The pressure, densities and first order phase transitions associated with multidimensional SOFT

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Outline of the talk

Motivation: Ising model - 1925

On lattice $\mathbb{Z}^d$ two kinds of particles: spin up 1 and spin down 2. Each neighboring particles located on $(\mathbf{i}, \mathbf{i}+\mathbf{e}_j)$ interact with energy $-J$ if both locations are occupied by the same particles, and with energy J if the two sites are occupied by two different particles. In addition each particle has a magnetization due to the external magnetic field. The energy of the particle of type 1 is H while the energy of the particle of type 2 is $-H$. The energy of $E(\phi)$ of a given finite configuration of particles in $\mathbb{Z}^d$ is the sum of the energies of the above type.

Ferromagnetism $J > 0$: all spins are up or down.

Antiferromagnetism $J < 0$ half spins up and down

(Lowest free energy)

Phase transition:

from one state to another as the temperature varies

Energy: $\frac{k}{T}E(\phi)$

$$< n > := \{1, 2, 3, ..., n\}$$

ALPHABET ON n LETTERS - COLORS.

$$\underbrace{\bullet \bullet \bullet \ldots \bullet \bullet \bullet}_{m \text{ times}}$$

Coloring of $\mathbb{Z}^d$ in n coloring =
Full $\mathbb{Z}^d$ shift on n symbols
Example of SOFT: $(0 - 1)$ LIMITED CHANNEL
HARD CORE LATTICE or NEAR NEIGHBOR EXCLUSION
$n = 2, < 2 > = \{1, 2\} = \{1, 0\}$ $(2 \equiv 0)$.
NO TWO $1'$s ARE NEIGHBORS.

One dimensional SOFT

$\Gamma \subseteq < n > \times < n >$ directed graph on n vertices $C_\Gamma(< m >)$-all Γ allowable configurations of length m:

$\{a = a_1 ... a_m = (a_i)_1^m :< m > \to < n > (a_i, a_{i+1}) \in \Gamma\}$

$C_\Gamma(\mathbb{Z})$-all Γ allowable configurations (tilings) on $\mathbb{Z}$:

$\{a = (a_i)_{i \in \mathbb{Z}} : \mathbb{Z} \to < n >, (a_i, a_{i+1}) \in \Gamma\}$

Hard core model:

$n = 2$, $\Gamma = \{$●●, ●●, ●●$\}$

MD SOFT=Potts Models

Dimension $d \geq 2$. For $\mathbf{m} \in \mathbb{N}^d$

$< \mathbf{m} > := < m_1 > \times \ldots \times < m_d >$

$\mathrm{vol}(\mathbf{m}) := |m_1| \times \ldots \times |m_d|$

$\Gamma := (\Gamma_1, \ldots, \Gamma_d), \Gamma_i \subset \langle n \rangle \times \langle n \rangle$

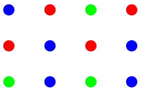
$C_\Gamma(< \mathbf{m} >)$-all Γ allowable configurations of $\mathbf{m}$:

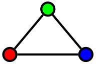
$a = (a_\mathbf{i})_{\mathbf{i} \in \langle \mathbf{m} \rangle} : < \mathbf{m} > \to < n >$

s.t. $(a_\mathbf{i}, a_{\mathbf{i}+\mathbf{e}_j}) \in \Gamma_j$ if $\mathbf{i}, \mathbf{i} + \mathbf{e}_j \in \langle \mathbf{m} \rangle$

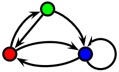
$\mathbf{e}_j = (\delta_{1j}, \ldots, \delta_{dj}), j = 1, \ldots, d$.

Example:

$< (4, 3) > :=$ 



Γ_1



Γ_2

For $\phi \in C_\Gamma(\langle \mathbf{m} \rangle)$ - $\mathbf{c}(\phi) := (c_1(\phi), \ldots, c_n(\phi))$
denotes coloring distribution of configuration ϕ
$c_i(\phi)$-the number of times the particle i appears in ϕ
$\frac{1}{\mathrm{vol}(\mathbf{m})}\mathbf{c}(\phi) \in \Pi_n$ - coloring frequency of ϕ
$\Pi_n(\mathrm{vol}(\mathbf{m}))$ all $\mathbf{c} \in \mathbb{Z}_+^n$ s.t. $\frac{1}{\mathrm{vol}(\mathbf{m})}\mathbf{c} \in \Pi_n$
$C_\Gamma(\langle \mathbf{m} \rangle, \mathbf{c})$ denotes all $\phi \in C_\Gamma(\langle \mathbf{m} \rangle)$ with $\mathbf{c}(\phi) = \mathbf{c}$.
$C_{\Gamma,\mathrm{per}}(\langle \mathbf{m} \rangle) \subseteq C_\Gamma(\langle \mathbf{m} \rangle)$-$\mathbf{m}$-periodic configurations
$C_\Gamma(\mathbb{Z}^d)$-are-Γ allowable configurations of $\mathbb{Z}^d$
Assumption: $C_\Gamma(\mathbb{Z}^d) \neq \emptyset$
$u_i \in \mathbb{R}$ energy of particle $i \in \langle n \rangle$
$\mathbf{u} := (u_1, \ldots, u_n) \in \mathbb{R}^n$ energy vector
$E(\phi) = \mathbf{c}(\phi) \cdot \mathbf{u}$ Energy of configuration ϕ
Near neighbor interaction model, can be fit to the above noninteraction model by considering the coloring of the cube $\langle (3, \ldots, 3) \rangle$ as one particle
Similarly short range interaction model

Pressure

Grand partition function

$Z_\Gamma(\mathbf{m},\mathbf{u}) := \sum_{\phi\in C_\Gamma(\langle\mathbf{m}\rangle)} e^{\mathbf{c}(\phi)\cdot\mathbf{u}}$

$\log Z_\Gamma(\mathbf{m},\mathbf{u})$ subadditive in each component of $\mathbf{m}$ and convex in $\mathbf{u}$

$\frac{1}{\mathrm{vol}(\mathbf{m})}\log Z_\Gamma(\mathbf{m},\mathbf{u})$ - average energy or pressure

$P_\Gamma(\mathbf{u}) := \lim_{\mathbf{m}\to\infty} \frac{1}{\mathrm{vol}(\mathbf{m})}\log Z_\Gamma(\mathbf{m},\mathbf{u})$

Pressure of Γ-SOFT, (Pressure of the Potts model)

$h_\Gamma := P_\Gamma(\mathbf{0})$-ENTROPY of Γ-SOFT

$P_\Gamma(\mathbf{u})$ is a convex Lipschitz function on $\mathbb{R}^n$

$|P_\Gamma(\mathbf{u}) - P_\Gamma(\mathbf{v})| \le ||\mathbf{u}-\mathbf{v}||_\infty := \max |u_i - v_i|$

$P_\Gamma(\mathbf{u}+t\mathbf{1}) = P_\Gamma(\mathbf{u}) + t$

P_Γ has the following properties:

Has subdifferential $\partial P_\Gamma(\mathbf{u})$ for each $\mathbf{u}$

$\partial P_\Gamma(\mathbf{u}) \subseteq \Pi_n$ for each $\mathbf{u}$

Has differentiable $\nabla P_\Gamma(\mathbf{u})$ a.e.

Density points and density entropy

$\mathbf{p} \in \Pi_n$ density point of $C_\Gamma(\mathbb{Z}^d)$ when there exist sequences of boxes $\langle \mathbf{m}_q \rangle \subseteq \mathbb{N}^d$ and color distribution vectors $\mathbf{c}_q \in \Pi_n(\mathrm{vol}(\mathbf{m}_q))$
$\mathbf{m}_q \to \infty, \ \ C_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q) \neq \emptyset \ \ \forall q \in \mathbb{N}, \quad$ and $\lim_{q\to\infty} \frac{\mathbf{c}_q}{\mathrm{vol}(\mathbf{m}_q)} = \mathbf{p}$

Π_Γ the set of all density points of $C_\Gamma(\mathbb{Z}^d)$

Π_Γ is a closed set

For $\mathbf{p} \in \Pi_\Gamma$ the color density entropy
$h_\Gamma(\mathbf{p}) := \sup_{\mathbf{m}_q, \mathbf{c}_q} \limsup_{q\to\infty} \frac{\log \#C_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q)}{\mathrm{vol}(\mathbf{m}_q)} \geq 0$
where the supremum is taken over all sequences satisfying the above conditions

h_Γ is upper semi-continuous on Π_Γ

Convex functions

$f : \mathbb{R}^n \to [-\infty, \infty]$ convex.
$\mathrm{dom}\, f := \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) < \infty\}$
f proper if $f : \mathbb{R}^n \to \bar{\mathbb{R}} := (-\infty, \infty]$ and $f \not\equiv \infty$
f closed if f is lower semi-continuous.
$\mathbf{q}$ subgradient: $f(\mathbf{x}) \ge f(\mathbf{u}) + \mathbf{q}^\top(\mathbf{x} - \mathbf{u})\ \forall \mathbf{x}$
$\partial f(\mathbf{u}) \subset \mathbb{R}^n$ the subset of subgradients of f at $\mathbf{u}$
ASSUMPTION: f is proper and closed
$\partial f(\mathbf{u})$ is a closed nonempty set for each $\mathbf{u} \in \mathrm{ri}\ \mathrm{dom}\, f$
f is differentiable at $\mathbf{u} \iff \partial f(\mathbf{u}) = \{\nabla f(\mathbf{u})\}$
$\mathrm{diff}\, f$ - the set of differentiability points of f
∇f continuous on $\mathrm{diff}\, f$ and $\overline{\mathrm{diff}\, f} \supseteq \mathrm{dom}\, f$
The conjugate, (Legendre transform) f^* defined:
$f^*(\mathbf{y}) := \sup_{\mathbf{x}\in\mathbb{R}^n} \mathbf{x}^\top \mathbf{y} - f(\mathbf{x})$ for each $\mathbf{y} \in \mathbb{R}^m$
f^* is a proper closed function and $f^{**} = f$

P_Γ^* and color density entropy

Thm 1: $h_\Gamma(\mathbf{p}) \leqslant -P_\Gamma^*(\mathbf{p})\ \forall \mathbf{p} \in \Pi_\Gamma$.
$P_\Gamma(\mathbf{u}) = \max_{\mathbf{p}\in\Pi_\Gamma}(\mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})), \mathbf{u} \in \mathbb{R}^n$
$\Pi_\Gamma(\mathbf{u}) := \arg\max_{\mathbf{p}\in\Pi_\Gamma}(\mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})) = \{\mathbf{p} \in \Pi_\Gamma : P_\Gamma(\mathbf{u}) = \mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})\}$
For each $\mathbf{p} \in \Pi_\Gamma(\mathbf{u})$, $h_\Gamma(\mathbf{p}) = -P_\Gamma^*(\mathbf{p})$.
$\Pi_\Gamma(\mathbf{u}) \subseteq \partial P_\Gamma(\mathbf{u})$.
$\mathbf{u} \in \mathrm{diff}\, P_\Gamma \Rightarrow \Pi_\Gamma(\mathbf{u}) = \{\nabla P_\Gamma(\mathbf{u})\}$.
Therefore $\partial P_\Gamma(\mathrm{diff}\, P_\Gamma) \subseteq \Pi_\Gamma$.
$S(\mathbf{u}), \mathbf{u} \in \mathbb{R}^n \setminus \mathrm{diff}\, P_\Gamma$-
are all the limits of sequences
$\nabla P_\Gamma(\mathbf{u}_i), \mathbf{u}_i \in \mathrm{diff}\, P_\Gamma$ and $\mathbf{u}_i \to \mathbf{u}$.
Then $S(\mathbf{u}) \subseteq \Pi_\Gamma(\mathbf{u})$.
$\mathrm{conv}\ \Pi_\Gamma(\mathbf{u}) = \mathrm{conv}\ S(\mathbf{u}) = \partial P_\Gamma(\mathbf{u})$.
$\partial P_\Gamma(\mathbb{R}^n) \subseteq \mathrm{conv}\ \Pi_\Gamma \subseteq \Pi_n$.
$\mathrm{conv}\ \Pi_\Gamma = \mathrm{dom}\, P_\Gamma^*$.

Outline of proof

From the definitions of $P_\Gamma(\mathbf{u}), \mathbf{p}$,

$h_\Gamma(\mathbf{p}) := \sup_{\mathbf{m}_q, \mathbf{c}_q} \limsup_{q\to\infty} \frac{\log \# C_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q)}{\mathrm{vol}(\mathbf{m}_q)} \geq 0$

$P_\Gamma(\mathbf{u}) \geqslant \mathbf{p}^\top \mathbf{u} + h_\Gamma(\mathbf{p}) \Rightarrow P_\Gamma(\mathbf{u}) \geqslant \sup_{\mathbf{p}\in\Pi_\Gamma} \mathbf{p}^\top \mathbf{u} + h_\Gamma(\mathbf{p}) \Rightarrow$

$-h_\Gamma(\mathbf{p}) \geqslant P_\Gamma^*(\mathbf{p}) \Rightarrow \Pi_\Gamma \subseteq \mathrm{dom}\, P_\Gamma^*$

$\mathbf{C}(\mathbf{m}, \mathbf{u}) := \arg\max_{\mathbf{c}\in\Pi_n(\mathrm{vol}(\mathbf{m}))} \# C_\Gamma(\langle \mathbf{m} \rangle, \mathbf{c}) e^{\mathbf{c}^\top \mathbf{u}}$

$Z_\Gamma(\mathbf{m}, \mathbf{u}) = O(\mathrm{vol}(\mathbf{m})^{n-1}) \# C_\Gamma(\langle \mathbf{m} \rangle, \mathbf{c}(\mathbf{m}, \mathbf{u})) e^{\mathbf{c}(\mathbf{m},\mathbf{u})^\top \mathbf{u}}$

Let $\mathbf{m}_q \to \infty$ s.t. $\frac{\mathbf{c}(\mathbf{m}_q, \mathbf{u})}{\mathrm{vol}(\mathbf{m}_q)} \to \mathbf{p}(\mathbf{u}) \Rightarrow$

$P_\Gamma(\mathbf{u}) \leqslant \mathbf{p}(\mathbf{u})^\top \mathbf{u} + \limsup_{q\to\infty} \frac{\log \# C_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}(\mathbf{m}_q, \mathbf{u}))}{\mathrm{vol}(\mathbf{m}_q)} \leqslant \mathbf{p}(\mathbf{u})^\top \mathbf{u} + h_\Gamma(\mathbf{p}(\mathbf{u}))$

For $p \in \Pi_\Gamma(\mathbf{u})$ use maximal characterization

$P_\Gamma(\mathbf{u}+\mathbf{v}) \geqslant \mathbf{p}^\top(\mathbf{u}+\mathbf{v}) + h_\Gamma(\mathbf{p}) = \mathbf{p}^\top \mathbf{v} + P_\Gamma(\mathbf{u})$

So $\mathbf{p} \in \partial P_\Gamma(\mathbf{u}) \Rightarrow \Pi_\Gamma(\mathbf{u}) \subseteq \partial P_\Gamma(\mathbf{u}) \Rightarrow$

$\mathbf{u} \in \mathrm{diff}\, P_\Gamma \Rightarrow \Pi_\Gamma(\mathbf{u}) = \{\nabla P_\Gamma(\mathbf{u})\}$

First order phase transition

Claim: For $\mathbf{u} \in \mathbb{R}^n$ each $\mathbf{p} \in \Pi_\Gamma(\mathbf{u})$ is the set of possible density of n colors in an allowable configurations from $C_\Gamma(\mathbb{Z}^d)$ with the potential $\mathbf{u}$.
For $\mathbf{u} \in \operatorname{diff} P_\Gamma$ $\mathbf{p}(\mathbf{u}) = \nabla P_\Gamma(\mathbf{u})$ is a unique density.
Claim: Any point of nondifferentiabity of P_Γ is a point of the phase transition.
Proof Let $\mathbf{u} \in \mathbb{R}^n \setminus \operatorname{diff} P_\Gamma$ Then ∂P_Γ consists of more than one point.
Thm 1 yields that $\partial P_\Gamma(\mathbf{u}) = \operatorname{conv} S(\mathbf{u}) \subseteq \Pi_\Gamma(\mathbf{u})$. $S(\mathbf{u})$ consists of more than one point. Hence $\Pi_\Gamma(\mathbf{u})$ consists of more than one density for $\mathbf{u}$.
$\mathbf{u} \in \mathbb{R}^n \setminus \operatorname{diff} P_\Gamma$ is called a point of phase transition, or a phase transition point of the first order.

$C_\Gamma(\mathbb{Z}^d)$-a compact metric space.
It is invariant under the shifts
$\sigma_i : C_\Gamma(\mathbb{Z}^d) \to C_\Gamma(\mathbb{Z}^d), i = 1, \ldots, d$
$\sigma_i(\phi)$ is obtained by shifting the allowable configuration $\phi \in C_\Gamma(\mathbb{Z}^d)$ using the transformation $\mathbf{x} \mapsto \mathbf{x} - \mathbf{e}_i$.
Let $\mathcal{M}_\Gamma$ be the compact set of invariant measures on $C_\Gamma(\mathbb{Z}^d)$ with respect to $\sigma_i, i = 1, \ldots, d$.
$h_\Gamma(\mu)$ -Kolmogorov-Sinai entropy for $\mu \in \mathcal{M}_\Gamma$
$h_\Gamma(\mu) = \lim_{m\to\infty} \frac{1}{(2m+1)^d} H_\mu(\vee_{-m\le i_1,\ldots,i_d\le m} \sigma_1^{i_1} \ldots \sigma_d^{i_d} \mathcal{A})$
where $\mathcal{A} = \{A_1, \ldots, A_n\}$
a cylinder partition of $\mathbb{C}_\Gamma(\mathbb{Z}^d)$.
A_i - the set of all configurations $\phi \in C_\Gamma(\mathbb{Z}^d)$ s.t. $\mathbf{0} \in \mathbb{Z}^d$ colored by color i in ϕ.

The maximum principle

$f_{\mathbf{u}} : C_\Gamma \to \mathbb{R}$ be given by
$f_{\mathbf{u}}(\phi) = u_i$ for $\phi \in A_i, \mathbf{u} = (u_1, \dots, u_n)$.
$P_\Gamma(\mathbf{u}) = \max_{\mu \in \mathcal{M}_\Gamma} h_\Gamma(\mu) + \int f_{\mathbf{u}}(\mathbf{x}) d\mu(x)$
$\mu_{\mathbf{u}} \in \mathcal{M}_\Gamma$ is maximal if
$P_\Gamma(\mathbf{u}) = h_\Gamma(\mu) + \int f_{\mathbf{u}}(\mathbf{x}) d\mu(x)$
$\mathbf{u}$ -ergodic phase transition
if there are at least two maximal $\mu_{\mathbf{u}}$ measures
Conjecture If $\mathbf{u} \in \mathbb{R}^n \setminus \mathrm{diff}\, P_\Gamma$ then $\mathbf{u}$ is an ergodic phase transition
Special case studied case in the literature $\mathbf{u} = 0$:
The entropy
$h_\Gamma = P_\Gamma(\mathbf{0}) = \max_{\mathbf{p} \in \Pi_\Gamma} h_\Gamma(\mathbf{p}) = \max_{\mu \in \mathcal{M}_\Gamma} h_\Gamma(\mu)$

d-Dimensional Monomer-Dimers

Dimer: $(\mathbf{i},\mathbf{j}),\ \mathbf{j} = \mathbf{i} + \mathbf{e}_k \in \mathbb{Z}^d$.
any partition of $\mathbb{Z}^d$ to dimers (1-factor).
Monomer: occupies $\mathbf{i} \in \mathbb{Z}^d$.
any partition of $\mathbb{Z}^d$ to monomer-dimers
is 1-factor of a subset of $\mathbb{Z}^d$.

Dimer and Monomer-Dimer are SOFT

$$0 = \tilde{h}_1 \le \tilde{h}_2 \le \ldots \le \tilde{h}_d \le \ldots (\mathrm{dimers})$$

$$\log \frac{1+\sqrt{5}}{2} = h_1 \le h_2 \le \ldots \le h_d \le \ldots$$
$$(\mathrm{monomer} - \mathrm{dimer})$$

Fisher, Kasteleyn and Tempreley 61

$$\tilde{h}_2 = \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.29156090\ldots$$

Hammersley's results

Hammersley in 60's studied extensively the monomer-dimer model. He showed $\Pi_\Gamma = \Pi_{d+1}$ for d-dimensional model $\mathbf{p} = (p_1, \ldots, p_d, p_{d+1})$
p_i-the dimer density in $\mathbf{e}_i$-direction $i = 1, \ldots, d$ p_{d+1}-the monomer density Hammersley studied $p := p_1 + \ldots + p_d$-the total dimer density
$h_d(p)$-the p-dimer density in $\mathbb{Z}^d$, $p \in [0, 1]$
He showed $h_d(p)$-concave continuous function on $[0, 1]$
Heilman and Lieb 72: $h_d(p)$ analytic on $(0, 1)$
No phase transition in parameter $p \in (0, 1)$
Au-Yang and Perk 84: Phase transition at $p = 1$
Friedland-Krop-Lundow-Markstrom 08
$h_d(p) + \frac{1}{2}(p \log p + (1 - p) \log(1 - p))$ concave

The Graphs for $h_2(p)$

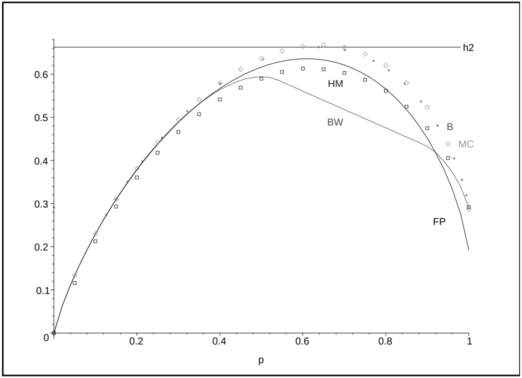
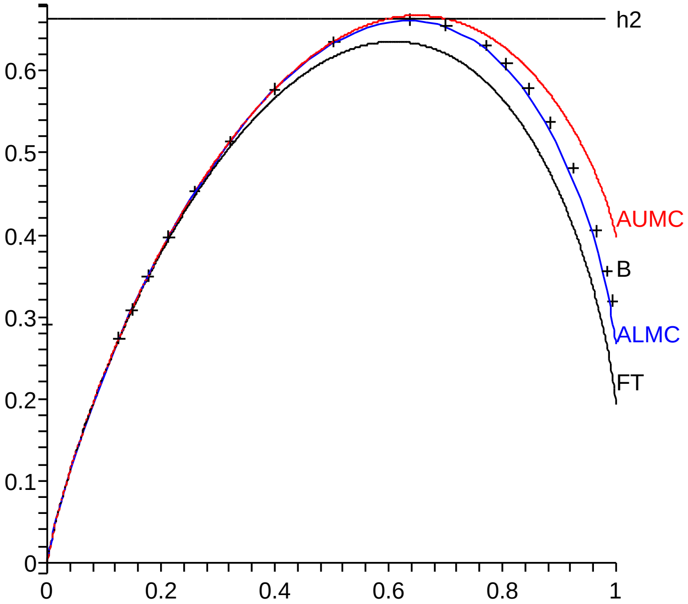


Figure: HM is the lower bound of Hammersley-Menon, BW is the lower bound of Bondy-Welsh, FP is the lower bound of Friedland-Peled, MC is the Monte Carlo estimate of Hammersley-Menon, B are Baxter's estimates, and h2 is the true value of $h_2 = \max h_2(p)$.

Graph



Friendly colorings

Thm 1 implies:

For any Potts model $h_\Gamma(\cdot) : \Pi_\Gamma \to \mathbb{R}_+$ is concave on every convex subset of $\Pi_\Gamma(\mathbb{R}^n)$.

To get the exact analog of Hammersley's result

$\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ on $\langle n \rangle$

$\mathcal{F} = \cup_{\mathbf{m} \in \mathbb{N}^d} \widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$, where $\widetilde{C}_\Gamma(\langle \mathbf{m} \rangle) \subseteq C_\Gamma(\langle \mathbf{m} \rangle)$ for each $\mathbf{m} \in \mathbb{N}^d$,

friendly: if whenever a box $\langle \mathbf{m} \rangle$ is cut in two and each part is colored by a coloring in $\mathcal{F}$, the combined coloring is in $\mathcal{F}$.

Γ friendly if there exist a friendly set $\mathcal{F} = \cup_{\mathbf{m} \in \mathbb{N}^d} \widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$ and a constant vector $\mathbf{b} \in \mathbb{N}^d$ such that if any box $\langle \mathbf{m} \rangle$ is padded with an envelope of width b_i in the direction of $\mathbf{e}_i$, then each Γ-allowed coloring of $\langle \mathbf{m} \rangle$ can be extended in the padded part to a coloring in $\mathcal{F}$.

Examples of friendly colorings

Γ has a friendly color $f \in \langle n \rangle$, i.e., for each $i \in \langle d \rangle$ $(f, j), (j, f) \in \Gamma_i$ for all $j \in \langle n \rangle$

Then $\widetilde{C}_\Gamma(\mathbf{m})$ are Γ-allowed colorings of $\langle \mathbf{m} \rangle$ whose boundary points are colored with f

Hard-core model: $\Gamma_i = \{(1,1),(1,2),(2,1)\}$, has friendly color $f = 1$.

Γ associated with the monomer-dimer covering

$\widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$ the set of tilings of $\langle \mathbf{m} \rangle$ by monomers and dimers, i.e., the coverings in which no dimer protrudes out of $\langle \mathbf{m} \rangle$, as in Hammersley

P^*_Γ for friendly colorings

Thm 2: Let $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ be a friendly coloring digraph. Then

1. Π_Γ is convex. Hence $\Pi_\Gamma = \mathrm{dom}\, P^*_\Gamma$.
2. $h_\Gamma(\cdot) : \Pi_\Gamma \to \mathbb{R}_+$ is concave.
3. For each $\mathbf{u} \in \mathbb{R}^n$, $\Pi_\Gamma(\mathbf{u}) = \partial P_\Gamma(\mathbf{u})$.
4. For each $\mathbf{u} \in \mathbb{R}^n$, $h_\Gamma(\cdot)$ is an affine function on $\partial P_\Gamma(\mathbf{u})$.
5. $h_\Gamma(\mathbf{p}) = -P^*_\Gamma(\mathbf{p})$ for each $\mathbf{p} \in \Pi_\Gamma$.

Outline of proof

(1). Let $\alpha \in \widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$, $\mathbf{c}(\alpha) = (c_1, \ldots, c_n) \in \Pi_n(\mathrm{vol}(\mathbf{m}))$ color frequency vector of α, and $\mathbf{p} := \frac{1}{\mathrm{vol}(\mathbf{m})}\mathbf{c}(\alpha)$.

For $\mathbf{j} = (k_1, \ldots, k_d) \in \mathbb{N}^d$ let $\mathbf{j} \cdot \mathbf{m} := (k_1 m_1, \ldots, k_d m_d)$. View $\langle \mathbf{j} \cdot \mathbf{m} \rangle$ as a box composed of $\mathrm{vol}(\mathbf{j})$ boxes isomorphic to $\langle \mathbf{m} \rangle$ color each box by α obtaining a coloring $\alpha(\mathbf{j} \cdot \mathbf{m}) \in \widetilde{C}_\Gamma(\mathbf{j} \cdot \mathbf{m})$. Clearly $\mathbf{p} = \frac{1}{\mathrm{vol}(\mathbf{j} \cdot \mathbf{m})} c(\alpha(\mathbf{j} \cdot \mathbf{m}))$. Choose $\mathbf{j}_q \to \infty$ to deduce $\mathbf{p} \in \Pi_\Gamma$.

Let $\beta \in \widetilde{C}_\Gamma(\langle \mathbf{n} \rangle)$. So $\mathbf{q} := \frac{1}{\mathrm{vol}(\mathbf{n})}\mathbf{c}(\beta) \in \Pi_\Gamma$.

Claim: For $i, j \in \mathbb{N}$ $\frac{i}{i+j}\mathbf{p} + \frac{j}{i+j}\mathbf{q} \in \Pi_\Gamma$.

Let $\alpha(\mathbf{n} \cdot \mathbf{m}), \beta(\mathbf{m} \cdot \mathbf{n}) \in \widetilde{C}_\Gamma(\mathbf{n} \cdot \mathbf{m})$ defined as above. Let $\mathbf{j} := (m_1 n_1, \ldots, m_{d-1} n_{d-1}, (i+j) m_d n_d)$ view box $\langle \mathbf{j} \rangle$ composed of $i + j$ boxes isomorphic to $\langle \mathbf{m} \cdot \mathbf{n} \rangle$ aligned side-by-side along the direction of $\mathbf{e}_d$. Color the first i of these boxes by $\alpha(\mathbf{m} \cdot \mathbf{n})$ and the last j by $\beta(\mathbf{n} \cdot \mathbf{m})$, to get $\gamma \in \widetilde{C}_\Gamma(\langle \mathbf{j} \rangle)$ with $\frac{1}{\mathrm{vol}(\mathbf{j})}\mathbf{c}(\gamma) = \frac{i}{i+j}\mathbf{p} + \frac{j}{i+j}\mathbf{q}$. Hence $\frac{i}{i+j}\mathbf{p} + \frac{j}{i+j}\mathbf{q} \in \Pi_\Gamma$. Since Π_Γ is closed $a\mathbf{p} + (1-a)\mathbf{q} \in \Pi_\Gamma$ for all $a \in [0, 1]$.

Outline of proof-II

Let $\widetilde{\Pi}_\Gamma$ be the convex hull of $\frac{1}{\mathrm{vol}(\mathbf{m})}\mathbf{c}(\alpha)$ for some $\mathbf{m}$ and some $\alpha \in \widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$. So $\widetilde{\Pi}_\Gamma \subseteq \Pi_\Gamma$.

The padding part of definition of Γ friendly implies

$\widetilde{\Pi}_\Gamma \subseteq \Pi_\Gamma \subseteq \mathrm{Cl}\, \widetilde{\Pi}_\Gamma \Rightarrow \Pi_\Gamma = \mathrm{Cl}\, \widetilde{\Pi}_\Gamma$

Equality $\Pi_\Gamma = \mathrm{dom}\, P^*_\Gamma$ follows from last part of Thm 1.

Outline of proof-III

(b) The padding part of definition of Γ friendly implies
For $\mathbf{p}, \mathbf{q} \in \Pi_\Gamma, \varepsilon > 0\ \exists$
$\mathbf{m}_q := (m_{1,q}, \ldots, m_{d,q}), \mathbf{n}_q := (n_{1,q}, \ldots, n_{d,q}) \in \mathbb{N}^d, q \in \mathbb{N},\ \mathbf{m}_q, \mathbf{n}_q \to \infty$
s.t.

$$\widetilde{C}_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q), \widetilde{C}_\Gamma(\langle \mathbf{n}_q \rangle, \mathbf{d}_q) \neq \emptyset, q \in \mathbb{N},$$

$$\lim_{\mathbf{m}_q \to \infty} \frac{1}{\mathrm{vol}(\mathbf{m}_q)} \mathbf{c}_q = \mathbf{p}, \lim_{\mathbf{n}_q \to \infty} \frac{1}{\mathrm{vol}(\mathbf{n}_q)} \mathbf{d}_q = \mathbf{q},$$

$$\lim_{q \to \infty} \frac{\log \# \widetilde{C}_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q)}{\mathrm{vol}(\mathbf{m}_q)} \geq h_\Gamma(\mathbf{p}) - \varepsilon,$$

$$\lim_{q \to \infty} \frac{\log \# \widetilde{C}_\Gamma(\langle \mathbf{n}_q \rangle, \mathbf{d}_q)}{\mathrm{vol}(\mathbf{n}_q)} \geq h_\Gamma(\mathbf{q}) - \varepsilon.$$

Outline of proof - IV

Observation that for any $\mathbf{m}, \mathbf{n} \in \mathbb{N}^d, \mathbf{c} \in \Pi_n(\mathrm{vol}(\mathbf{m}))$:
$\#\widetilde{C}_\Gamma(\langle \mathbf{n} \cdot \mathbf{m} \rangle, \mathrm{vol}(\mathbf{n})\mathbf{c}) \ge (\#\tilde{C}_\Gamma(\langle \mathbf{m} \rangle, \mathbf{c}))^{\mathrm{vol}(\mathbf{n})}$ yields:
For $i, j \in \mathbb{N}$ $h_\Gamma(\frac{i}{i+j}\mathbf{p} + \frac{j}{i+j}\mathbf{q}) \ge \frac{i}{i+j}h_\Gamma(\mathbf{p}) + \frac{j}{i+j}h_\Gamma(\mathbf{q}) - \varepsilon$
which proves the concavity of h_Γ.
(c-d): Let $\mathbf{u} \in \mathrm{diff}\, P_\Gamma$.
Then $\Pi_\Gamma(\mathbf{u}) = \{\nabla P_\Gamma(\mathbf{u})\} = \partial P_\Gamma(\mathbf{u})$ and (c-d) trivially hold.
Recall $S(\mathbf{u}) \subseteq \Pi_\Gamma(\mathbf{u})$, $\mathrm{conv}\, S(\mathbf{u}) = \partial P_\Gamma(\mathbf{u}) \supseteq \Pi_\Gamma(\mathbf{u})$ Let
$\mathbf{p}_i \in S(\mathbf{u}), i = 1, \ldots, j$. So $P_\Gamma(\mathbf{u}) = \mathbf{p}_i^\top \mathbf{u} + h_\Gamma(\mathbf{p}_i), i = 1, \ldots, j$
Since Π_Γ convex, for $\mathbf{a} = (a_1, \ldots, a_j) \in \Pi_j$ $\mathbf{p} := \sum_{i=1}^j a_i \mathbf{p}_i \in \Pi_\Gamma$.
As h_Γ concave $P_\Gamma(\mathbf{u}) = \sum_{i=1}^j a_i \mathbf{p}_i^\top \mathbf{u} + h_\Gamma(\mathbf{p}_i) \le \mathbf{p}^\top \mathbf{u} + h_\Gamma(\mathbf{p})$
The maximal characterization of $P_\Gamma(\mathbf{u})$ implies $P_\Gamma(\mathbf{u}) = \mathbf{p}^\top \mathbf{u} + h_\Gamma(\mathbf{p})$.
So $\mathbf{p} \in \Pi_\Gamma(\mathbf{u})$ and $h_\Gamma(\mathbf{p}) = \sum_{i=1}^j a_i h_\Gamma(\mathbf{p}_i)$.
(e) Follows from Thm 1 and extra arguments using convexity of P_Γ^*

Reduction of one parameter

$P_\Gamma(\mathbf{u}) = t + P_\Gamma(\mathbf{u} - t\mathbf{1}) \Rightarrow \partial P_\Gamma(\mathbf{u}) \in \Pi_n$
It is enough to compute $\hat{P}_\Gamma(\hat{\mathbf{u}}) := P_\Gamma(\hat{\mathbf{u}}), \hat{\mathbf{u}} = (u_1, \ldots, u_{n-1}, 0)$
Hard core model: $\hat{P}_\Gamma(t)$ depends on the energy $t \in \mathbb{R}$.
(It is known that for $d \geq 2$ hard core model has phase transition)
For the dimer problem the pressure $P_d(\mathbf{v})$ depends on $\mathbf{v} = (v_1, \ldots, v_d)$,
where v_i is the energy of the dimer in the direction $\mathbf{e}_i, i = 1, \ldots$.
(Non-isotropic model)
Dimer isotropic model in $\mathbb{Z}^d$:
pressure $P_d(v)$, where v is the energy of the dimer in any direction.
(Standard model-No phase transition for $v \in \mathbb{R}$)

Computation of pressure

Using the scaled transfer matrices on the torus $T(\mathbf{m}'), \mathbf{m}' = (m_1, \ldots, m_{d-1})$ as in Friedland-Peled 2005 [3].
Assume for simplicity $d = 2, \Gamma = (\Gamma_1, \Gamma_2)$, where Γ_1 symmetric digraph.
Let Δ transfer digraph induced by Γ_2 between the allowable Γ_1 coloring of the circle $T(m)$.
Then $V := C_{\Gamma_1,\mathrm{per}}(m)$ are the set of vertices of $\Delta(m)$. For $\alpha, \beta \in C_{\Gamma_1,\mathrm{per}}(m)$ the directed edge (α, β) is in $\Delta(m)$ iff the configuration $[(\alpha, \beta)]$ is an allowable configuration on $C_\Gamma((m, 2))$.
Adjacency matrix $D(\Delta(m)) = (d_{\alpha\beta})_{\alpha,\beta \in C_{\Gamma_1,\mathrm{per}}(m)}$ is $N \times N$ matrix, where $N := \#C_{\Gamma_1,\mathrm{per}}(m)$.
One dimensional SOFT is $C_\Gamma(T(m) \times \mathbb{Z})$:
all Γ allowable coloring of the infinite torus in the direction $\mathbf{e}_2$ with the basis $T(m)$.
The pressure corresponding to this one dimensional SOFT is denoted by $\tilde{P}_{\Delta(m)}(\mathbf{u})$. Its formula:

Computation of pressure II

Let $\tilde{D}(\Delta(m),\mathbf{u}) = (\tilde{d}_{\alpha\beta}(\mathbf{u}))_{\alpha,\beta\in\mathcal{C}_{\Gamma_1,\mathrm{per}}(m)}$ $\tilde{d}_{\alpha\beta}(\mathbf{u}) = d_{\alpha\beta}e^{\frac{1}{2}(\mathbf{c}(\alpha)+\mathbf{c}(\beta))^\top\mathbf{u}}$

Then $\tilde{P}_\Delta(\mathbf{u}) := \frac{\theta(\mathbf{u},m)}{m}, \theta(\mathbf{u},m) := \log\rho(\tilde{D}(\Delta(m),\mathbf{u}))$

We divide $\log\rho(\tilde{D}(\Delta,\mathbf{u}))$ by m, to have

$\tilde{P}_\Delta(\mathbf{u}+t\mathbf{1}) = \tilde{P}_\Delta(\mathbf{u}) + t \quad \text{for any } t\in\mathbb{R}$

Main inequalities

$\frac{1}{p}(\theta(\mathbf{u},p+2q)-\theta(\mathbf{u},2q)) \le P_\Gamma(\mathbf{u}) \le \frac{1}{2m}(\theta(\mathbf{u},2m))$

for any $m,p\ge 1$ and $q\ge 0$.

Automorphism Subgroups

$A = (a_{ij})_1^N$ nonnegative matrix
$\mathcal{A}(A) := \{\pi \in S_N : a_{\pi(i)\pi(j)} = a_{ij}, i, j \in < N >\}$
Let $G \leq \mathcal{A}(A), \mathcal{O}(G) :=< N > /G, M = \#\mathcal{O}(G)$
$\hat{A} = (\hat{a}_{\alpha\beta})_{\alpha,\beta\in\mathcal{O}(G)}, \hat{a}_{\alpha\beta} =: \sum_{j\in\beta} a_{ij},\ i \in \alpha, \rho(A) = \rho(\hat{A}),$
If $A = A^T$ then $\hat{A}$ symmetric for
$< \mathbf{x}, \mathbf{y} >= \sum_{\alpha\in\mathcal{O}(G)} (\#\alpha) x_\alpha y_\alpha . M \geq N/\#G,$
In our computations $M \sim N/\#G$
Using these tools we confirmed Baxter's computations with nine digits of precision of $P_2(v)$ and of $h_2(p)$.
We also computed the non-isotropic $P_2((v_1, v_2))$.

Graph 1 of pressure 2-dimensional dimers $P(v_1, v_2)$

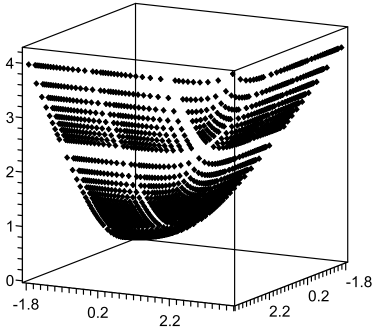


Figure: The graph of $\frac{\bar{P}_1(12,(v_1,v_2))}{12}$ for angle $\theta = 28^o, \varphi = 78^o$

Graph 2 of pressure 2-dimensional dimers $P(v_1, v_2)$

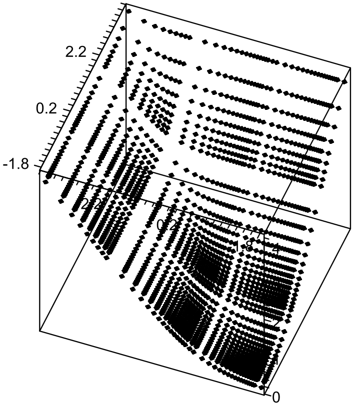


Figure: The graph of $\frac{\bar{P}_1(12,(v_1,v_2))}{12}$ for angle $\theta = -159^o, \varphi = 42^0$

Graph 1 of 2-dimensional entropy $h_2(v_1, v_2)$

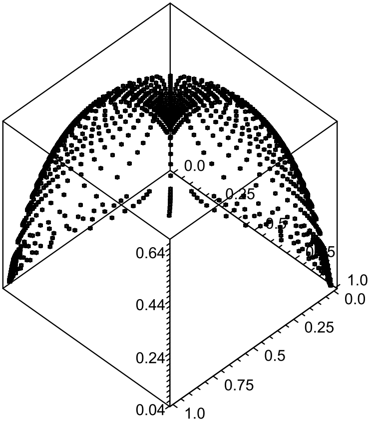


Figure: The graph of an approximation of $\bar{h}_2(p_1, p_2)$ for angle $\theta = 45^o, \varphi = 45^o$

Graph 2 of 2-dimensional entropy $h_2(v_1, v_2)$

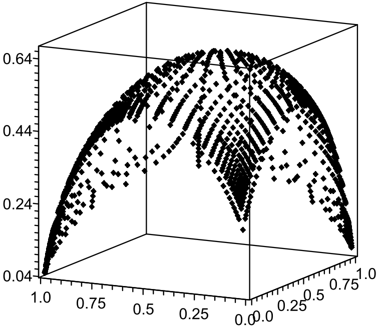


Figure: $\bar{h}_2(p_1, p_2)$ for angle $\theta = -153^o, \varphi = 78^o$

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