. \tag{5.6}$$

6 Second-Order Approximations of Present Value

The **second-order modified approximation** of the present value function is:

$$P(i) \approx P(i_0) \cdot \left(1 - (i - i_0) \cdot D_{mod}(i_0) + \frac{(i-i_0)^2}{2} \cdot C_{mod}(i_0) \right). \tag{6.1}$$

This approximation can be found in most of the texts.

Letting $T = D_{mac}(i_0)$ and $Q = C_{mac}(i_0) - T^2$, the **second-order Macaulay approximation** of the present value function is:

$$P(i) \approx P(i_0) \cdot \left(\frac{1+i_0}{1+i} \right)^T \cdot \left(1 + \left(\frac{i-i_0}{1+i_0} \right)^2 \cdot \frac{Q}{2} \right). \tag{6.2}$$

A derivation of this formula can be found in Appendix D.

To illustrate these two approximations, we will apply them to the 10-year annuity example. Using the convexity values from Section 5 and the duration values from Section 3, we can calculate the two second-order approximations of $P(0.065)$, with the following results.

First, for the second-order modified approximation, we get

$$\begin{aligned} P(0.065) &\approx P(0.07) \cdot \left(1 - (0.065 - 0.07) \cdot D_{mod}(0.07) + \frac{(0.065-0.07)^2}{2} \cdot C_{mod}(0.07) \right) \\ &= 7023.5815 \cdot \left(1 + 0.005 \cdot 4.6224963 + \frac{(0.005)^2}{2} \cdot 32.729830 \right) \\ &= 7023.5815 \cdot (1 + 0.0231125 + 0.0004091) \\ &= 7188.7874. \end{aligned} \tag{6.3}$$

Because $P(0.065)$ = 7188.8302, the percent error is –0.00060%.

Next we calculate the second-order Macaulay approximation:

$$
\begin{aligned}
P(0.065) \approx P(0.07)\cdot\left(\frac{1+0.07}{1+.065}\right)^{4.9460719} \\
\cdot\left(1+\left(\frac{0.065-0.07}{1+.07}\right)^{2}\cdot\frac{32.526311-4.9460710^{2}}{2}\right) \qquad (6.4) \\
=7023.5815\cdot 1.02343708\cdot 1.0000881 \\
=7188.8266.
\end{aligned}
$$

Here the percent error is –0.00005%. For this example, the error for the second-order Macaulay approximation is less than 10% of the error of the second-order modified approximation.

Table (B.3) of Appendix B shows that the error from the second-order Macaulay approximation is less than 20% of the error from the second-order modified approximation over 180 different scenarios.

As a final observation about the second-order methods, we note that the Macaulay approximation gives the exact present value at the new interest rate in the case of a single cash flow, because in this case, using (3.5) and (5.4), $Q=t_1^2-t_1^2=0$.

Appendix A: Derivation of First-Order Macaulay Approximation

To derive this approximation of $P(i)$ we reason as follows. For each time T, we define a function V_T to represent the **current value** of the given cash flow series at time T:

$$V_T(i) = P(i) \cdot (1+i)^T. \tag{A.1}$$

Note that if we set $T = 0$ in (A.1), we obtain the present-value function. It is important to understand that each function V_T is a function of a single real variable, which we think of as representing an effective rate of interest. Below, when we take the derivative of one of these functions, it is with respect to that variable.

For the moment, let us consider a specific interest rate, i_0, and consider current-value functions for various values of T. If T is small enough, for example before the time of the first payment, then a small increase in the interest rate will decrease the current value, i.e., $V_T'(i_0) < 0$. However, if T is large enough, then a small increase in the interest rate will increase the current value, i.e., $V_T'(i_0) > 0$. This suggests that there is some value of T such that the function V_T is neither increasing nor decreasing at i_0. That is, for this value of T, we would have $V_T'(i_0) = 0$. We solve for this value:

$$0 = V_T'(i_0) = P(i_0) \cdot T \cdot (1+i_0)^{T-1} + P'(i_0) \cdot (1+i_0)^T .$$

Thus,

$$T = \frac{-P'(i_0) \cdot (1+i_0)^T}{P(i_0) \cdot (1+i_0)^{T-1}} = \frac{-P'(i_0) \cdot (1+i_0)}{P(i_0)} = D_{\text{mod}}(i_0) \cdot (1+i_0) = D_{\text{mac}}(i_0).$$

It is easily checked that, in fact, $0 = V_T'(i_0)$ if $T = D_{\text{mac}}(i_0)$. Let us now define the function V, with no subscript, as V_T with $T = D_{\text{mac}}(i_0)$. Thus,

$$V(i) = P(i) \cdot (1+i)^{D_{\text{mac}}(i_0)}$$

and $V'(i_0) = 0$. By applying the first-order Taylor approximation to $V(i)$ about i_0 we see

$$\begin{aligned} V(i) &\approx V(i_0) + (i - i_0) \cdot V'(i_0) = V(i_0) \\ P(i) \cdot (1+i)^{D_{\text{mac}}(i_0)} &\approx P(i_0) \cdot (1+i_0)^{D_{\text{mac}}(i_0)} \qquad \text{(A.2)} \\ P(i) &\approx P(i_0) \cdot \left(\frac{1+i_0}{1+i} \right)^{D_{\text{mac}}(i_0)} . \end{aligned}$$

Appendix B: Comparisons of Approximations

The percent error has been analyzed for both the modified duration approximation and the Macaulay duration approximation under a variety of scenarios. We have considered nine different cash flow series, each with up to 25 cash flows at times 1 through 25. The series are defined as follows.

(B.1) Table of Cash Flow Series Scenarios

Time	Level-5	Level-10	Level-15	Level-20	Level-25	Increasing	Decreasing	Inc/Dec	Dec/Inc
1	1,000	1,000	1,000	1,000	1,000	1,000	26,000	1,000	26,000
2	1,000	1,000	1,000	1,000	1,000	2,000	25,000	2,000	25,000
3	1,000	1,000	1,000	1,000	1,000	3,000	24,000	3,000	24,000
4	1,000	1,000	1,000	1,000	1,000	4,000	23,000	4,000	23,000
5	1,000	1,000	1,000	1,000	1,000	5,000	22,000	5,000	22,000
6	0	1,000	1,000	1,000	1,000	6,000	21,000	6,000	21,000
7	0	1,000	1,000	1,000	1,000	7,000	20,000	7,000	20,000
8	0	1,000	1,000	1,000	1,000	8,000	19,000	8,000	19,000
9	0	1,000	1,000	1,000	1,000	9,000	18,000	9,000	18,000
10	0	1,000	1,000	1,000	1,000	10,000	17,000	10,000	17,000
11	0	0	1,000	1,000	1,000	11,000	16,000	11,000	16,000
12	0	0	1,000	1,000	1,000	12,000	15,000	12,000	15,000
13	0	0	1,000	1,000	1,000	13,000	14,000	13,000	14,000
14	0	0	1,000	1,000	1,000	14,000	13,000	12,000	15,000
15	0	0	1,000	1,000	1,000	15,000	12,000	11,000	16,000
16	0	0	0	1,000	1,000	16,000	11,000	10,000	17,000
17	0	0	0	1,000	1,000	17,000	10,000	9,000	18,000
18	0	0	0	1,000	1,000	18,000	9,000	8,000	19,000
19	0	0	0	1,000	1,000	19,000	8,000	7,000	20,000
20	0	0	0	1,000	1,000	20,000	7,000	6,000	21,000
21	0	0	0	0	1,000	21,000	6,000	5,000	22,000
22	0	0	0	0	1,000	22,000	5,000	4,000	23,000
23	0	0	0	0	1,000	23,000	4,000	3,000	24,000
24	0	0	0	0	1,000	24,000	3,000	2,000	25,000
25	0	0	0	0	1,000	25,000	2,000	1,000	26,000

For each cash flow series, the present value was approximated at 20 interest rates that differed from the initial interest rate of 7.0% by multiples of 0.2% between 5.0% and 9.0%. The percent errors were averaged using a subjectively selected weighting of $e^{-|i-i_0|}$ to give greater value to rates nearer the initial rate.

(B.2) Table of average weighted percent errors for first-order approximations

Cash Flow Series	1st-order modified	1st-order Macaulay	Macaulay err/ modified err
Level-5	0.0820%	0.0125%	15.24%
Level-10	0.2351%	0.0506%	21.52%
Level-15	0.4402%	0.1112%	25.26%
Level-20	0.6765%	0.1905%	28.16%
Level-25	0.9266%	0.2837%	30.62%
Increasing	1.6473%	0.2601%	15.79%
Decreasing	0.5313%	0.1776%	33.43%
Inc/Dec	1.0181%	0.1689%	16.59%
Dec/Inc	0.8984%	0.3138%	34.93%

Table (B.2) shows that the first-order Macaulay approximation is consistently markedly better than the first-order modified approximation. Overall, the error from the Macaulay approximation is about 1/3 or less of the error from the modified approximation.

(B.3) Table of weighted-average percent errors for second-order approximations

Cash Flow Series	2^{nd}-Order modified	2^{nd}-Order Macaulay	Macaulay err/ modified err
Level-5	0.0023%	0.0002%	8.70%
Level-10	0.0107%	0.0009%	8.41%
Level-15	0.0272%	0.0024%	8.82%
Level-20	0.0522%	0.0051%	9.77%
Level-25	0.0851%	0.0095%	11.16%
Increasing	0.1666%	0.0028%	1.68%
Decreasing	0.0405%	0.0071%	17.53%
Inc/Dec	0.0844%	0.0034%	4.03%
Dec/Inc	0.0853%	0.0122%	14.30%

Table (B.3) shows that the second-order Macaulay approximation is consistently markedly better than the second-order modified approximation. Overall, the error from the Macaulay approximation is about 1/5 or less of the error from the modified approximation.

We can use the second-order results to measure the success of the Macaulay first-order approximation. For the Level-5 cash flow series, the difference between the first-order modified average error and the second-order modified average error is 0.0820% - 0.0023%, or 0.0797%. The difference between the first-order modified average error and the first-order Macaulay average error is 0.0820% – 0.0125%, or 0.0695%. Thus the first-order Macaulay approximation takes you 87% of the way from the first-order modified to the second-order modified approximation. This percentage varies between 72% and 94% over the nine different cash flow series studied.

Appendix C: Demonstration that the First-Order Macaulay Approximation is More Accurate than the First-Order Modified Approximation

We assume in this appendix that the cash flow amounts are positive. We first establish some notation. We are given an initial periodic effective interest rate, i_0. For our given cash flow series, we set

$$T = D_{\text{mac}}(i_0)$$

$$F_1(i) = P(i_0)\cdot(1-(i-i_0)\cdot D_{\text{mod}}(i_0)) = P(i_0)\cdot\left(1-\frac{i-i_0}{1+i_0}\cdot T\right)$$

$$F_2(i) = P(i_0)\cdot\left(\frac{1+i_0}{1+i}\right)^T = P(i_0)\cdot\left(\frac{1+i}{1+i_0}\right)^{-T}$$

so that $F_1(i)$ is the first-order modified approximation to $P(i)$, and $F_2(i)$ is the first-order Macaulay approximation to $P(i)$.

In Theorem (C.5) below, we show that, the first-order modified approximation is less than or equal to the first-order Macaulay approximation which is less than or equal to the actual present value. Thus the first-order Macaulay approximation is always a better approximation.

We begin by showing that the first-order modified approximation is less than or equal to the first-order Macaulay approximation.

(C.1) Theorem: $F_1(i) \le F_2(i)$

Proof:

We have

$$F_2'(i) = -T\cdot P(i_0)\cdot\left(\frac{1+i}{1+i_0}\right)^{-T-1}\cdot\frac{1}{1+i_0}$$

$$F_2''(i) = T\cdot(T+1)\cdot P(i_0)\cdot\left(\frac{1+i}{1+i_0}\right)^{-T-2}\cdot\frac{1}{(1+i_0)^2} > 0.$$

By Taylor's Theorem with remainder there is j between i_0 and i such that

$$\begin{aligned}
F_2(i) &= F_2(i_0) + (i - i_0) \cdot F_2'(i_0) + \frac{(i-i_0)^2}{2} \cdot F_2''(j) \\
&\geq F_2(i_0) + (i - i_0) \cdot F_2'(i_0) \\
&= P(i_0) + (i - i_0) \cdot \frac{-T \cdot P(i_0)}{1 + i_0} \\
&= F_1(i).
\end{aligned}$$

Theorems (C.2) through (C.5) are devoted to showing that the first-order Macaulay approximation is less than or equal to the present value. While Theorems (C.2) through (C.4) are important in their own right, the reader may wish to think of these as Lemmas. For these theorems, our argument is simplified by using a continuously compounded rate of interest, δ, as the independent variable. Thus we will define the present value function, Macaulay duration, Macaulay convexity, and the first-order Macaulay approximation in terms of this variable. We begin with an initial $\delta_0 = \ln(1 + i_0)$ and make the following definitions.

$$\begin{aligned}
&P_\infty(\delta) = P\left(\mathrm{e}^\delta - 1\right) = \sum_{k \in N} \left(a_k \cdot \mathrm{e}^{-t_k \cdot \delta}\right); \\
&D_\infty(\delta) = D_{\mathrm{mac}}\left(\mathrm{e}^\delta - 1\right) = \frac{\sum_{k \in N} \left(t_k \cdot a_k \cdot \mathrm{e}^{-t_k \cdot \delta}\right)}{P_\infty(\delta)}; \\
&C_\infty(\delta) = C_{\mathrm{mac}}\left(\mathrm{e}^\delta - 1\right) = \frac{\sum_{k \in N} \left(t_k^2 \cdot a_k \cdot \mathrm{e}^{-t_k \cdot \delta}\right)}{P_\infty(\delta)}; \text{ and} \\
&F_\infty(\delta) = F_2\left(\mathrm{e}^\delta - 1\right) = P_\infty(\delta_0) \cdot \mathrm{e}^{-(\delta - \delta_0) \cdot T}.
\end{aligned}$$

(C.2) Theorem: If $D = D_\infty(\delta)$ and $C = C_\infty(\delta)$ then $C - D^2 \geq 0$.

Proof:

For each $k \in N$, set $q_k = \dfrac{a_k \cdot \mathrm{e}^{-t_k \cdot \delta}}{P_\infty(\delta)}$, and note that $q_k > 0$ and

$1 = \sum_{k \in N} q_k$ and $D_\infty(\delta) = \sum_{k \in N} \left(t_k \cdot q_k\right)$ and $C_\infty(\delta) = \sum_{k \in N} \left(t_k^2 \cdot q_k\right)$. Then

$$
\begin{aligned}
C - D^2 &= C - 2 \cdot D \cdot D + D^2 \cdot 1 \\
&= \sum_{k \in N} \left(t_k^2 \cdot q_k \right) - 2 \cdot D \cdot \sum_{k \in N} \left(t_k \cdot q_k \right) + D^2 \cdot \sum_{k \in N} q_k \\
&= \sum_{k \in N} \left(\left(t_k^2 - 2 \cdot D \cdot t_k + D^2 \right) \cdot q_k \right) \\
&= \sum_{k \in N} \left(\left(t_k - D \right)^2 \cdot q_k \right) \\
&\geq 0.
\end{aligned}
$$

(C.3) Theorem: $D'_\infty(\delta) \leq 0$

Proof: We first note that

$$
\begin{aligned}
P'_\infty(\delta) &= \sum_{k \in N} \left(-t_k \cdot a_k \cdot \mathrm{e}^{-t_k \cdot \delta} \right) \\
D_\infty(\delta) &= \frac{-P'_\infty(\delta)}{P(\delta)} \\
C_\infty(\delta) &= \frac{P''_\infty(\delta)}{P(\delta)}.
\end{aligned}
$$

We can now see that

$$
\begin{aligned}
D'_\infty(\delta) &= \frac{-P_\infty(\delta) \cdot P''_\infty(\delta) + P'_\infty(\delta) \cdot P'_\infty(\delta)}{P_\infty(\delta)^2} \\
&= -C_\infty(\delta) + D_\infty(\delta) \cdot D_\infty(\delta) \\
&\leq 0.
\end{aligned}
$$

Theorem (C.3) shows that Macaulay duration decreases as the interest rate increases.

(C.4) Theorem: $F_\infty(\delta) \leq P_\infty(\delta)$

Proof: Set $V_\infty(\delta) = P_\infty(\delta) \cdot \mathrm{e}^{T \cdot \delta}$. Then

$$
V'_\infty(\delta) = P_\infty(\delta) \cdot T \cdot \mathrm{e}^{T \cdot \delta} + P'_\infty(\delta) \cdot \mathrm{e}^{T \cdot \delta} = P_\infty(\delta) \cdot \mathrm{e}^{T \cdot \delta} \cdot \left(T - D_\infty(\delta) \right).
$$

Using Taylor's Theorem with Remainder, there is *j* between δ and δ_0 such that

$$
V_\infty(\delta) = V_\infty(\delta_0) + (\delta - \delta_0) \cdot V'_\infty(j)
$$

and hence

$$
\begin{aligned}
P_\infty(\delta) \cdot \mathrm{e}^{T \cdot \delta} &= P_\infty(\delta_0) \cdot \mathrm{e}^{T \cdot \delta_0} + (\delta - \delta_0) \cdot P_\infty(j) \cdot \mathrm{e}^{T \cdot j} \cdot (T - D_\infty(j)) \\
&= P_\infty(\delta_0) \cdot \mathrm{e}^{T \cdot \delta_0} + (\delta - \delta_0) \cdot P_\infty(j) \cdot \mathrm{e}^{T \cdot j} \cdot (D_\infty(\delta_0) - D_\infty(j)).
\end{aligned}
$$

If $\delta \le \delta_0$ then $\delta \le j \le \delta_0$, and because of (C.3), $D_\infty(j) \ge D_\infty(\delta_0)$, and

$$\left(\delta - \delta_0\right) \cdot (D_\infty(\delta_0) - D_\infty(j)) \ge 0.$$

Similarly, if $\delta \ge \delta_0$, then $\left(\delta - \delta_0\right) \cdot (D_\infty(\delta_0) - D_\infty(j)) \ge 0$.

Thus

$$\begin{aligned} P_\infty(\delta) \cdot \mathrm{e}^{T \cdot \delta} &\ge P_\infty(\delta_0) \cdot \mathrm{e}^{T \cdot \delta_0} \\ P_\infty(\delta) &\ge P_\infty(\delta_0) \cdot \mathrm{e}^{-T \cdot (\delta - \delta_0)} \\ &= F_\infty(\delta). \end{aligned}$$

(C.5) Theorem: $F_1(i) \le F_2(i) \le P(i)$

Proof:

$$\begin{aligned} F_1(i) &\le F_2(i) \\ &= F_\infty(\ln(1+i)) \\ &\le P_\infty(\ln(1+i)) \\ &= P(i). \end{aligned}$$

Appendix D: Derivation of Second-Order Macaulay Approximation

As in Appendix A, we let $V(i) = P(i)\cdot(1+i)^T$ where $T = D_{\text{mac}}(i_0)$, and we remember that $V'(i_0) = 0$. We will use a second-order Taylor approximation for *V*, and therefore we compute the first and second derivatives of *V*:

$$V'(i) = P(i)\cdot T\cdot(1+i)^{T-1} + P'(i)\cdot(1+i)^T \tag{D.1}$$

and

$$\begin{aligned} V''(i) &= P(i)\cdot T\cdot(T-1)\cdot(1+i)^{T-2} + 2\cdot P'(i)\cdot T\cdot(1+i)^{T-1} + P''(i)\cdot(1+i)^T \\ &= P(i)\cdot(1+i)^{T-2}\cdot\left(T\cdot(T-1) + 2\cdot\frac{P'(i)}{P(i)}\cdot T\cdot(1+i) + \frac{P''(i)}{P(i)}\cdot(1+i)^2\right) \\ &= P(i)\cdot(1+i)^{T-2}\cdot\left(T\cdot(T-1) - 2\cdot D_{\text{mod}}(i)\cdot T\cdot(1+i) + C_{\text{mod}}(i)\cdot(1+i)^2\right) \\ &= P(i)\cdot(1+i)^{T-2}\cdot\left(T\cdot(T-1) - 2\cdot D_{\text{mac}}(i)\cdot T + C_{\text{mac}}(i) + D_{\text{mac}}(i)\right). \end{aligned} \tag{D.2}$$

In particular, for $i = i_0$, we have

$$\begin{aligned} V''(i_0) &= P(i_0)\cdot(1+i_0)^{T-2}\cdot\left(T\cdot(T-1) - 2\cdot D_{\text{mac}}(i_0)\cdot T + C_{\text{mac}}(i_0) + D_{\text{mac}}(i_0)\right) \\ &= P(i_0)\cdot(1+i_0)^{T-2}\cdot\left(T\cdot(T-1) - 2\cdot T\cdot T + C_{\text{mac}}(i_0) + T\right) \\ &= P(i_0)\cdot(1+i_0)^{T-2}\cdot\left(C_{\text{mac}}(i_0) - T^2\right). \end{aligned} \tag{D.3}$$

We now use the second-order Taylor approximation for *V*(*i*) about i_0:

$$V(i) \approx V(i_0) + (i-i_0)\cdot V'(i_0) + \frac{(i-i_0)^2}{2}\cdot V''(i_0)$$

This translates to

$$\begin{aligned} P(i)\cdot(1+i)^T &\approx P(i_0)\cdot(1+i_0)^T + 0 + \frac{(i-i_0)^2}{2}\cdot\left(P(i_0)\cdot(1+i_0)^{T-2}\cdot\left(C_{\text{mac}}(i_0) - T^2\right)\right) \\ &= P(i_0)\cdot(1+i_0)^T\cdot\left(1 + \frac{(i-i_0)^2}{(1+i_0)^2}\cdot\frac{C_{\text{mac}}(i_0) - T^2}{2}\right) \end{aligned}$$

from which we obtain the second-order Macaulay approximation:

$$P(i) \approx P(i_0)\cdot\left(\frac{1+i_0}{1+i}\right)^T\cdot\left(1 + \frac{(i-i_0)^2}{(1+i_0)^2}\cdot\frac{C_{\text{mac}}(i_0) - T^2}{2}\right). \tag{D.4}$$

Acknowledgements

The author thanks Steve Kossman, David Cummings, and Stephen Meskin for their suggestions during the preparation of this note and for their review of drafts containing various errors. Of course, any errors that remain in the note are the responsibility of the author.

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