Discrete Lyapunov exponents and Hausdorff dimension *

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§0. Introduction

A major concept in differentiable dynamics is the Lyapunov exponents of a given map f. It combines the results of ergodic theory with differential properties of f. Consider the following two closely related examples which motivate our paper. Let M be a compact surface and $f : M \to M$ a smooth diffeomorphism. Let $\mathcal{E}$ denote the set of Borel f-invariant ergodic measures on M. Assume that $\mu \in \mathcal{E}$. Let $h(\mu)$ be the μ-entropy of f and $\lambda_1(\mu) \ge \lambda_2(\mu)$ be the Lyapunov exponents of f. Suppose that $h(\mu) > 0$. The well known result of L.S. Young [**You**] yields that $\frac{h(\mu)}{\lambda_1(\mu)}$ is the Hausdorff dimension of μ unstable manifold $W^u(\mu)$ associated with $\lambda_1(\mu)$. Suppose furthermore that f is an Axiom A diffeomorphism. Then for each x in the nonwandering set $\Omega(f)$ one has the unstable manifold $W^u(x)$. The result of McCluskey-Manning [**M-M**] yields that the Hausdorff dimension of any $W^u(x) \cap \Omega(f)$ is equal to $\sup_{\mu \in E} \frac{h(\mu)}{\lambda_1(\mu)}$. The supremum is achieved for a unique Gibbs measure μ^*.

Let $f : \mathbf{CP} \to \mathbf{CP}$ be a rational map of the Riemann sphere $\mathbf{CP}$ of degree at least two. Denote by $J(f)$ the Julia set of f. Let $\mathcal{E}$ be all f-invariant ergodic measures supported on $J(f)$. For $\mu \in \mathcal{E}$, f has two equal Lyapunov exponents $\lambda_1(\mu) = \lambda_2(\mu) \ge 0$. Assume that $h(\mu) > 0$. Then the μ-Hausdorff dimension of $J(f)$, given by $\inf_{X \subset J(f), \mu(X)=1} dim_H X$, is equal to $\frac{h(\mu)}{\lambda(\mu)}$. Suppose furthermore that f is hyperbolic. That is, $|(f^{\circ m})'(z)| > 1,\ z \in J(f)$, for some integer $m \ge 1$. Then $dim_H J(f) = \sup_{\mu \in \mathcal{E}} \frac{h(\mu)}{\lambda(\mu)}$. The above supremum is achieved for a unique Gibbs measure μ^* which is equivalent to the Hausdorff measure on $J(f)$ [**Rue**]. In these two examples the proof of the variational formula for the Hausdorff dimension is based on the notion of the topological pressure and the Bowen equation [**Bow1-2**].

In this paper we generalize these results to a discrete setting as follows. Let $< n >= \{1, ..., n\}$ be an alphabet on n symbols. Denote by $\mathbf{N}$ and $< n >^{\mathbf{N}}$ the set of natural numbers and the space of all infinite sequences on $< n >$ symbols, equipped with the Tychonoff topology, respectively. Let $\sigma :< n >^{\mathbf{N}} \to < n >^{\mathbf{N}}$ be the (one sided) shift map. Then a σ invariant closed set $\mathcal{S} \subset < n >^{\mathbf{N}}, \sigma\mathcal{S} = \mathcal{S}$, is called a subshift. A subshift $\mathcal{S}$ is called a subshift of finite type (SFT) if $\mathcal{S}$ can be described by a finite number of conditions. A standard representation of SFT is given by a digraph $\Gamma \subset < n > \times < n >$ as follows:

$$\Gamma^\infty := \{(a_i)_1^\infty \in < n >^{\mathbf{N}}: \quad (a_i, a_{i+1}) \in \Gamma, \quad i = 1, ..., \}.$$

It is well known that any SFT $\mathcal{S} \subset < n >^{\mathbf{N}}$ can be represented in the above way by enlarging the given alphabet. Recall that if f an Axiom A diffeomorphisms (hyperbolic rational map) then the $\Omega(f)$ $(J(f))$ has a Markov partition ([**Shu**, Thm. 10.28], [**Rue**]). The action f on $\Omega(f)$ $(J(f))$ induces a SFT, such that the map f on $\Omega(f)$ $(J(f))$ induces the shift map on SFT. If $\Omega(f)$ $(J(f))$ is totally disconnected then $\Omega(f)$ $(J(f))$ is homeomorphic to SFT. If $\Omega(f)$ $(J(f))$ is not totally disconnected then all the points of $\Omega(f)$ $(J(f))$, whose orbit stays in the interior of the Markov partition, are in one to one correspondance with a "big" subset of SFT.

Let $\mathcal{S} \subset < n >^{\mathbf{N}}$ be a subshift. Then every point $a = (a_i)_1^\infty \in \mathcal{S}$ corresponds to an infinite walk on the digraph $< n > \times < n >$. For convenience, we view our walk starting from o: $o - a_1 - a_2 - ...$. To each finite path $o - a_1 - ... - a_p$, viewed as $(a_i)_1^p$, we assign a positive weight $\phi_p(a_1, ..., a_p)$. Assume that

$$\phi_{p+q}(a_1, ..., a_{p+q}) \le \phi_p(a_1, ..., a_p) + \phi_q(a_{p+1}, ..., a_{p+q}), \quad \forall (a_i)_1^\infty \in \mathcal{S}. \tag{0.1}$$

$\phi_p(a_1, ..., a_p)$ can be viewed as the distance of the end of the path $(a_i)_1^p$ to o. Then (0.1) is equivalent to the triangle inequality. For each $p \ge 1$ let $\psi_p : \mathcal{S} \to \mathbf{R}_+$ be the random variable such that $\psi_p((a_i)_1^\infty) = \phi_p(a_1, ..., a_p)$. Denote by ϕ and ψ the sequences $\{\phi\}_1^\infty$ and $\{\psi\}_1^\infty$ respectively. Let μ be a σ-invariant ergodic

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measure on $\mathcal{S}$. Kingman's subadditive ergodic theorem yields that that the sequence $\frac{\psi_p}{p}, p = 1, ...,$ converges μ a.e. to $\alpha(\mu)$. We show that $\alpha(\mu)$ is the discrete version of the Lyapunov exponent of the distance function induced by ϕ. Assume furthermore that for each $t > 0$ there exists $A(t)$ so that

$$\phi_p(a_1, ..., a_p) > t, \quad p > A(t), \quad a \in \mathcal{S}. \tag{0.2}$$

Let $d : \mathcal{S} \to \mathbf{R}_+$ be the following metric on $\mathcal{S}$:

$$\begin{aligned}
&d(a,a) = 0, \quad a \in \mathcal{S}, \\
&d(a,b) = 1, \quad a = (a_i)_1^\infty, b = (b_i)_1^\infty, \quad a, b \in \mathcal{S}, \quad a_1 \neq b_1, \\
&d(a,b) = e^{-\phi_p(a_1,...,a_p)}, \quad a = (a_i)_1^\infty, b = (b_i)_1^\infty, \quad a, b \in \mathcal{S}, \quad a_i = b_i, \quad i = 1, ..., p, \quad a_{p+1} \neq b_{p+1}.
\end{aligned} \tag{0.3}$$

The condition (0.2) yields that $\mathcal{S}$ is a complete metric space. For $X \subset \mathcal{S}$ denote by $dim_H X$ the Hausdorff dimension of X with respect to the metric (0.3). Set $\delta(\phi) := dim_H \mathcal{S}$. In [**Fri**] we give an explicit formula for $\delta(\phi)$ for certain sequences ϕ on SFT. The purpose of this paper is treat a much broader class of sequences ϕ on $\mathcal{S}$ than in [**Fri**], e.g. $\mathcal{S}$ not have to be SFT, and to relate $\delta(\phi)$ to the Hausdorff dimension of σ invariant probability measures on $\mathcal{S}$.

Let μ be a probability measure on $\mathcal{S}$. Then

$$\delta(\phi, \mu) := \inf_{X \subset \mathcal{S}, \mu(X)=1} dim_H(X)$$

is called the Hausdorff dimension of μ. Assume that μ is σ-invariant ergodic measure. Denote by $h(\mu)$ the μ entropy of the shift. We prove that

$$\delta(\phi, \mu) = \frac{h(\mu)}{\alpha(\mu)}, \tag{0.4}$$

if either $h(\mu)$ or $\alpha(\mu)$ are positive. Compare this equality with the formulas for the Hausdorff dimension of $W^u(\mu)$ and the μ Hausdorff dimension of $J(f)$ to deduce that $\alpha(\mu)$ is the analog of the Lyapunov exponent. Set $\alpha_m(\mu) = \int \frac{\psi_m}{m} d\mu$. Then $\alpha_m \geq \alpha(\mu), m = 1, ...,$. Moreover, $\lim_{m\to\infty} \alpha_m = \alpha(\mu)$. Hence,

$$\delta(\phi, \mu) \geq \frac{h(\mu)}{\alpha_m(\mu)}, \quad m = 1, ..., \tag{0.5}$$

For certain SFT and corresponding Markov chains one can compute explicitly the right-hand side of (0.5). Thus we can obtain explicit lower bounds for $\delta(\phi, \mu)$. Set

$$\hat{\delta}(\phi) := \sup_{\mu \in \mathcal{E}, h(\mu) > 0} \frac{h(\mu)}{\lambda(\mu)}. \tag{0.6}$$

Then $\delta(\phi) \geq \hat{\delta}(\phi)$. We give a condition on ϕ for which $\delta(\phi) = \hat{\delta}(\phi)$. Under this condition the subshift $\mathcal{S}$, which does not have be a SFT, behaves as a rational map f on its Julia set.

The main motiviation of this paper is the Hausdorff dimension of a Kleinian group $F \leq PSL(2, \mathbf{C})$. (See Maskit [**Mas**] for a reference on the Kleinian groups.) Let o be a point in three dimensional hyperbolic space H^3 on which F acts as a group of hyperbolic isometries. Then Fo, the F orbit of o, accumilates to $\Lambda(F)$, the limit set of F. Here $\Lambda(F)$ is located on the Riemann sphere $\mathbf{CP}$. We want to give computable *lower bounds* on $dim_H \Lambda(F)$, which can be arbitrary close to $dim_H \Lambda(F)$, using only the hyperbolic distances between the points in Fo. We do that for geometrically finite, purely loxodromic, Kleinian groups.

Suppose that F is a Schottky group. Then F is a free group on r generators. The orbit of F is $2r$ regular tree which correspond to a standard SFT Γ^∞, where $\Gamma \subset < 2r > \times < 2r >$ is the graph induced by a free group on r generators. Then the sequence ϕ is the sequence induced by the hyperbolic distances between o and other points of the orbit. We show

$$dim_H \Lambda(F) = \delta(\phi) = \hat{\delta}(\phi). \tag{0.7}$$

Hence we can use lower bounds (0.5) to get lower bounds on $dim_H \Lambda(F)$. Moreover, we show that these lower bounds are arbitrary close to $dim_H \Lambda(F)$. Our results complement the results of Bowen [**Bow2**], who showed how to apply the thermodynamics formalism to the action of F on **CP** to find $dim_H \Lambda(F)$.

In the last section we show how to apply our results to a geometrically finite, purely loxodromic, Kleinian group F. We construct a subshift $\mathcal{S}$ corresponding to $\Lambda(F)$. We do not know if $\mathcal{S}$ is a SFT. (For certain Fuchsian group F, $\Lambda(F)$ has a coding as a SFT, e.g. [**B-S**].) We prove (0.7) in this case.

We now survey briefly the contents of the paper. In §1 we discuss examples of ϕ on subshifts. We define $\kappa(\phi)$ - an analog of the Poincaré exponent. We show the inequality $\delta(\phi) \le \kappa(\phi)$. (It is an opposite to inequality $dim_H \Lambda(F) \ge \kappa(F)$ for Kleinian groups [**B-J**].) In §2 we prove the characterization (0.4). We also show that for topologically transitive SFT the inequality (0.5) gives computational lower bounds to $\hat{\delta}(\phi)$, which can be arbtrary close to $\hat{\delta}(\phi)$. Section 3 is devoted to the nonadditive topological pressure, see [**Fal**] and [**Bar**]. We give a sufficient condition which ensures the variational characterization of the topological pressure. This condition generalizes a condition of Barreira [**Bar**]. This condition on ϕ implies the equality $\delta(\phi) = \hat{\delta}(\phi)$. We show that this condition is satisfied in the context of geometrically finite, purely loxodromic, Kleinian groups. In §4 and §5 we apply our results to the Hausdorff dimension of the limit sets of Schottky groups and geometrically finite, purely loxodromic, Kleinian groups respectively.

§1. Metrics on subshifs

Let $\mathcal{S} \subset < n >^{\mathbf{N}}$ be a subshift. Associate with $\mathcal{S}$ the following infinite tree $\mathcal{T}(\mathcal{S}) := \mathcal{T} = (V, E)$. Let o be the root of the tree. Then each $a = (a_i)_1^\infty \in \mathcal{S}$ represents a chain (geodesic) in $\mathcal{T}$ starting from o. The vertices of this chain are o and $(a_i)_1^m, m = 1, ...,$. The chain is given by $o - (a_1) - \cdots - (a_i)_1^m - \cdots$. Let

$$a = (a_i)_1^\infty, \quad b = (b_i)_1^\infty \in \mathcal{S}, \quad a_i = b_i, \quad i = 1, ..., p, \quad a_{p+1} \ne b_{p+1}.$$

Then the two chains induced by a, b have a common chain $o - a_1 - \cdots - a_p$. If $a_1 \ne b_1$ (p=0), then the two induced chains by a, b have only a common vertex o. Thus

$$V = \{v : \quad v = o, \quad v = (a_i)_1^m, \quad m = 1, ..., \quad a = (a_i)_1^\infty \in \mathcal{S}\}.$$

Let $dist : V \times V \to \mathbf{R}_+$ be a metric on $\mathcal{T}$ which satisfies the following conditions: First,

$$0 < dist((a_i)_1^m, o) = \phi_m(a_1, ..., a_m), \quad m = 1, ..., . \tag{1.1}$$

Second,

$$dist((a_i)_1^{p+q}, (a_i)_1^p) = dist((a_i)_{p+1}^{p+q}, o) = \phi_q(a_{p+1}, ..., a_{p+q}), \quad 1 \le p, q. \tag{1.2}$$

Then the triangle inequality

$$dist((a_i)_1^{p+q}, o) \le dist((a_i)_1^{p+q}, (a_i)_1^p) + dist((a_i)_1^p, o)$$

is equivalent to

$$\phi_{p+q}(a_1, ..., a_{p+q}) \le \phi_p(a_1, ..., a_p) + \phi_q(a_{p+1}, .., a_{p+q}), \quad p, q = 1, ..., . \tag{1.3}$$

Third,

$$a_1 = b_1 \Rightarrow dist((a_i)_1^p, (b_j)_1^q) = dist((a_i)_2^p, (b_j)_2^q), \tag{1.4}$$

$$a_1 \ne b_1 \Rightarrow dist((a_i)_1^p, (b_j)_1^q) = \phi_p(a_1, ..., a_p) + \phi_q(b_1, ..., b_q). \tag{1.5}$$

It is straightforward to show that (0.3) defines a metric d on $\mathcal{S}$. Assume furthermore that (0.2) holds. Then $\mathcal{S}$ is a compact metric with respect to the metric (0.3). Furthermore, the Tychonoff (coordinatewise) topology is induced by the metric d. We let $\delta(\phi)$ to be the Hausdorff dimension of $\mathcal{S}$ with respect to the metric d.

We discuss a few examples that motivate the above definitions. Assume that $\phi_p(a_1,...,a_p) = p$. Then the induced metric is the standard graph metric $dg(\cdot,\cdot)$ on $\mathcal{T}$. That is, $dg(u,v)$ is the number of edges in the chain connecting u,v. Let $h > 0$ and assume that $\phi_p(a_1,...,a_p) = ph$. Then the induced metric is dg_h, the weighted graph metric. The distance between the adjacent vertices is h, i.e. the weight of each edge in $\mathcal{T}$ is h. The choice $h = \log 2$ in (0.3) gives the standard metric on $< n >^{\mathbf{N}}$ and on any of its subshifts.

We now consider examples related to SFT. Let $C = (c_{ij})_1^n \in M_n(\mathbf{R}_+)$ be a nonnegative $n \times n$ matrix. Denote by $\Gamma(C) \subset < n > \times < n >$ is the digraph induced by C. That is,

$$(i,j) \in \Gamma(C) \iff c_{ij} > 0.$$

Denote by $\rho(C)$ the spectral radius of C. For any $\Gamma \subset < n > \times < n >$, let $A(\Gamma) \in M_n(\mathbf{R})$ denote the $0-1$ matrix such that $\Gamma(A(\Gamma)) = \Gamma$. Set $\rho(\Gamma) = \rho(A(\Gamma))$.

Let $\Gamma \subset < n > \times < n >$ be a digraph which has a cycle, i.e. $\rho(\Gamma) > 0$. Set

$$\begin{aligned}
&\Gamma^1 = < n >, \quad \Gamma^2 = \Gamma, \\
&\Gamma^k = \{a: \quad a = (a_i)_1^k, \quad a_i \in < n >, \quad i = 1,...,n, \quad (a_i, a_{i+1}) \in \Gamma, \quad i = 1,...,k-1\}, \quad k = 3,...,. \\
&\Gamma^\infty = \{a: \quad a = (a_i)_1^\infty, \quad a_i \in < n >, \quad i = 1,..., \quad (a_i, a_{i+1}) \in \Gamma, \quad i = 1,...,\}.
\end{aligned}$$

Then Γ^∞ is a nonempty SFT induced by Γ. Let $h > 0$ and consider the weighted graph metric dg_h. It is well known (e.g. [**Fri**]) that

$$\delta(\phi) = \frac{\log \rho(\Gamma)}{h}. \tag{1.6}$$

Let $C = (c_{ij})_1^n \in M_n(\mathbf{R}_+)$. Suppose that $\Delta \subset \Gamma(C), \rho(\Delta) > 0$. On $\mathcal{T}(\Delta^\infty)$ we define a metric using the following functions:

$$\phi_1(i) = t, \quad i \in < n >, \quad \max_{1 \le i,j} c_{ij} \le t, \quad \phi_p(a_1,...,a_p) = t + \sum_{i=1}^{p-1} c_{a_i a_{i+1}}, \quad (a_i)_1^p \in \Delta^p, \quad p = 2,...,. \tag{1.7}$$

In [**Fri**] we give the following formula for $\delta(\phi)$. Let

$$\begin{aligned}
&B(x) = (b_{ij}(x))_1^n, \quad \rho(x) = \rho(B(x)), \quad x > 0, \\
&b_{ij}(x) = e^{-xc_{ij}}, \quad (i,j) \in \Gamma(C), \quad b_{ij}(x) = 0, \quad (i,j) \notin \Gamma(C).
\end{aligned}$$

Then $\delta(\phi) \ge 0$ is the unique nonnegative number so that

$$\rho(\delta(\phi)) = 1, \quad \rho(x) < 1, \quad \text{for} \quad x > \delta(\phi).$$

This characterization of $\delta(\phi)$ appears in Mauldin and Williams [**M-W**] for certain geometrical constructions in $\mathbf{R}^m$. If $\frac{C}{h}$ is a matrix with rational entries for some $h > 0$ then (1.6) holds for an appropriate Γ ([**Fri**]).

Consider the special linear group $SL(N,\mathbf{C})$ of $N \times N$ complex valued matrices. Let I denote the identity matrix. For $A \in SL(N,\mathbf{C})$ let $A^* \in SL(N,\mathbf{C})$ denote the conjugate transpose of A. Recall that the spectral norm $||A||$ is given by the formula $||A||^2 = \rho(AA^*) = \rho(A^*A)$, where $\rho(B)$ is the spectral radius of $B \in SL(N,\mathbf{C})$. Note that $A \in SL(N,\mathbf{C}) \Rightarrow ||A|| \ge 1$. Let $SU(N,\mathbf{C}) \subset SL(N,\mathbf{C})$ be the special unitary group, i.e. the maximal compact subgroup of $SL(N,\mathbf{C})$. It is easy to show that

$$B \in SL(N,\mathbf{C}), \quad ||B|| = 1 \iff B \in SU(N,\mathbf{C}).$$

Assume that $A_1,...,A_n \in SL(N,\mathbf{C})$. Suppose furthermore that the following conditions are satisfied:

$$A_i \notin SU(N,\mathbf{C})), \quad i = 1,...,n, \quad A_{a_1} \cdots A_{a_m} \notin SU(N,\mathbf{C}), \quad (a_i)_1^m \in \Gamma^m, \quad m = 1,...,. \tag{1.8}$$

The above conditions hold if $A_1,...,A_n$ generate a torsion free and discrete semigroup (in the standard topology). Set

$$\phi_p(a_1,...,a_p) = 2\log ||A_{a_1} \cdots A_{a_p}||, \quad (a_i)_1^p \in \Gamma^p, \quad p = 1,...,. \tag{1.9}$$

Then (1.1) holds. Furthermore, the submultiplicativity of the norm yields (1.3). If (1.8) does not hold, replace (1.1) by

$$0 \leq \phi_m(a_1, ..., a_m), \quad m = 1, ..., . \tag{1.1$'$}$$

Then the conditions $(1.1)'$ and $(1.2) - (1.5)$ yield that $dist(\cdot, \cdot)$ is a semimetric on $\mathcal{T}(\Gamma^\infty) = (V, E)$. Set

$$V(v) = \{u : \quad u \in V, \quad dist(u, v) = 0\}.$$

For each $v \in V$ we identify the vertices $V(v)$ with one vertex v'. We thus obtain a new graph $\mathcal{T}'$ with the metric $dist(\cdot, \cdot)$.

The following inverse problem arises naturally: Let ϕ be defined as above and assume that $(1.1)'$ and (1.3) hold. Does there exist a separable Hilbert space $\mathcal{H}$ and bounded linear operators $A_i : \mathcal{H} \to \mathcal{H}$ with the the operator norm $||A_i||$ for $i = 1, ..., n$, so that (1.9) holds?

Let $F =< f_1, ..., f_r >$ be a free group on r generators. We identify f_i, f_i^{-1} with $i, i + r$ for $i = 1, ..., r$. That is, $g_i = f_i, g_{i+r} = f_i^{-1}, i = 1, ..., r$. A word $g_{i_1} g_{i_2} \cdots g_{i_m}$ is called a reduced word if $|i_j - i_{j+1}| \neq r, j = 1 = 1, ..., m - 1$. Then F induces the graph $\Gamma =< 2r > \times < 2r > \setminus \cup_{i=1}^r ((i, i + r) \cup (i + r, i))$. That is, Γ^m gives the set of all reduced words in F of length m. Furthermore, Γ^∞ corresponds to all (half) infinite reduced words, which are standardly identified with the limit set $\Lambda(F)$ of F (e.g. [**Fri**]). For each $i \in< 2r >$ let $\bar{i} \in< 2r >$ be the unique number so that $|\bar{i} - i| = r$.

Assume that $\Gamma \subset< 2r > \times < 2r >$ is the graph induced by a free group on r generators. Consider the SFT Γ^∞. Suppose that we have a sequence of functions ϕ satisfying (1.3). We then define the distance function $dist : V \times V \to \mathbf{R}_+$ using (1.1), (1.2) and (1.4). We replace the condition (1.5) by

$$a_1 \neq b_1 \Rightarrow dist((a_i)_1^p, (b_j)_1^q) = \phi_{p+q}(\bar{b}_q, \bar{b}_{q-1}, ..., \bar{b}_1, a_1, ..., a_p). \tag{1.5f}$$

To ensure the equality $dist(u, v) = dist(v, u)$ we assume

$$\phi_p(a_1, ..., a_p) = \phi_p(\bar{a}_p, ..., \bar{a}_1), p = 1, ..., . \tag{1.10}$$

We give a natural set of examples of metrics satisfying (1.1)-(1.4), (1.5f) and (1.10). Let $F =< A_1, ..., A_r >, A_1, ..., A_r \in SL(N, \mathbf{C})$ be a free discrete group. Recall that for $i \in< r >, \bar{i} = i + r$ and $A_{\bar{i}} = A_i^{-1}$. Set

$$\begin{aligned} dist((a_i)_1^p, (b_i)_1^q) &= \log ||A_{b_q}^{-1} \cdots A_{b_1}^{-1} A_{a_1} \cdots A_{a_p}|| + \log ||A_{a_p}^{-1} \cdots A_{a_1}^{-1} A_{b_1} \cdots A_{b_q}|| \\ &= \log ||A_{\bar{b}_q} \cdots A_{\bar{b}_1} A_{a_1} \cdots A_{a_p}|| + \log ||A_{\bar{a}_p} \cdots A_{\bar{a}_1} A_{b_1} \cdots A_{b_q}||, \quad 0 \leq p, q. \end{aligned} \tag{1.11}$$

The above definition implies (1.1)-(1.4), (1.5f) and (1.10). Consider the special case $N = 2$. As for any $B \in SL(2, \mathbf{C})$ we have the equality $||B^{-1}|| = ||B||$, we deduce that for $SL(2, \mathbf{C})$, (1.9) is equivalent to (1.11).

We now show, that the action of a free group $F =< f_1, ..., f_r >$ of hyperbolic isometries on n dimensional hyperbolic space H^n, induces a metric (1.11) on the corresponding orbit of F. For simplicity we consider the cases $n = 2, 3$. Let

$$\begin{aligned} H^2 = SL(2, \mathbf{R})/SO(2, \mathbf{R}), &\quad PSL(2, \mathbf{R}) = SL(2, \mathbf{R})/\{\pm I\}, \\ H^3 = SL(2, \mathbf{C})/SU(2, \mathbf{C}), &\quad PSL(2, \mathbf{C}) = SL(2, \mathbf{C})/\{\pm I\}. \end{aligned}$$

Then $PSL_2(\mathbf{R})$ ($PSL_2(\mathbf{C})$) is the group of orientation preserving isometries acting on H^2 (H^3) by the left multiplication. Assume that $F =< A_1, ..., A_r >\leq PSL_2(\mathbf{R})$ ($PSL_2(\mathbf{C})$) be a discrete free group. Then F acts on H^2 (H^3). Let $o \in H^2$ (H^3) be the point corresponding the coset $SO(2, \mathbf{R})$ ($SU(2, \mathbf{C})$). Then Fo, the F-orbit of o, corresponds to the vertices of the tree $\mathcal{T}(\Gamma^\infty)$. The hyperbolic distance $d_h(u, v), u, v \in Fo$ coincides with the distance given by (1.11). Moreover, if F is a Schottky group then Γ^∞ is isomorphic $\Lambda(F)$-the limit set of F. Furthermore, the metric d on Γ^∞ is Lipschitz equivalent to the metric on $\Lambda(F)$ viewed as a subset of the sphere S^n. (This point is discussed in detail in §4.) In this case, $\delta(\phi)$ is the the Hausdorff dimension of $\Lambda(F)$.

Assume that $\mathcal{S} \subset < n >^{\mathbf{N}}$ is a subshift and suppose that we have a sequence of positive functions ϕ satisfying (1.3). Let $dist(\cdot, \cdot)$ be the distance function defined by (1.1),(1.2),(1.4) and either (1.5) or (1.5f) on the induced tree $\mathcal{T}(\mathcal{S}) = (V, E)$. For any $t > 0$ let

$$B(o,t) = \{v : \quad v \in V, \quad dist(v,o) \le t\}.$$

Assume the condition (0.2). Then $B(o,t)$ is a finite set and let $|B(o,t)|$ be the number of vertices in $B(o,t)$. $|B(o,t)|$ can be considered as the "volume" of $B(o,t)$. The volume growth of the metric $dist$ is given by

$$\kappa(\phi) = \limsup_{t \to \infty} \frac{\log |B(o,t)|}{t}. \tag{1.12}$$

It is straightforward to show that the volume growth of $dist$ is independent of the choice of the root, i.e. in (1.12) we can replace o by any $o' \in V$. In context of discrete groups of hyperbolic motions $\kappa(\phi)$ is identified with the Poincaré exponent of the Poincaré series:

$$\sum_{v \in V} e^{-sdist(o,v)}. \tag{1.13}$$

It is straightforward to show that the above series converge for $s > \kappa(\phi)$ and diverge for $s < \kappa(\phi)$. (See for example the arguments in [**Nic**].)

Let $G \le PSL(2, \mathbf{C})$ be a discrete group of hyperbolic isometries. Denote by $\kappa(G)$ and $dim_H \Lambda(G)$ the Poincaré exponent of G and the Hausdorff of the limit set of G respectively. Then it is known that $\kappa(G)$ is the Hausdorff dimension of the conical limit set of G [**B-J**]. Hence $dim_H \Lambda(G) \ge \kappa(G)$. If G is finitely generated and the area of $\Lambda(G)$ is zero then $dim_H \Lambda(G) = \kappa(G)$. In our context we have the opposite inequality:

Theorem 1.14. *Let $\mathcal{S} \subset < n >^{\mathbf{N}}$ be a subshift. Assume that a positive sequence of functions ϕ satisfies $(0.1) - (0.2)$. Then*

$$\delta(\phi) \le \kappa(\phi).$$

Proof. Let $\Gamma = < n > \times < n >$ be the complete graph on n vertices. For $(a_i)_1^m \in \Gamma^m$ let

$$C((a_i)_1^m) = \{x : \quad x = (x_i)_1^\infty \in \Gamma^\infty, \quad x_i = a_i, \quad i = 1, ..., m, \} \tag{1.15}$$

be the cylindrical set corresponding to $(a_i)_1^m$. Note that $C((a_i)_1^m)$ is open and closed set in the product topology on $\Gamma^\infty = < n >^{\mathbf{N}}$. Then $C((a_i)_1^m) \cap \mathcal{S}$ is an open and closed set of $\mathcal{S}$ (and may be $\emptyset$). For $t > 0$ set

$$S(o,t) = \{v : \quad v = (a_i)_1^m \in B(o,t), \quad \phi(a_1, ..., a_{m+p}) > t, \quad p = 1, ..., \quad (a_i)_1^\infty \in \mathcal{S}\}, \tag{1.16}$$

to be the "boundary sphere" of the ball $B(o,t)$. Clearly, $|S(o,t)| \le |B(o,t)|$. Suppose that $(a_i)_1^m \in S(o,t)$. Then diam $(C((a_i)_1^m) \cap \mathcal{S})$, the diameter of $C((a_i)_1^m) \cap \mathcal{S})$, is less than e^{-t}.

Fix $1 > \epsilon > 0$. Then

$$\cup_{(a_i)_1^m \in S(o, -\log \epsilon)} C((a_i)_1^m) \cap \mathcal{S}$$

is a closed cover of $\mathcal{S}$ with sets of diameters less than ϵ. Hence

$$\sum_{(a_i)_1^m \in S(o, -\log \epsilon)} \text{diam } (C((a_i)_1^m) \cap \mathcal{S})^x \le |S(o, -\log \epsilon)| \epsilon^x \le |B(o, -\log \epsilon)| \epsilon^x, \quad x > 0.$$

Fix $a > 0$. From the definition of $\kappa(\phi)$ it follows that we have a positive constant $K(a) > 0$ so that

$$|B(o, -\log \epsilon)| < K(a) \epsilon^{-\kappa(\phi) - a}, \quad 0 < \epsilon < 1.$$

Then

$$\lim_{\epsilon \to 0^+} |B(o, -\log \epsilon)| \epsilon^x = 0, \quad \text{for} \quad x > \kappa(\phi) + a.$$

Hence $\delta(\phi) \le \kappa(\phi) + a$. As a was an arbitrary positive number we deduce the theorem. $\diamond$

§2. Hausdorff dimension of the invariant measures

Let $\mathcal{B}$ the Borel sigma-algebra on $< n >^{\mathbf{N}}$ generated by cylindrical sets (1.15). Denote by Π the set of probability measures on $< n >^{\mathbf{N}}$ which are invariant under the shift σ. We view $< n >^{\mathbf{N}}$ as a compact metric space equipped with the standard metric (induced by the graph metric dg on $\mathcal{T}(< n >^{\mathbf{N}})$ as in §1). Let $\mathcal{E} \subset \Pi$ be the set of ergodic measures. It is well known that $\mathcal{E}$ is the set of the extreme points of Π in the w^* topology, e.g. [**Wal**, §6.2]. For each $\mu \in \Pi$ let $h(\mu)$ denote the measure entropy of σ. As σ is expansive it follows that $h(\mu)$ is an upper semicontinuous function on Π [**Wal**, §8.2].

Assume that $\mathcal{S} \subset < n >^{\mathbf{N}}$ is a subshift. Let

$$\Pi(\mathcal{S}) := \{\mu : \quad \mu \in \Pi, \quad \mu(\mathcal{S}) = 1\}$$

to be the set of all σ-invariant probability measures supported on $\mathcal{S}$. Let $\mathcal{E}(\mathcal{S}) = \mathcal{E} \cap \Pi(\mathcal{S})$ be the set of the extreme points of $\Pi(\mathcal{S})$. Assume that the nonnegative functions ϕ satisfy the assumptions (0.1)-(0.2). Let d the metric (0.3) on $\mathcal{S}$. Set

$$\delta(\mu, \phi) = \inf_{X \subset \mathcal{S}, \mu(X)=1} dim_H X, \quad \mu \in \Pi(\mathcal{S}),$$

to be the μ-Hausdorff dimension of $\mathcal{S}$ with respect to d. Let $\psi_p : \mathcal{S} \to \mathbf{R}_+, p = 0, ...,$ be defined as follows:

$$\begin{aligned} &\psi_0((a_i)_1^\infty) = 0, \\ &\psi_p((a_i)_1^\infty) = \phi_p(a_1, ..., a_p), \quad p = 1, ..., \quad (a_i)_1^\infty \in \mathcal{S}. \end{aligned} \tag{2.1}$$

That is, ψ_p is the random variable which describes the length of the path on $\mathcal{T}(\mathcal{S})$, of the random variable X, travelled in p units of time starting o so that $X(p) = (a_i)_1^p$. As ψ_p is a continuous function it follows that ψ_p is μ measurable for any $\mu \in \Pi(\mathcal{S})$ and $p = 0, ...,$. From the inequality (0.1) we deduce

$$0 \leq \psi_{p+q} \leq \psi_p + \psi_q \circ \sigma^p, \quad p, q \geq 0, \quad x \in \mathcal{S}.$$

Assume that $\mu \in \Pi(\mathcal{S})$. Kingman's subadditive ergodic theorem claims that the sequence $\frac{\psi_m}{m}, m = 1, ...,$ converges $\mu - a.e.$ to $\alpha(x, \mu) \geq 0$. Furthermore, $\alpha(\sigma(x), \mu) = \alpha(x, \mu)$ $\mu - a.e.$ and

$$\begin{aligned} &\alpha_m(\mu) := \int \frac{\psi_m(x)}{m} d\mu(x), \quad m = 1, ..., \\ &\lim_{m \to \infty} \alpha_m(\mu) = \int \alpha(x, \mu) d\mu(x). \end{aligned} \tag{2.2}$$

The above inequality on ψ implies that $\alpha_{km}(\mu) \leq \alpha_m(\mu), k = 1, ...,$. Hence

$$\int \alpha(x, \mu) d\mu(x) \leq \alpha_m(\mu), \quad m = 1, ..., .$$

If $\mu \in \mathcal{E}(\mathcal{S})$ then $\alpha(x, \mu)$ is a constant function $\alpha(\mu)$ $\mu - a.e.$. See for example [**Wal**, §10.2]. In that case we have:

$$\begin{aligned} &\alpha_m(\mu) \geq \alpha(\mu), \quad m = 1, ..., \\ &\lim_{m \to \infty} \alpha_m(\mu) = \alpha(\mu), \quad \mu \in \mathcal{E}(\mathcal{S}). \end{aligned} \tag{2.3}$$

We will show that $\alpha(x, \mu)$ is the discrete analog of the Lyapunov exponent for the family ψ with respect to μ. We will only consider ergodic μ.

Theorem 2.4. *Let $\mathcal{S} \subset < n >^{\mathbf{N}}$ be a subshift. Assume that the sequence of positive functions ϕ satisfies the conditions* $(0.1) - (0.2)$. *Suppose that dist is the distance function on the vertices of the induced tree $\mathcal{T}(\mathcal{S})$ given by* $(1.1), (1.2), (1.4)$ *an either* (1.5) *or* $(1.5f)$. *Let d be the metric on $\mathcal{S}$ given by* (0.3). *Suppose that ψ is given by* (2.1). *Let $\mu \in \mathcal{E}(\mathcal{S})$ and assume that* $\max(\alpha(\mu), h(\mu)) > 0$. *Then*

$$\delta(\mu, \phi) = \frac{h(\mu)}{\alpha(\mu)}.$$

Proof. Assume first that $\alpha(\mu) > 0$. Let $B(x,r) = \{y : d(x,y) \le r\}$ be the closed ball of radius $r > 0$ centered at $x \in \mathcal{S}$. Assume that $Y \subset \mathcal{S}$ is a Borel set and $\mu(Y) > 0$. Suppose furthermore that for each $y \in Y$ the following inequality holds:

$$\underline{\delta} \le \liminf_{r\to 0} \frac{\log \mu(B(y,r))}{\log r} \le \limsup_{r\to 0} \frac{\log \mu(B(y,r))}{\log r} \le \bar{\delta}.$$

Then $\underline{\delta} \le dim_H Y \le \bar{\delta}$, where $dim_H Y$ is the Hausdorff dimension of Y with respect to the metric d. See for example [**You**, Prop. 2.1]. Then our theorem is implied by

$$\lim_{m\to\infty} \frac{\log \mu(B(y, r_m(y)))}{\log r_m(y)} = \frac{h(\mu)}{\alpha(\mu)}, \quad \lim_{m\to\infty} r_m(y) = 0, \tag{2.5}$$

for μ-almost all $y \in \mathcal{S}$ and a corresponding sequence $r_m(y), m = 1, ...,$. See for example the Remark after Prop. 2.1 in [**You**].

We prove (2.5). Assume that $y = (y_i)_1^\infty \in \mathcal{S}$. Let $B_m(y)$ be the cylinder $C((y_i)_1^m)$. The Shannon-McMilan-Breiman theorem (e.g. [**Wal**]) claims that

$$\lim_{m\to\infty} \frac{\log \mu(B_m(y))}{m} = -h(\mu)$$

for μ-almost all $y \in < n >^{\mathbf{N}}$. (Here we may assume that the finite partition ξ is given by $\{C((1)), ..., C((n))\}$.)

The Kingman subadditive ergodic theorem claims that $\frac{\psi_m(y)}{m}$ converges μ-almost everywhere in $\mathcal{S}$ to $\alpha(\mu)$. Hence

$$\lim_{m\to\infty} \frac{\log \mu(B_m(y))}{\psi_m(y)} = -\frac{h(\mu)}{\alpha(\mu)}$$

μ-almost everywhere. The assumption that $\alpha(\mu) > 0$ and the definition of the metric d by (0.3) means that

$$B_m(y) = B(y, r_m), \quad r_m = e^{-\psi_m(y)} \approx e^{-\alpha(\mu)m}$$

for μ-almost all y. Combine the above equalities to deduce (2.5). This proves the theorem for $\alpha(\mu) > 0$.

Assume that $\alpha(\mu) = 0, h(\mu) > 0$. It is left to show that $\delta(\mu,\phi) = \infty$. Fix $\epsilon > 0$ and let

$$\phi_{p,\epsilon}((a_i)_1^m) = \phi_p((a_i)_1^m) + p\epsilon, \quad (a_i)_1^\infty \in \mathcal{S}, \quad m = 1, ...,.$$

Clearly the functions ϕ_ϵ satisfy (0.1)-(0.2). Denote by $\psi_{p,\epsilon}, p = 1, ...,$ the corresponding functions on $\mathcal{S}$. Let d_ϵ be the induced metric on $\mathcal{S}$. As $\epsilon > 0$ it follows that $d_\epsilon(a,b) < d(a,b), a, b \in \mathcal{S}$. Then for any $X \subset \mathcal{S}$ the Hausdorff dimension of X with respect to d_ϵ does not exceed the Hausdorff dimension of X with respect to d. Hence $\delta(\mu,\phi) \ge \delta(\mu,\phi_\epsilon)$. Clearly,

$$\alpha_\epsilon(\mu) = \lim_{m\to\infty} \int \frac{\psi_{m,\epsilon}}{m} d\mu = \lambda(\mu) + \epsilon = \epsilon.$$

The previous arguments show that $\delta(\mu,\phi_\epsilon) = \frac{h(\mu)}{\epsilon}$. Hence $\delta(\mu,\phi) \ge \frac{h(\mu)}{\epsilon}$ for any $\epsilon > 0$. Thus $\delta(\mu,\phi) = \infty$ and the proof of the theorem is complete. $\diamond$

Comparing the formula for $\delta(\phi,\mu)$ in (2.4) with the the formula for the μ - Hausdorff dimension of $J(f)$ discussed in §0, we realize that $\alpha(\mu)$ is a discrete analog of the Lyapunov exponent for the family ϕ satisfying the assumptions (0.1)-(0.2). One also can view $\alpha(\mu)$ as an average weight of an edge in the tree $\mathcal{T}(\mathcal{S})$.

Corollary 2.6. *Under the assumptions of Theorem 2.4,*

$$\delta(\mu,\phi) \ge \frac{h(\mu)}{\alpha_m(\mu)}, \quad m = 1, ...,$$
$$\delta(\mu,\phi) = \lim_{m\to\infty} \frac{h(\mu)}{\alpha_m(\mu)}.$$

Let $\Gamma \subset < n > \times < n >$ be a strongly connected graph on n vertices. That is, the $0-1$ matrix $A(\Gamma)$ is irreducible. Assume that $S = (s_{ij})_1^n$ is a stochastic matrix whose graph $\Gamma(S) = \Gamma$. Let $\pi = (\pi_1, ..., \pi_n)$ be the unique probability left eigen-vector of S. That is, π is a positive vector whose coordinates add to one and $\pi S = \pi$. Define the probability measure ν_S on $< n >^{\mathbf{N}}$ by its value on the cylindrical sets:

$$\nu_S(C((i))) = \pi_i, \quad i \in < n >, \quad \nu_S(C((a_i)_1^m)) = \pi_{a_1} s_{a_1 a_2} \cdots s_{a_{m-1} a_m}, \quad (a_i)_1^m \in \Gamma^m, \quad m > 1. \tag{2.7}$$

It is well known that ν_S is shift invariant. As Γ was assumed to be strongly connected, we deduce that the shift is ergodic with respect to ν_S, e.g. [**Wal**, Thm. 1.13]. Recall that ([**Wal**, §4.8])

$$h(\nu_S) = -\sum_{1 \le i,j \le n} \pi_i s_{ij} \log s_{ij}, \tag{2.8}$$

and the topological entropy h_{top} of the shift restricted to Γ^∞ is equal to $\log \rho(\Gamma)$. Furthermore, our assumption that Γ is strongly connected implies that there exists a unique ergodic invariant measure μ so that the Kolmogorov-Sinai measure entropy $h(\mu)$ is equal to h_{top}. This is so called Parry measure [**Par**]. This measure is ν_P where P is the unique stochastic matrix of the form

$$P = \rho(\Gamma)^{-1} D^{-1} A(\Gamma) D, \quad D = diag(u_1, ..., u_n), \quad u = (u_1, ..., u_n)^T > 0, \quad A(\Gamma)u = \rho(\Gamma)u. \tag{2.9}$$

See for example [**Wal**, Thm 8.10].

Corollary 2.10. *Let $\Gamma \subset < n > \times < n >$ be a strongly connected graph on n vertices. Assume that the sequence of positive functions ϕ satisfies the conditions* $(0.1) - (0.2)$ *for $\mathcal{S} = \Gamma^\infty$. Suppose that ν_S is an ergodic measure given by* (2.7)*. Then*

$$\alpha(\nu_S) \le \frac{1}{m} \sum_{(a_i)_1^m \in \Gamma^m} \pi_{a_1} s_{a_1 a_2} \cdots s_{a_{m-1} a_m} \phi_m(a_1, ..., a_m) = \alpha_m(\nu_S) \le \sum_{i=1}^n \pi_i \phi_1(i) = \alpha_1(\nu_S), \quad m = 2, ...,$$
$$\delta(\nu_S, \phi) \ge \frac{h(\nu_S)}{\alpha_m(\nu_S)}, \quad m = 1, ..., .$$

Suppose furthermore that $n = 2r$ and Γ is the graph induced by a free group on r generators. Then for the Parry measure ν_P we have the following:

$$\lambda(\nu_P) \le \frac{\sum_{(a_i)_1^m \in \Gamma^m} \phi(a_1, ..., a_m)}{m 2r (2r-1)^{m-1}} = \alpha_m(\nu_P), \quad m = 1, ...,$$
$$\delta(\nu_P, \phi) \ge \frac{\log(2r-1)}{\alpha_m(\nu_P)}, \quad m = 1, ..., .$$

Let the assumption of Theorems 2.4 hold. Set

$$\hat{\delta}(\phi) := \sup_{\mu \in \mathcal{E}(\mathcal{S}), h(\mu) > 0} \frac{h(\mu)}{\alpha(\mu)}. \tag{2.11}$$

As $\delta(\phi) \ge \delta(\mu, \phi)$ we obtain that $\delta(\phi) \ge \hat{\delta}(\phi)$. Combine this observation with Theorem 1.14 to obtain

$$\kappa(\phi) \ge \delta(\phi) \ge \hat{\delta}(\phi). \tag{2.12}$$

We now show how to use Corollary 2.11 to obtain lower bounds for the $\hat{\delta}(\phi)$ which converge to $\hat{\delta}(\phi)$. Let $\Gamma \subset < n > \times < n >$ be a strongly connected digraph. Set $\Gamma(1) := \Gamma$ and define $\Gamma(l) \subset \Gamma^l \times \Gamma^l$ for $l > 1$ as follows:

$$\Gamma(l) = \{(a, b) : \quad a = (a_i)_1^l, \quad b = (b_i)_1^l \in \Gamma^l, \quad (a_l, b_1) \in \Gamma\}, \quad l = 2, ..., .$$

Suppose that $A(\Gamma)^p$ is a positive matrix for some $p > 1$. It is straightforward to show that there exists $q > 1$ so that $A(\Gamma(l))^q$ is a positive matrix. Hence $\Gamma(l)$ is strongly connected for $l = 1, ...,$. Assume that $A(\Gamma)^p$ is never positive. Then there exists $1 < p \le n$ so that $A(\Gamma)$ has exactly p distinct eigenvalues of modulus $\rho(\Gamma)$, e.g. [**Min**]. It is straightforward to show that if l and p are coprime than $\Gamma(l)$ is strongly connected. (For any $l \ge 1$ $\Gamma(l)$ is a disjoint union of strongly connected graphs.) Denote by $\Pi(\Gamma^l), \Sigma(\Gamma(l))$ the space of probability measures on Γ^l and the space of stochastic matrices induced by $\Gamma(l)$:

$$\Pi(\Gamma^l) := \{\pi : \quad \pi = (\pi_i)_{i\in\Gamma^l} \ge 0, \quad \sum_{i\in\Gamma^l} \pi_i = 1\},$$
$$\Sigma(\Gamma(l)) := \{B = (b_{ij})_{i,j\in\Gamma^l} : \quad B \ge 0, \quad b_{ij} = 0 \quad \forall (i,j) \notin \Gamma(l), \quad \sum_{j\in\Gamma^l} b_{ij} = 1, \quad i \in \Gamma^l\},$$
$$l = 1, ..., .$$

For each $B \in \Sigma(\Gamma(l))$ let

$$\Pi(B) := \{\pi : \quad \pi \in \Pi(\Gamma^l), \quad \pi B = \pi\}.$$

Note that if B is irreducible then $\Pi(B)$ consists of a unique probability eigenvector of B.

Theorem 2.13. *Let $\Gamma \subset < n > \times < n >$ be a strongly connected graph on n vertices. Assume that the sequence of positive functions ϕ satisfies the conditions* $(0.1) - (0.2)$ *for $\mathcal{S} = \Gamma^\infty$. Fix $l \ge 1$ an let*

$$\delta_l(\phi) = \max_{(\pi_i)_{i\in\Gamma^l}\in\Pi(B), B=(b_{ij})_{i,j\in\Gamma^l}\in\Sigma(\Gamma(l))} \frac{-\sum_{i,j\in\Gamma^l} \pi_i b_{ij} \log b_{ij}}{\sum_{i\in\Gamma^l} \pi_i \phi_l(i)}, \quad l = 1, ..., .$$

Then

$$\begin{aligned} &\hat{\delta}(\phi) \ge \delta_l(\phi), \quad l = 1, ..., \\ &\lim_{l\to\infty} \delta_l(\phi) = \hat{\delta}(\phi). \end{aligned} \tag{2.14}$$

Proof. Let $\mathcal{T}(\Gamma(l)) = (V_l, E_l)$ be the tree induced by $\Gamma(l)$. Then V_l is a subset of V with the same root o and the vertices induced by $\Gamma^{pl}, p = 1, ...,$. Any metric $dist : V \times V : \mathbf{R}_+$ restricts to a metric on V_l. In particular, a sequence of positive functions ϕ satisfying (0.1)-(0.2) will restrict to a metric on V_l satisfying the finiteness assumption. Clearly, $\Gamma(l)^\infty$ is equal to Γ^∞ for $l = 1, ...,$. Furthermore, the action of the shift $\sigma_l : \Gamma(l)^\infty \to \Gamma(l)^\infty$ is identical with $\sigma^l : \Gamma^\infty \to \Gamma^\infty$. Let $\mu \in \mathcal{E}(\Gamma^\infty)$. We view μ as a measure $\tilde{\mu} \in \mathcal{E}(\Gamma(l)^\infty)$. Then

$$\begin{aligned} &h(\tilde{\mu}) = l h(\mu), \\ &\alpha_m(\tilde{\mu}) = l\alpha_{ml}(\mu), \quad m = 1, ..., \\ &\alpha(\tilde{\mu}) = \lim_{m\to\infty} \alpha_m(\tilde{\mu}) = l\alpha(\mu), \\ &\frac{h(\tilde{\mu})}{\alpha(\tilde{\mu})} = \frac{h(\mu)}{\alpha(\mu)}, \\ &\sup_{\tilde{\mu}\in\mathcal{E}(\Gamma(l)^\infty)} \frac{h(\tilde{\mu})}{\alpha(\tilde{\mu})} = \sup_{\mu\in\mathcal{E}(\Gamma^\infty)} \frac{h(\mu)}{\alpha(\mu)} = \hat{\delta}(\phi). \end{aligned}$$

Fix $l \ge 1$ and choose $B = (b_{ij})_{i,j\in\Gamma^l} \in \Sigma(\Gamma(l))$. Suppose first that B is irreducible. Then $\Pi(B) = \{\pi\}$. Let $\nu_B \in \mathcal{E}(\Gamma(l)^\infty)$ be given by (2.7). According to Corollary 2.10 (with $m = 1$),

$$\begin{aligned} &\delta(\phi, \nu_B) \ge \frac{-\sum_{i,j\in\Gamma^l} \pi_i b_{ij} \log b_{ij}}{\sum_{i\in\Gamma^l} \pi_i \phi_l(i)} \Rightarrow \\ &\hat{\delta}(\phi) \ge \frac{-\sum_{i,j\in\Gamma^l} \pi_i b_{ij} \log b_{ij}}{\sum_{i\in\Gamma^l} \pi_i \phi_l(i)}. \end{aligned} \tag{2.15}$$

Assume that B is reducible. Then $\Pi(B)$ is a convex hull of its extreme points $\pi^1, ..., \pi^k \in \Pi(B)$. Each π^j induces an ergodic measure ν^j on $\Gamma(l)^\infty$. Set

$$\delta(\phi, \nu_B) := \max_{1 \le j \le k} \delta(\phi, \nu^j).$$

It is straightforward to show that for any $\pi \in \Pi(B)$ (2.15) holds. Hence $\hat{\delta}(\phi) \ge \delta_l(\phi), l = 1, ...,$.

We now show the second part of (2.14). Assume first that $\hat{\delta}(\phi) < \infty$. Fix $1 > \epsilon > 0$ and assume that

$$\hat{\delta}(\phi) < \frac{h(\mu)}{\alpha(\mu)}(1 + \epsilon), \quad \mu \in \mathcal{E}(\Gamma^\infty), \quad h(\mu) > 0.$$

Since $\sigma : \Gamma^\infty \rightarrow \Gamma^\infty$ is expansive, $h(\mu)$ can be computed with respect to the partition $C(i), i = 1, ..., n$. Recall that the sequence $\frac{1}{m} \sum_{i \in \Gamma^m} -\mu(C(i)) \log \mu(C(i)), m = 1, ...,$ decreases to $h(\mu)$. There exists $N(\epsilon)$ so that

$$(1 + \epsilon)^{-1} \le \frac{m h(\mu)}{\sum_{i \in \Gamma^m} -\mu(C(i)) \log \mu(C(i))} \le 1,$$
$$\alpha(\mu) \le \alpha_m(\mu) \le \alpha(\mu)(1 + \epsilon), \quad m \ge N(\epsilon).$$

Fix $m \ge N(\epsilon)$. Let $\pi_i = \mu(C(i)), i \in \Gamma^m$. Then $\pi = (\pi_i)_{i \in \Gamma^m} \in \Pi(\Gamma^m)$. Assume first that π is a positive vector. Set

$$B = (b_{ij})_{i,j \in \Gamma^m}, \quad b_{ij} = \pi_i^{-1} \mu(C((i,j))), \quad (i,j) \in \Gamma^{2m}, \quad b_{ij} = 0, \quad (i,j) \notin \Gamma^{2m}.$$

Since μ is a probability measure on Γ^∞, it follows that B is a stochastic matrix, i.e. $B \in \Sigma(\Gamma^m)$. As μ is σ invariant we deduce that $\pi \in \Pi(B)$. Let ν be σ-invariant measure on $\Gamma(m)^\infty$ given by (2.7). Use (2.8) and the above inequalities to obtain

$$h(\nu) = - \sum_{i,j \in \Gamma^m} \pi_i b_{ij} \log b_{ij} =$$
$$- \sum_{(i,j) \in \Gamma^{2m}} \mu(C((i,j)) \log \mu(C((i,j)) + \sum_{i \in \Gamma^m} \pi_i \log \pi_i \ge$$
$$2m h(\mu) - m h(\mu)(1 + \epsilon) = (1 - \epsilon) m h(\mu),$$
$$m \alpha_m(\mu) \le m \alpha(\mu)(1 + \epsilon), \quad m \ge N(\epsilon).$$

Hence

$$\hat{\delta}(\phi) \ge \delta_m(\phi) \ge \frac{- \sum_{i,j \in \Gamma^m} \pi_i b_{ij} \log b_{ij}}{\sum_{i \in \Gamma^m} \pi_i \phi_m(i)} \ge \frac{(1 - \epsilon) h(\mu)}{(1 + \epsilon) \alpha(\mu)} > (1 - \epsilon)^3 \hat{\delta}(\phi), \quad m \ge N(\epsilon).$$

These inequalities remain valid for a nonnegative probability vector π. Hence the second part of (2.14) follows. In a similar way one shows the second part of (2.14) when $\hat{\delta}(\phi) = \infty$. $\diamond$

§3. Topological pressure

In this section we give sufficient conditions for the equality $\delta(\phi) = \hat{\delta}(\phi)$ by using the topological pressure. We state our results for subshifts. Let $\mathcal{S} \subset < n >^{\mathbf{N}}$. Let $C(\mathcal{S})$ denote the Banach space of real continuous functions on $\mathcal{S}$ with the max norm $|| \cdot ||$. Assume that

$$\psi_m \in C(\mathcal{S}), \quad ||\frac{\psi_m}{m}|| \le K, \quad m = 1, ...,. \tag{3.1}$$

Set

$$
\begin{aligned}
&M(m) := \{(a_i)_1^m : \quad C((a_i)_1^m) \cap \mathcal{S} \neq \emptyset\}, \quad n = 1, ..., \\
&P_m := \sum_{(a_i)_1^m \in M(m)} \max_{x \in C((a_i)_1^m) \cap \mathcal{S}} e^{\psi_m(x)}, \quad m = 1, ..., \\
&P := \limsup_{m \to \infty} \frac{1}{m} \log P_m.
\end{aligned} \tag{3.2}
$$

Use (3.1) to deduce that $-K \leq P \leq \log n + K$. Let $q : \mathcal{S} \to \mathbf{R}$ be a continuous function. Set

$$
S_m(q)(x) = \sum_{i=0}^{m-1} q(\sigma^i(x)), \quad m = 1, ...,
$$

Then for the sequence $\psi_m = S_m(q), m = 1, ...,$ P is the topological pressure associated with q. See for example [**Wal**, Ch.9]. (We use here the fact that σ is expansive on $\mathcal{S}$.) Set $P(q) := P$. Recall that in this case one has the maximal characterization:

$$
P(q) = \sup_{\mu \in \mathcal{E}(\mathcal{S})} (h(\mu) + \int q d\mu). \tag{3.3}
$$

As σ is expansive, $h(\mu)$ is upper semicontinuous, e.g. [**Wal**, Thm 8.2, pp. 184]. Hence the supremum in (3.3) is achieved for at least one ergodic measure

$$
P(q) = h(\nu) + \int q d\nu, \quad \nu \in \mathcal{E}(\mathcal{S}). \tag{3.4}
$$

If $\mathcal{S}$ is a topologically transitive SFT and q is Hölder continuous, then ν is a unique (Gibbs) measure [**Bow1**]. We call P the topological pressure of $\{\psi_m\}_1^\infty$. See [**Fal**] and [**Bar**] for similar definitions.

Theorem 3.5. *Let $\mathcal{S} \subset < n >^{\mathbf{N}}$ be a subshift and assume that $\psi_m \in C(\mathcal{S}), m = 1, ...,$ satisfy the following condition. There exists $q \in C(\mathcal{S})$ so that*

$$
\lim_{m \to \infty} || \frac{1}{m} (\psi_m - S_m(q)) || = 0. \tag{3.6}
$$

Then the topological pressure P associated with ψ is equal to $P(q)$. For any σ-invariant probability measure set $\alpha(\mu) = \int q d\mu$. Then

$$
\begin{aligned}
&\lim_{m \to \infty} \int \frac{\psi_m}{m} d\mu = \alpha(\mu), \\
&P = \sup_{\mu \in \mathcal{E}(\mathcal{S})} (h(\mu) + \alpha(\mu)).
\end{aligned} \tag{3.7}
$$

The supremum is achieved for some ν satisfying (3.4)

Proof. Let $P_m, P_m(q)$ be defined by (3.2) for ψ_m and $S_m(q)$ respectively. Set $||\frac{1}{m}(\psi_m - S_m(q)|| = \epsilon_m$. Then

$$
\frac{1}{m} \log P_m(q) - \epsilon_m \leq \frac{1}{m} \log P_m \leq \frac{1}{m} \log P_m(q) + \epsilon_m.
$$

Hence $P = P(q)$. The condition (3.6) implies that $\alpha_m(\mu) := \int \frac{\psi_m}{m} d\mu, m = 1, ...,$ converge to $\int q d\mu = \alpha(\mu)$. $\diamond$

Let ϕ be given by (1.7). Define $q \in C(\Delta^\infty)$ to be a piecewise constant function on the cylinders of length two:

$$
q(x) = c_{ij}, \quad x \in C((i,j)) \quad (i,j) \in \Delta.
$$

It is straighforward to show that the induced sequence ψ satisfies (3.6). As q is Hölder continuous, we deduce that ν is a unique Gibbs measure [**Bow1**]. We will show in the next sections that the condition (3.6) satisfied for Schottky groups and geometrically finite, purely loxodromic, Kleinian groups. Recall Bareira's condition [**Bar**, Thm 1.7]:

$$\lim_{m\to\infty} ||\psi_{m+1}(x) - \psi_m(\sigma(x) - q(x)|| = 0 \tag{3.8}$$

for some $q \in C(\mathcal{S})$. It is straightfoward to show that the above condition implies (3.6). However, (3.6) does not have to imply (3.8). Indeed, (3.6) is equivalent to

$$\psi_m = S_m(q) + me_m, \quad e_m \in C(\mathcal{S}), \quad \lim_{m\to\infty} ||e_m|| = 0. \tag{3.9}$$

Clearly, one can choose $\{e_m\}_1^\infty$ such that (3.8) does not hold. We give a simple intrinsic condition for a subadditive sequence $\{\psi_m\}_1^\infty$ which yields (3.7).

Lemma 3.10. *Let $\{\psi_m\}_1^\infty \subset C(\mathcal{S})$ and assume that (3.6) holds for some $q \in C(\mathcal{S})$. Then for each $\epsilon > 0$ there exists $N(\epsilon)$ so that*

$$||\frac{1}{m}(\psi_m - S_m(\frac{\psi_l}{l}))|| \le \epsilon, \quad l > N(\epsilon), \quad m > p(l), \tag{3.11}$$

where $p(l) = l^2$.

Proof. Let $q \in C(\mathcal{S})$. Then

$$S_m(S_l(q)) = lS_m(q) + r_{m,l}(q), \quad ||r_{m,l}(q)|| \le l(l-1)||q||.$$

The condition (3.6) is equivalent to (3.9). Hence

$$\begin{aligned}
&S_m(\psi_l) = S_m(S_l(q) + le_l) = S_m(S_l(q)) + S_m(le_l) = lS_m(q) + r_{m,l}(q) + lS_m(e_l),\\
&\frac{1}{m}(\psi_m - S_m(\frac{\psi_l}{l})) = e_m - \frac{r_{m,l}}{ml} - \frac{S_m(e_l)}{m},\\
&||\frac{1}{m}(\psi_m - S_m(\frac{\psi_l}{l}))|| \le ||e_m|| + \frac{(l-1)||q||}{m} + ||e_l||.
\end{aligned}$$

Choose $N(\epsilon) > \frac{3||q||}{\epsilon}$ so that $||e_m|| \le \frac{\epsilon}{3}$ for $m > N(\epsilon)$. Then (3.11) holds for $p(l) = l^2$. $\diamond$

Theorem 3.12. *Let $\{\psi_m\}_1^\infty \subset C(\mathcal{S})$. Assume that for each $\epsilon > 0$, there exists $N(\epsilon) > 0$, such that the condition (3.11) holds for an increasing sequence of positive integers $\{p(l)\}_1^\infty$. Let μ be a σ-invariant probability measure on $\mathcal{S}$. Then the sequence*

$$\alpha_m(\mu) := \int \frac{\psi_m}{m} d\mu, \quad m = 1, ...,$$

converges to a limit denoted by $\alpha(\mu)$. The topological pressure P associated with ψ has the variational characterization (3.7). Moreover,

$$P = \lim_{l\to\infty} \sup_{\mu\in\mathcal{E}(\mathcal{S})} (h(\mu) + \alpha_l(\mu)) = h(\mu^*) + \alpha(\mu^*), \tag{3.13}$$

for some $\mu^* \in \mathcal{E}(\mathcal{S})$.

Proof. The variational characterization of $P(\frac{\psi_l}{l})$ gives

$$P(\frac{\psi_l}{l}) = \sup_{\mu\in\mathcal{E}(\mathcal{S})} (h(\mu) + \alpha_l(\mu)), \quad l = 1, ..., . \tag{3.14}$$

Assume that μ is an invariant probablity measure. Then (3.11) implies

$$|\alpha_m(\mu) - \alpha_l(\mu)| \le \epsilon, \quad l > N(\epsilon), \quad m > p(l). \tag{3.15}$$

Fix ϵ and $l > N(\epsilon)$. Then

$$|\alpha_m(\mu) - \alpha_k(\mu)| \le 2\epsilon, \quad k, m > p(l).$$

Hence $\{\alpha_m(\mu)\}_1^\infty$ is a Cauchy sequence which converges to $\alpha(\mu)$. Furthermore,

$$|\alpha(\mu) - \alpha_l(\mu)| \le \epsilon, \quad l > N(\epsilon). \tag{3.16}$$

In view of (3.11), $|P - P(\frac{\psi_l}{l})| \le \epsilon,\ l > N(\epsilon)$. Let Q denote the right-hand side of the second equality in (3.7). Use (3.14) and (3.16) to obtain

$$P(\frac{\psi_l}{l}) - \epsilon \le Q \le P(\frac{\psi_l}{l}) + \epsilon, \quad l > N(\epsilon).$$

Then $P = Q$ and P has the characterization (3.7). It is left to show that the supremum in (3.7) is achieved. Let μ_l be an ergodic measure which maximizes (3.14). Pick up a weakly convergent subsequence $\{\mu_{l_k}\}$ which converges to a probability measure ν. Since the shift σ is expansive on $\mathcal{S}$, the Kolmogorov-Sinai measure of $\sigma|\mathcal{S}$ is upper semicontinuous (e.g. [**Wal**, Ch.8]). Hence, for each $\epsilon > 0$ there exists $N_1(\epsilon)$ so that $h(\nu) > h(\mu_{l_k}) - \epsilon, k > N_1(\epsilon)$. Use (3.15) to deduce that

$$h(\nu) + \alpha_m(\mu_{l_k}) > h(\mu_{l_k}) + \alpha_{l_k}(\mu_{l_k}) - 2\epsilon, \quad k > \max(N(\epsilon), N_1(\epsilon)), \quad m > p(l).$$

Let $k \to \infty$ and use the assumption that $\{\mu_{l_k}\}$ converges weakly to deduce that $h(\nu) + \alpha_m(\nu) \ge P - 2\epsilon$. Let $m \to \infty$ to obtain $h(\nu) + \alpha(\nu) \ge P - 2\epsilon$. Hence $h(\nu) + \alpha(\nu) \ge P$. Use the ergodic decomposition of ν and (3.7) to obtain that $h(\nu) + \alpha(\nu) = P$. Moreover, almost all ergodic components μ^* of ν satisfy the equality $h(\mu^*) + \alpha(\mu^*)$. $\diamond$

Theorem 3.17. *Let $\mathcal{S} \subset < n >^{\mathbf{N}}$ be a subshift. Assume that the sequence of positive functions ϕ satisfies the conditions* $(0.1) - (0.2)$. *Suppose that dist is the distance function on the vertices of the induced tree $\mathcal{T}(\mathcal{S})$ given by* $(1.1), (1.2), (1.4)$ *an either* (1.5) *or* $(1.5f)$. *Let d be the metric on $\mathcal{S}$ given by* (0.3) *and let $\delta(\phi)$ denote the Hausdorff dimension of $\mathcal{S}$ with respect to d. Suppose that ψ is given by* (2.1). *Assume that for each $\epsilon > 0$ there exists $N(\epsilon) > 0$ such that the condition (3.11) holds for an increasing sequence of positive integers $\{p(l)\}_1^\infty$. Let $P_m(t) := P(\frac{-t\psi_m}{m}), t \in \mathbf{R}$, be the topological pressure corresponding to $\frac{-t\psi_m}{m}, m = 1, ...,$. Then $P(t) = \lim_{m\to\infty} P_m(t)$ is the topological pressure associated with $\{-t\psi_m\}_1^\infty$. Assume that the topological entropy of $\sigma|\mathcal{S}$ is positive, i.e. $P(0) > 0$. If $P(t) > 0, \forall t > 0$ then $\delta(\phi) = \infty$. Assume that there exists $t > 0$ so that $P(t) < 0$. Then $\infty > \delta(\phi) > 0$ is the unique solution of the Bowen equation $P(t) = 0$. Furthermore, $\delta(\phi) = \hat{\delta}(\phi)$, where $\hat{\delta}(\phi)$ is given by* (2.11). *There exists $\mu^* \in \mathcal{E}(\mathcal{S})$ such that $\hat{\delta}(\phi) = \delta(\phi, \mu^*)$.*

Proof. Clearly, for each $t \in \mathbf{R}$ the sequence $\{-t\psi_m\}_1^\infty$ satisfies the condition (3.11). Hence $P(t) = \lim_{m\to\infty} P_m(t)$. Use Theorem (3.12) to get

$$P(t) = \sup_{\mu \in \mathcal{E}(\mathcal{S})} (h(\mu) - t\alpha(\mu)) = \lim_{m\to\infty} P_m(t). \tag{3.18}$$

As $h(\mu) - t\alpha(\mu)$ an affine decreasing function in t we deduce that $P(t)$ is a decreasing continuous convex function on $\mathbf{R}$. Hence $P(t) \ge P(0) > 0$ for $t \le 0$. Suppose first that $P(t) > 0$ for $t > 0$. Then there exists $\mu \in \mathcal{E}(\mathcal{S})$ so that $h(\mu) - t\alpha(\mu) > 0$. Use Theorem 2.4 to obtain

$$\delta(\phi) \ge \delta(\phi, \mu) \ge t.$$

Hence $\delta(\phi) = \infty$ if $P(t) > 0$ for all $t > 0$. Assume now that $P(t) < 0$ for some $t > 0$. Hence $\alpha(\mu) > 0$ for any $\mu \in \mathcal{E}(\mathcal{S})$, i.e. $\alpha(\mu) \ge a > 0, \mu \in \mathcal{E}(\mathcal{S})$. Let $P(t_0) = 0, t_0 > 0$. Use (3.7) to deduce that $t_0 = \hat{\delta}(\phi)$. Hence, for $P(t) < 0, t > t_0$. We now show that $\delta(\phi) \le t$ for any $t > t_0$. Recall that $P(t)$ is the topological pressure associated with the sequence $\{-t\psi_m\}_1^\infty$. The equality $\psi_m((a_i)_1^\infty) = \phi_m((a_i)_1^m)$ and the inequality $P(t) < 0$ yield

$$\lim_{m\to\infty} \sum_{(a_i)_1^m \in M(m)} e^{-t\phi_m((a_i)_1^m)} = 0.$$

Let $\epsilon > 0$. Then the condition (0.2) yields that

$$\mathrm{diam}\,(C((a_i)_1^m) \cap \mathcal{S}) < \epsilon, \quad m > A(\epsilon).$$

The definition of the Hausdorff dimension implies that $\delta(\phi) \le t$. Hence $\delta(\phi) \le t_0 = \hat{\delta}(\phi)$. Use (2.12) to deduce $\delta(\phi) = \hat{\delta}(\phi)$. $\diamond$

§4. Schottky groups

Let $F =< f_1, ..., f_r >\le PSL(2, \mathbf{C}), r > 1$ be a discrete free group. View F as a discrete group of Möbius transformations of the extended complex plane $\mathbf{CP}$ (complex projective line) which acts as the group of isometries on H^3. As F is free it follows that F acts freely on H^3. Fix a point $o \in H^3$ and consider the orbit Fo. Let $\Gamma \subset< 2r > \times < 2r >$ be the F-induced graph. Then Fo are the vertices in the tree $\mathcal{T}(\Gamma^\infty)$. Set

$$\phi_p(a_1, ..., a_p) = d_h(f_{a_1} \cdots f_{a_p} o, o), \quad (a_i)_i^p \in \Gamma^p, \quad p = 1, ...,$$
$$f_{i+r} = f_i^{-1}, \quad i = 1, ..., r.$$

Here $d_h(x, y)$ is the hyperbolic distance between $x, y \in H^3$. If F is a Fuchsian group, choose $o \in H^2$ so that $Fo \subset H^2$. Let $\kappa(F)$ be the Poincaré exponent of F. Comparing the standard definition of $\kappa(F)$ ([**Nic**]), and the definition (1.12) of $\kappa(\phi)$ we deduce that $\kappa(F) = \kappa(\phi)$. Use the inequality of Bishop-Jones [**B-J**] and (2.12) to obtain

$$dim_H(\Lambda(F)) \ge \kappa(F) = \kappa(\phi) \ge \delta(\phi) \ge \hat{\delta}(\phi). \tag{4.1}$$

Theorem 2.13 (when applicable) gives nontrivial lower bounds on any quantity appearing in (4.1). It is known that $dim_H\Lambda(G) = \kappa(G)$ for the following discrete $G \le PSL(2, \mathbf{C})$: G is a lattice, i.e. the volume of the fundamental domain is finite; G is geometrically finite or convex cocompact (e.g. [**Nic**]); $\Lambda(G)$ has zero Lebesgue area [**Bis**]. It seems that one has equality signs in (4.1) in many cases. We show the equality $dim_H(\Lambda(F)) = \hat{\delta}(\phi)$ for a finitely generated, free Kleinian group F ($\Lambda(F) \ne \mathbf{CP}$) without parabolic elements.

A finitely generated free group $F =< f_1, ..., f_r >\le PSL(2, \mathbf{C})$ is called a classical Schottky group of rank r if the following conditions hold: There exists $2r$ disjoint circles $C_1, , ..., C_{2r}$ in $\mathbf{CP}$ with a common exterior and f_i maps the inside of C_i onto the outside of C_{r+i} for $i = 1, ..., r$. It is well known that F is discrete. Furhtermore, F is purely loxodromic, i.e. does not contain parabolic elements. See for example [**Mas**]. View F as the group of hyperbolic isometries of H^3. Then F has a following fundamental domain $D(F)$. Assume for simplicity that each C_i is a standard Euclidean circle in $\mathbf{C}$ with the center o_i and radius r_i. Let B_i be the open three dimensional ball centered at o_i with the radius r_i. Then $D(F) = H^3 \backslash \cup_{i=1}^{2r} B_i$. Denote by $D_i = B_i \cap \mathbf{C}$ the open disk centered at o_i with radius r_i. Then $\Lambda(F) \subset \cup_{i=1}^{2r} D_i$.

A finitely generated free group $F =< f_1, ..., f_r >\subset PSL(2, \mathbf{C})$ is called Schottky group if we replace in the above definition the disjoint circles $C_1, ..., C_{2r}$ by simple closed curves in $\mathbf{CP}$. Let T be the closed connected component bounded by $C_1, ..., C_{2r}$. Let D_i denote the open connected component of the complement of D bounded by C_i for $i = 1, ..., 2r$. Then F is discrete; F does not have parabolic elements; T is the fundamental domain for the action of F on $\mathbf{CP}$; F (viewed as a discrete group of isometries of H^3) is geometrically finite; $\Lambda(F) \subset \cup_{i=1}^{2r} D_i$. Vice versa, assume that F is a finitely generated, free, purely loxodromic Kleinian group. Then F is a Schottky group. See [**Mas**, X.H].

We now recall the results of Bowen [**Bow2**], who applies the tools of thermodynamics formalism to compute $dim_H\Lambda(F)$ for a Schottky group F. For convenience, we assume that the curves $C_1, ..., C_{2r}$ lie in $\mathbf{C}$. Then $dim_H\Lambda(F)$ is computed with respect to the Euclidean metric on $\mathbf{C}$. The sets $D_i \cap \Lambda(F), i = 1, ..., 2r$, form a Markov partition for $\Lambda(F)$. Let $f : \Lambda(F) \to \Lambda(F)$ be

$$f|\Lambda(F) \cap D_i := f_i|\Lambda(F) \cap D_i, \quad i = 1, ..., 2r. \tag{4.2}$$

There is a natural homeomorphism $\Theta : \Gamma^\infty \to \Lambda(F)$, where $\Gamma < 2r > \times < 2r >$ is the digraph induced by the free group on r generators. Furthermore, $\Theta^{-1} f \Theta$ is a shift σ on Γ^∞. Set

$$q(z) = q(f)(z) := \log |f_i'(z)|, \quad z \in \Lambda(F) \cap D_i, \quad i = 1, ..., 2r.$$

Then f expands uniformly on $\Lambda(F)$. That is, $q(f^k)(z) \geq a > 0, z \in \Lambda(F)$, for some $k \geq 1$. Let $\Pi(\Lambda(F))$ be the set of f-invariant measures supported on $\Lambda(F)$ and let $\mathcal{E}(\Lambda(F))$ denote the subset of ergodic measures. Let $P(-tq)$ be the topological pressure associated with $-tq$ as defined in §3. Then $dim_H \Lambda(F)$ is the unique solution of $P(-tq) = 0$. Furthermore, the unique maximal Gibbs measure μ^* corresponding to $t = dim_H \Lambda(F)$ is equivalent to a $dim_H \Lambda(F)$-Hausdorff measure. From the definition of q it follows that $\int q d\mu$ is the Lyapunov exponent $\lambda(\mu)$ of f. As f expands uniformly it follows that $\lambda(\mu) \geq a > 0$. Thus

$$dim_H \Lambda(F) = \sup_{\mu \in \mathcal{E}(\Lambda(F))} \frac{h(\mu)}{\lambda(\mu)} = \frac{h(\mu^*)}{\lambda(\mu^*)}. \tag{4.3}$$

Theorem 4.4. *Let $F =< f_1, ..., f_r >$ be a finitely generated, free, purely loxodromic Kleinian group. Assume that $o \in H^3$ and let $\mathcal{T}$ be the induced tree whose vertices are the elements of the F-orbit of o. Define a metric on $\mathcal{T}$ using the hyperbolic metric on the F-orbit of o. Let $\Gamma \subset < 2r > \times < 2r >$ be the graph induced by F. Define $\phi_m, \psi_m, m = 1, ...,$ by* (1.1) *and* (2.1) *respectively. Suppose that the metric d on Γ^∞ is given by* (0.3). *Then*

$$\dim_H \Lambda(F) = \delta(\phi) = \hat{\delta}(\phi).$$

Proof. As F is finitely generated, free, purely loxodromic Kleinian group, it follows that F is Schottky [**Mas**, X.H.]. Without loss of generality we assume that $C_i, D_i \subset \mathbf{C}, i = 1, ..., 2r$.

We first consider the case where F is a classical Schottky group. It is more convenient to consider the open ball $B^3 \subset \mathbf{R}^3$ of radius one centered at the origin 0 as a model for three dimensional hyperbolic space. Recall that the hyperbolic metric ds is given by $\frac{2|dx|}{1-|x|^2}, x \in B^3$ where $|dx|$ is the Euclidean metric. For $x, y \in B^3$ we denote by $d_h(x, y)$ the hyperbolic distance between x, y. Let $S^2 = \partial B^3$ and we identify S^2 with the Riemann sphere using the stereographic projection. The fundamental domain $D(F)$ for the action of F is given by $B^3 \backslash \cup_{i=1}^{2r} B_i$, where B_i are open balls centered at o_i and $\partial B_i \cap S^2 = D_i$ for $i = 1, ..., 2r$. Assume furthermore that o is in the interior of $D(F)$. Then $o \in \mathcal{T}$ and o is connected to vertices $f_j o \in f_j D(F) \subset B_{j+r}, j \in < 2r >= \Gamma^1$. (Here $j + r$ is taken modulo $2r$). Note that $\Lambda(F) \cap B_i = \Lambda(F) \cap D_i, i = 1, ..., 2r$. Other vertices of F-orbit of o are of the form

$$x_k = f_{a_1} f_{a_2} \cdots f_{a_k} o \in f_{a_1} f_{a_2} \cdots f_{a_k} D(F) \subset f_{a_1} f_{a_2} \cdots f_{a_{k-1}} B_{a_k + r} \subset B_{a_1 + r}, \quad (a_i)_1^k \in \Gamma^k, \quad k = 2, ...,. \tag{4.5}$$

It is straightforward to show that there exists $0 < \rho < 1$ and $K > 0$ so that the diameter of the ball $f_{a_1} f_{a_2} \cdots f_{a_{k-1}} B_{a_k + r}$ is less than $K\rho^k, k = 2, ...,$. Observe that the balls $f_{a_1} f_{a_2} \cdots f_{a_{k-1}} B_{a_k + r}, k = 1, ...,$ form a sequence of nested balls. Hence there exists a unique point x so that

$$|x_k - x| \leq K\rho^k, \quad x \in \Lambda(F) \cap D_{a_1 + r}, \quad k = 0, ...,. \tag{4.6}$$

Let

$$\Theta : \Gamma^\infty \to \Lambda(F), \quad \Theta((a_i)_1^\infty) = \lim_{k \to \infty} f_{a_1} f_{a_2} \cdots f_{a_k} o, \quad (a_i)_1^\infty \in \Gamma^\infty.$$

Then Θ is a homeomorphism and $\Theta^{-1} f \Theta = \sigma$. For $l \geq k$ let

$$\gamma((a_i)_k^l) := f_{a_k} f_{a_{k+1}} \cdots f_{a_l}, \quad 1 \leq k \leq l, \quad (a_i)_1^\infty \in \Gamma^\infty$$
$$x_{k,l} = \gamma((a_i)_k^l)(o) \in B^3,$$
$$x_{k,\infty} = \lim_{l \to \infty} x_{k,l} \in \Lambda(F) \cap D_{a_k + r}.$$

That is, $\Theta^{-1}x_{k,\infty} = (a_i)_k^\infty \in \Gamma^\infty$, while the point $x_{k,l}$ is the vertex in $\mathcal{T}$ given by $(a_i)_k^l$. The inequality (4.6) yields

$$\begin{aligned} |x_{k,l} - x_{k,\infty}| &\le K\rho^{l-k}, \\ |x_{k,l} - x_{k,m}| &\le 2K\rho^{\min(l,m)-k}. \end{aligned} \tag{4.7}$$

Assume that ω is a Möbius transformation which maps B^3 onto itself. Conjugating F with an appropriate ω we will assume that the reference point o (the root of the tree $\mathcal{T}$) is the origin $0 \in B^3$. Recall that

$$d_h(0,a) = \log\frac{1+|a|}{1-|a|}, \quad a \in B^3.$$

Furthermore, any Möbius transformation $\omega : B^3 \to B^3$ satisfies

$$|\omega'(x)| = \frac{1-|\omega(x)|^2}{1-|x|^2}, \quad x \in B^3.$$

For $a = \omega(0)$ we get

$$d_h(0,\omega(0)) = -\log|\omega'(0)| + 2\log(1+|\omega(0)|) = -\log|\omega'(0)| + e(\omega), \quad |e(\omega)| \le \log 4.$$

Let $x \in \Lambda(F)$ and assume that $\Theta^{-1}x = (a_i)_1^\infty \in \Gamma^\infty$. Then

$$\psi_k(\Theta^{-1}x) = d_h(0,x_k) = d_h(0,\gamma((a_i)_1^k)(0)) = -\log|\gamma((a_i)_1^k))'(0)| + e(\gamma((a_i)_1^k)).$$

Observe next that $0 = \gamma((a_i)_1^k)^{-1}(x_k)$. Hence

$$-\log|\gamma((a_i)_1^k)'(0)| = \log|(\gamma((a_i)_1^k)^{-1})'(x_k)|.$$

Recall that $f_{a_i}^{-1} = f_{a_i+r}, i = 1,...,$. Hence

$$\log|(\gamma((a_i)_1^k)^{-1})'(x_k)| = \sum_{i=1}^k \log|\gamma'_{a_i+r}(x_{i,k})|, \quad k = 1,....,.$$

Note that

$$x_{j+1,\infty} = \gamma((a_i)_1^j)^{-1}(x), \quad j = 1,...,.$$

Let $\bar{B}^3$ be the closed unit ball. As $f_j(\bar{B}^3) = \bar{B}^3, j = 1,...,2r$, we deduce that each $\log|f_j'|$ is Lipschitz on $\bar{B}^3$. Hence

$$|\log|f_j'(u)| - \log|f_j'(v)|| \le K_1|u-v|, \quad u,v \in \bar{B}^3, \quad j = 1,...,2r,$$

for some positive K_1. Using (4.7), we obtain

$$\begin{gathered} |\sum_{i=1}^k \log|f'_{a_i+r}(x_{i,k})| - \sum_{i=1}^k \log|f'_{a_i+r}(x_{i,\infty})|| \le \\ K_1\sum_{i=1}^k |x_{i,k} - x_{i,\infty}| \le K_1K\sum_{i=1}^k \rho^{k-i} \le \frac{K_1K}{1-\rho}. \end{gathered}$$

Observe next that

$$\begin{aligned} &x_{i+1,\infty} = f_{a_i+r}(x_{i,\infty}) = f(x_{i,\infty}) = f^i(x), \\ &\log|f'_{a_i+r}(x_{i,\infty})| = \log|f'(x_{i,\infty})| = \log|f'(f^{i-1}(x))|, \quad i = 1,...,. \end{aligned}$$

Combine all the above estimates to obtain

$$\frac{\psi_m(\Theta^{-1}x)}{m} = \frac{1}{m}\sum_1^m \log|f'(f^{i-1}(x))| + \frac{1}{m}\tilde{e}(x,m). \tag{4.8}$$

Here $\tilde{e}(x,m)$ is an error term whose absolute value is bounded by $\log 4 + \frac{K_1 K}{1-\rho}$. Let $\tilde{q} \in C(\Gamma^\infty)$ be given by $\tilde{q} = q \circ \Theta$. Then the sequence ψ satisfies the assumptions of Theorem 3.5. Combine Theorem 3.17 with (4.3) and (4.1) to deduce the theorem in the case $o \in D(F)$. Let o' be any point in H^3. As F is a group of isometries of H^3, we obtain

$$d_h(o',\gamma((a_i)_1^k))o') \le d_h(o',o) + d_h(o,\gamma((a_i)_1^k))o) + d_h(\gamma((a_i)_1^k))o,\gamma((a_i)_1^k))o') = d_h(o,\gamma((a_i)_1^k))o) + 2d_h(o,o').$$

Similarly,

$$d_h(o,\gamma((a_i)_1^k))o) \le d_h(o',\gamma((a_i)_1^k))o') + 2d_h(o,o').$$

Hence, for any $\mu \in \mathcal{E}(\Gamma^\infty)$, the value $\alpha(\mu)$ is independent of the choice of o'. This concludes the proof of the theorem for the classical Schottky group.

Assume that F is a Schottky group, i.e. $C_1,...,C_{2r}$ disjoint Jordan curves on $\mathbf{C}$. By a quasi-conformal change of variables F can be conjugated to a classical Schottky group, e.g. [**Mas**]. Hence each C_i has a finite length. It is straightforward to show that each $D_i, i = 1,...,r$, can be covered by an open union of disks $\tilde{D}_i := \cup_{j=1}^{n_i} D_{i,j}, i = 1,...,r$, with the following properties: Each $\tilde{D}_i, i = 1,...,r$ is simply connected, $\tilde{C}_1 := \partial\tilde{D}_1,...,\tilde{C}_r := \partial\tilde{D}_r$ are disjoint Jordan curves which do not intersect any of $C_{r+1},...,C_{2r}$. Then $\tilde{C}_{i+r} := f_i(\tilde{C}_i) \subset D_{i+r}, i = 1,...,r$. Hence F is Schottky with respect to $\tilde{C}_1,...,\tilde{C}_{2r}$. Let $B_{i,j}$ be the open ball centered at $o_{i,j} \in \mathbf{C}$ so that $B_{i,j} \cap \mathbf{C} = D_{i,j}, j = 1,...,n_i, i = 1,...,r$. Then $f_i(\partial D_{i,j})$ is another circle on $\mathbf{C}$ which bounds a disk $D_{i+r,j}$ centered at $o_{i+r,j}$. Let $B_{i+r,j}$ be the open ball centered in $o_{i+r,j}$ such that $B_{i+r,j} \cap \mathbf{C} = D_{i+r,j}, j = 1,...,n_i, i = 1,...,r$. Note that $f_i(\partial B_{i,j}) = \partial B_{i+r,j}, j = 1,...,n_i, i = 1,...,r$. Set

$$D(F) = H^3 \backslash \cup_{i=1}^r \cup_{j=1}^{n_i} (B_{i,j} \cup B_{i+r,j}).$$

Then $D(F) \subset H^3$ is a fundamental domain for the action of F. Repeat the arguments for the classical Schottky group to deduce the theorem in this case. $\diamond$

Corollary 4.9. *Let the assumption of Theorem 4.4 hold. Assume that $\delta_l(\phi), l = 1,...,$ are given as in Theorem 2.13. Then*

$$dim_H \Lambda(F) \ge \delta_l(\phi), \quad l = 1,...,$$
$$dim_H \Lambda(F) = \lim_{l\to\infty} \delta_l(\phi).$$

§5. Geometrically finite Kleinian groups

Let $F =< f_1,...,f_r >$ be a finitely generated infinite group. As in the case of a free group, we set $f_{i+r} = f_i^{-1}, i = 1,...,r$. Let $\Gamma \subset< 2r > \times < 2r >$ be the graph induced by a free group on r generators. For $(a_i)_1^k \in \Gamma^k$ we let $\gamma((a_i)_1^k) = f_{a_1}\cdots f_{a_k}$. We view $g \in PSL(2,\mathbf{C})$ as a Möbius transformation

$$g = \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbf{C}, \quad ad - bc = 1$$

of $\mathbf{CP}$. Assume that $g(\infty) \ne \infty$, i.e. $c \ne 0$. Recall that $|g'(z)| = 1$, i.e. $|cz+d|^2 = 1$, is called the isometric circle of g. Let I_g, R_g be the inside and the outside of the isometric circle of g:

$$I_g = \{z \in \mathbf{C} : \quad |g'(z)| > 1\},$$
$$R_g := \{z \in \mathbf{C} : \quad |g'(z)| < 1\} \cup \{\infty\}.$$

Then $g(R_g) = I_{g^{-1}}$. Let $\bar{I}_g, \bar{R}_g$ denote the closure of I_g, R_g. Let $\hat{g} : H^3 \to H^3$ be the induced isometry of H^3 by g. Call the upper part of the sphere centered at $-\frac{d}{c}$ with radius $\frac{1}{|c|}$ (located in H^3) as the isometric sphere of $\hat{g}$. Let $J_g, D_g \subset H^3$ denote the inside and the outside of the isometric sphere of $\hat{g}$. Then $\hat{g}(D_g) = J_{\hat{g}^{-1}}$. In what follows we shall idenitify $\hat{g}$ with g and no ambiguity will arise.

Let $F \leq PSL(2, \mathbf{C})$ be a finitely generated Kleinian group with the limit set $\Lambda(F)$. Then Selberg's theorem claims that F has a torsion free subgroup $G \leq F$ of a finite index. It is well known that $\Lambda(G) = \Lambda(F)$. In what follows we shall assume that F is torsion free. By conjugating F with some $g \in PSL(2, \mathbf{C})$ we can assume that $\Lambda(F) \subset \mathbf{C}$. Then the Ford fundamental region $R(F)$ is $\cap \bar{R}_f, f \in F \backslash \{e\}$. Similarly, if we view F as a group of discrete isometries of H^3, then the Ford domain $D(F)$ of F is given by $\cap \bar{D}_f, f \in F \backslash \{e\}$. See for example [**Mas**]. F is called geometrically finite if there exists a finite set $S \subset F \backslash \{e\}$, $S^{-1} = S$ so that $R(F) = \cap_{f \in S} \bar{R}_f$. Then F is generated by S. In what follows we assume that F is a geometrically finite, torsion free, Kleinian group F satisfying

$$F = < f_1, ..., f_r >, \quad S = \{f_1, f_1^{-1}, ..., f_r, f_r^{-1}\},$$

and S is a minimal set with respect to $R(F) = \cap_{f \in S} \bar{R}_f$.

We now construct a subshift of $\mathcal{S}(F) \subset \Gamma^\infty$ to which we can apply the results of the previous sections. We will assume in addition that F is PL (purely loxodromic). Some of the results will apply to F which are not PL. For $g \in PSL(2, \mathbf{C})$ and $0 \leq \epsilon$ set

$$I_g(\epsilon) = \{z \in \mathbf{C} : |g'(z)| > 1 + \epsilon\}.$$

Let $T_1, ..., T_m$ be given sets in a fixed space. For any nonvoid $U \subset < m >$ let $Y(U) := \cap_{i \in U} T_i$. $Y(U)$ is called a maximal intersection set of $T_1, ..., T_m$ if $Y(U) \neq \emptyset$ and $Y(U') = \emptyset$, for any $U' \subset < m >$ which strictly contains U.

Let $A_1(\epsilon), ..., A_{p(\epsilon)}(\epsilon)$ be the partition of $\cup_1^{2r} I_{f_i}(\epsilon)$ induced by $I_{f_i}(\epsilon), i = 1, ..., 2r$ as follows: First,

$$A_1(\epsilon) = Y(U_1(\epsilon)), ..., A_{p_1(\epsilon)}(\epsilon) = Y(U_{p_1(\epsilon)}(\epsilon)), \quad U_i(\epsilon) \subset < 2r >, \quad i = 1, ..., p_1(\epsilon),$$

are the maximal intersection sets corresponding to $I_{f_i}(\epsilon), i = 1, ..., 2r$. Let $I^1_{f_j}(\epsilon) = I_{f_j}(\epsilon) \backslash \cup_{i=1}^{p_1(\epsilon)} A_i(\epsilon)$. Then

$$A_{p_1+1}(\epsilon) = Y(U_{p_1+1}(\epsilon)), ..., A_{p_2(\epsilon)}(\epsilon) = Y(U_{p_2(\epsilon)}(\epsilon)), \quad U_i(\epsilon) \subset < 2r >, \quad i = p_1(\epsilon) + 1, ..., p_2(\epsilon),$$

are the maximal intersection sets corresponding to $I^1_{f_i}(\epsilon), i = 1, ..., 2r$. Repeat the above procedure a finite number of times to obtain the partition $A_1(\epsilon), ..., A_{p(\epsilon)}(\epsilon)$ of $\cup_1^{2r} I_{f_i}(\epsilon)$ to a finite number pairwise disjoint nonempty sets. Fix $\epsilon_0 > 0$ so that all the indices $p(\epsilon), p_1(\epsilon), ...,$ and the subset $U_i(\epsilon)$ do not depend on ϵ for $0 < \epsilon < \epsilon_0$. We assume that $\epsilon \in (0, \epsilon_0)$ and we drop the dependence on ϵ for all the indices.

Set

$$\Lambda_i(\epsilon) := \Lambda(F) \cap \bar{I}_{f_i}(\epsilon), \quad i = 1, ..., 2r.$$

We assume that $\epsilon_0 > 0$ is small enough so that

$$\Lambda(F) = \cup_{i=1}^{2r} \Lambda_i(\epsilon_0) = \cup_{i=1}^{2r} \Lambda_i(\epsilon), \quad 0 < \epsilon < \epsilon_0.$$

Let $\hat{\Lambda}(\epsilon)$ be the *disjoint* union of $\Lambda_1(\epsilon), ..., \Lambda_{2r}(\epsilon), 0 \leq \epsilon \leq \epsilon_0$. We define a metric $\hat{d}$ on $\hat{\Lambda}(\epsilon)$ as follows:

$$\hat{d}(x, y) = |x - y|, \quad x, y \in \Lambda_i(\epsilon), \quad i = 1, ..., 2r,$$
$$\hat{d}(x, y) = 2\text{diam } \Lambda(F), \quad x \in \Lambda_i(\epsilon), \quad y \in \Lambda_j(\epsilon), \quad 1 \leq i < j \leq 2r.$$

Then $\hat{\Lambda}(\epsilon)$ is a compact metric space and

$$dim_H \hat{\Lambda}(\epsilon) = dim_H \Lambda(F).$$

For $x, y \in \hat{\Lambda}(\epsilon)$ we let $|x - y|$ be the Euclidean distance between the two points x, y viewed as two points in $\Lambda(F)$. Thus $|x - y| < \hat{d}(x, y) \iff \hat{d}(x, y) = 2\text{diam } \Lambda(F)$. Let

$$B_{i,j}(\epsilon) = (f_i(\Lambda_i(\epsilon)) \backslash \bar{I}_{f_i}(\epsilon)) \cap A_j(\epsilon) \subset \Lambda(F), \quad j = 1, ..., p, \quad i = 1, ..., 2r.$$

If $B_{i,j}(\epsilon)$ is nonempty, choose $\eta_\epsilon(i,j) \in< 2r >$ so that $B_{i,j}(\epsilon) \subset \Lambda_{\eta_\epsilon(i,j)}(\epsilon)$. Note that in certain cases $\eta_\epsilon(i,j) \in< 2r >$ is *not* uniquely defined, and we make an arbitrary choice. Then

$$f_i(\Lambda_i(\epsilon)) = (\Lambda_i(\epsilon) \cap f_i(\Lambda_i(\epsilon))) \cup_{1 \le j \le p} B_{i,j}(\epsilon), \quad j = 1, ..., 2r.$$

We now define a measurable dynamical system $\hat{f} : \hat{\Lambda}(\epsilon) \to \hat{\Lambda}(\epsilon)$ as follows. Assume that x is in the component $\Lambda_i(\epsilon)$ for some $1 \le i \le 2r$. If $f_i(x) \in \Lambda_i(\epsilon)$ then $\hat{f}(x) := f_i(x)$ stays in the component $\Lambda_i(\epsilon)$. If $f_i(x) \notin \Lambda_i(\epsilon)$ then $f_i(x) \in B_{i,j}(\epsilon)$ for exactly one $1 \le j \le p$. We then view $\hat{f}(x) := f_i(x)$ as a point in the component $\Lambda_{\eta_\epsilon(i,j)}(\epsilon)$. We claim that $\hat{f} : \hat{\Lambda}(\epsilon) \to \hat{\Lambda}(\epsilon)$ is a measurable map with respect to its Borel sigma algebra. Let $X \subset \hat{\Lambda}(\epsilon)$. Then $X = \cup_{1 \le i \le 2r} X_i$, where X_i is a measurable subset of $\Lambda_i(\epsilon)$ and X_i is the $\Lambda_i(\epsilon)$ component of X for $i = 1, ..., 2r$. It is enough to show that $\hat{f}^{-1}(X_i)$ is a measurable set. Let $X_{i,k}$ be the $\Lambda_k(\epsilon)$ component of $\hat{f}^{-1}(X_i)$. Clearly, $X_{i,i} = f_i^{-1}(X_i) \cap X_i = f_i^{-1}(X_i)$ as f_i expands on $\Lambda_i(\epsilon)$. Hence $X_{i,i}$ is a measurable set. Observe next that

$$X_{i,k} = \cup_{1 \le j \le p, \eta_\epsilon(k,j) = i} \; f_k^{-1}(X_i \cap B_{k,j}(\epsilon)).$$

Hence $X_{i,k}$ is measurable. Therefore $\hat{f}$ is a measurable map. As $f_i^{-1}(\Lambda_i(\epsilon)) \subset \Lambda_i(\epsilon)$ we deduce that $\hat{f}(\hat{\Lambda}(\epsilon)) = \hat{\Lambda}(\epsilon)$.

For each $x \in \hat{\Lambda}(\epsilon)$ let

$$a(x,\epsilon) = (a_i(x,\epsilon))_1^\infty, \quad \hat{f}^{i-1}(x) \in \Lambda_{a_i(x,\epsilon)}(\epsilon), \quad i = 1, ...,$$

be the component coordinates of the forward orbit of x under the map $\hat{f}$. Then $\hat{f}$ induces the following set

$$\mathcal{U}_0(\epsilon) := \{a(x,\epsilon) : \quad x \in \hat{\Lambda}(\epsilon)\}.$$

Clearly, $a(\hat{f}(x),\epsilon) = \sigma(a(x,\epsilon))$. Hence $\sigma\mathcal{U}_0(\epsilon) \subset \mathcal{U}_0(\epsilon)$. As $\hat{f}\Lambda_i(\epsilon) \supset \Lambda_i(\epsilon), i = 1, ..., 2r$, it follows that $\sigma\mathcal{U}_0(\epsilon) = \mathcal{U}_0(\epsilon)$. As $\hat{f}$ is measurable but not necessary continuous, $\sigma\mathcal{U}_0(\epsilon)$ may not be a closed set, i.e. $\mathcal{U}_0(\epsilon)$ is not a subshift. However, $\mathcal{U}(\epsilon) = \bar{\mathcal{U}}_0(\epsilon)$ is a subshift.

Let $\hat{q} : \hat{\Lambda}(\epsilon) \to \mathbf{R}$ be given by

$$\hat{q}|\Lambda_i(\epsilon) = \log |f_i'||\Lambda_i(\epsilon), \quad i = 1, ..., 2r. \tag{5.1}$$

Since $\infty \in R(F)$ we deduce that $\hat{q}$ is Lipschitz on $\hat{\Lambda}(\epsilon)$. We will show that $\hat{q}$ induces a Hölder continuous function q on $\mathcal{U}_0(\epsilon)$. That is, there exists $0 < \rho < 1, 0 < K$ so that

$$|q(a) - q(b)| \le K\rho^n, \quad (a_i)_1^\infty, (b_i)_1^\infty \in \mathcal{U}_0(\epsilon), \quad a_i = b_i, \quad i = 1, ..., n, \quad n = 1, ..., . \tag{5.2}$$

Hence q extends to a Hölder continuous $q : \mathcal{U}(\epsilon) \to \mathbf{R}$. Set $\alpha(\mu) = \int q d\mu, \mu \in \mathcal{E}(\mathcal{U}(\epsilon))$, and let $P(-tq)$ denote the topological pressure defined in §3.

Theorem 5.3. *Let $F \le PSL(2,\mathbf{C})$ be a nonelementary, torsion free, geometrically finite, purely loxodromic Kleinian group. Let $\Lambda(F)$ be the limit set of F and assume that $\epsilon \in (0, \epsilon_0)$ satisfies the assumptions above. Then there is an injective, surjective map $\iota : \mathcal{U}_0(\epsilon) :\to \hat{\Lambda}(\epsilon)$ satisfying:*

$$\begin{aligned} &|\iota(a) - \iota(b)| \le C(1+\epsilon)^{-n+1}, \quad a = (a_i)_1^\infty, b = (b_i)_1^\infty \in \mathcal{U}_0(\epsilon), \quad a_i = b_i, \quad i = 1, ..., n, \\ &C = \max_{1 \le i \le 2r} \operatorname{diam} I_{f_i}. \end{aligned} \tag{5.4}$$

Let $q := \hat{q} \circ \iota$. Then (5.2) holds for $\rho = (1+\epsilon)^{-1}$. Extend q to a Hölder continuous function on $\mathcal{U}(\epsilon)$. The equation $P(-tq) = 0$ has a unique positive solution which is equal $dim_H\Lambda(F)$. Furthermore

$$dim_H\Lambda(F) = \sup_{\mu \in \mathcal{E}(\mathcal{U}(\epsilon))} \frac{h(\mu)}{\alpha(\mu)} = \frac{h(\mu^*)}{\alpha(\mu^*)}, \quad \mu^* \in \mathcal{E}(\mathcal{U}(\epsilon)). \tag{5.5}$$

Proof. Assume that $x, y \in \Lambda_j(\epsilon)$. Then $|f_j(x) - f_j(y)| \geq (1+\epsilon)|x-y|$. Hence

$$a(x,\epsilon) = (a_i)_1^\infty, \quad a(y,\epsilon) = (b_i)_1^\infty, \quad x,y \in \hat{\Lambda}(\epsilon), \quad a_i = b_i, i = 1,...,n+1, \Rightarrow$$
$$C \geq |\gamma((a_{n-i+1+r})_1^n)x - \gamma((a_{n-i+1+r})_1^n)y| \geq (1+\epsilon)^n|x-y|.$$

Therefore each $x \in \Lambda_j(\epsilon)$ induces a unique sequence $a(x,\epsilon) = (a_i)_1^\infty \in \mathcal{U}_0(\epsilon), a_1 = j$. Set $\iota((a(x,\epsilon)) = x \in \Lambda_j(\epsilon) \subset \hat{\Lambda}(\epsilon)$. Then (5.4) holds. For $x \in \Lambda_j(\epsilon)$ we let $q(a(x,\epsilon)) = \log|f_j'(x)|$. As $\Lambda(F)$ is a compact set in $\mathbf{C}$ and $f_i^{-1}(\infty) \notin \Lambda(F), i = 1,...,2r$, it follows that there exists $K_1 > 0$ so that

$$|\log f_i'(x) - \log f_i'(y)| \leq K_1|x-y|, \quad x,y \in \Lambda(F).$$

Combine (5.4) with the above inequality to deduce (5.2) with

$$\rho = (1+\epsilon)^{-1}, \quad K = CK_1(1+\epsilon)$$

and $a, b \in \mathcal{U}_0(\epsilon)$. Then q has a unique extension to $\mathcal{U}(\epsilon)$ satisfying (5.2). Note that from the definition of $I_g(\epsilon)$ it follows that

$$\log|f_i'(z)| \geq \log(1+\epsilon), \quad z \in I_{f_i}(\epsilon), \quad i = 1,...,2r, \quad \Rightarrow$$
$$q(a) \geq \log(1+\epsilon), \quad a \in \mathcal{U}(\epsilon) \quad \Rightarrow \quad \alpha(\mu) \geq \log(1+\epsilon), \quad \mu \in \mathcal{E}(\mathcal{U}(\epsilon).$$

Use the arguments of the proof of Theorem 3.17 to deduce that either $P(0) = h_{top} = 0, t_0 = 0$ or $h_{top} > 0$ and $P(-tq) = 0$ has a unique positive solution t_0. We will show $t_0 > 0$.

Set $\psi_m = -tS_m(q)$ and let $P_m(-tq)$ be given by (3.2) for $m = 1,...,$. Then

$$P(-tq) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(-tq). \tag{5.6}$$

Assume that $t > t_0$. Following Bowen [**Bow2**] we show that $dim_H \Lambda(F) \leq t$. (Since $\mathcal{U}(\epsilon)$ may not be a SFT we use (5.6) instead of using Gibbs measures as in [**Bow2**].) From (5.4) follows that each $C((a_i)_1^{n+1}) \cap \mathcal{U}(\epsilon), (a_i)_1^{n+1} \in M(n+1)$ corresponds to a set

$$\Theta((a_i)_1^{n+1}) := \text{closure } \iota(C((a_i)_1^{n+1}) \cap \mathcal{U}_0(\epsilon)) \subset \Lambda_{a_1}(\epsilon)$$

of diameter $C(1+\epsilon)^n$ at most. Furthermore $\cup_{(a_i)_1^{n+1} \in M(n+1)} \Theta((a_i)_1^{n+1}) \supset \Lambda(F)$. We claim that there exists $K_2 > 0$ so that

$$\text{diam } \Theta((a_i)_1^{n+1}) \leq K_2 \min_{x \in C((a_i)) \cap \mathcal{U}(\epsilon)} e^{\sum_{j=0}^{n-1} -q(\sigma^j x)}. \tag{5.7}$$

Let

$$x, y \in \Lambda_{a_1}(\epsilon), \quad a(x,\epsilon) = (b_i)_1^\infty, \quad a(y,\epsilon) = (c_i)_1^\infty, \quad , a_i = b_i = c_i, \quad i = 1,...,n+1,$$
$$g = f_{a_n} f_{a_{n-1}} \cdots f_{a_1}, \quad g^{-1}(u) = \frac{g_1 u + g_2}{g_3 u + g_4} \in PSL(2,\mathbf{C}), \quad g_1 g_4 - g_2 g_3 = 1,$$
$$z = g(x), \quad w = g(y), \quad z, w \in I_{f_{a_{n+1}}}(\epsilon).$$

Then

$$|x-y| = |g^{-1}(z) - g^{-1}(w)| = \frac{|z-w|}{|g_3 z + g_4|\,|g_3 w + g_4|} \leq$$
$$|z-w|\left(\frac{1}{2|g_3 z + g_4|^2} + \frac{1}{2|g_3 w + g_4|^2}\right) = |z-w|\frac{|(g^{-1})'(z)| + |(g^{-1})'(w)|}{2} \leq C\max\left(\frac{1}{|g'(x)|}, \frac{1}{|g'(y)|}\right).$$

Observe next

$$|g'(x)| = e^{\sum_{i=0}^{n-1} q(\sigma^i a(x,\epsilon))}, \quad |g'(y)| = e^{\sum_{i=0}^{n-1} q(\sigma^i a(y,\epsilon))}.$$

Hence

$$\text{diam }\Theta((a_i)_1^{n+1}) \le C \max_{u\in C((a_i)_1^{n+1})\cap\mathcal{U}(\epsilon)} e^{\sum_{j=0}^{n-1} -q(\sigma^j u)}.$$

As $\sigma^i a(x,\epsilon), \sigma^i a(y,\epsilon)$ agree in the places $1, ..., n+1-i$ (5.2) yields

$$|q(\sigma^i a(x,\epsilon)) - q(\sigma^i a(y,\epsilon))| \le K(1+\epsilon)^{-n-1+i}.$$

Hence

$$|\sum_{i=0}^{n-1} q(\sigma^i a(x,\epsilon)) - q(\sigma^i a(y,\epsilon))| \le K\sum_{i=0}^{n-1}(1+\epsilon)^{-n-1+i} < \frac{K}{\epsilon}.$$

Thus $\frac{1}{|g'(y)|} \le \frac{K}{\epsilon|g'(x)|}$. Hence

$$\max_{u\in C((a_i)_1^{n+1})\cap\mathcal{U}(\epsilon)} e^{\sum_{j=0}^{n-1} -q(\sigma^j u)} \le \frac{K}{\epsilon} \min_{u\in C((a_i)_1^{n+1})\cap\mathcal{U}(\epsilon)} e^{\sum_{j=0}^{n-1} -q(\sigma^j u)},$$

and (5.7) follows. Let $t > t_0 \ge 0$. Then

$$\sum_{(a_i)_1^{n+1}\in M(n+1)} (\text{diam }\Theta((a_i)_1^n))^t \le K_2 Q_n(tq), \quad n = 1, ..., .$$

As $\limsup_{n\infty} \frac{1}{n}\log P_n(-tq) = P(-tq) < P(-t_0 q) = 0$ we deduce that both sides of the above inequality tend to zero. Use the definition of $dim_H\Lambda(F)$ to deduce that $dim_H\Lambda(F) \le t$. Since t was an arbitrary number greater than t_0, $dim_H\Lambda(F) \le t_0$. Recall that any nonelementary Kleinian group has a positive Hausdorff dimension [**Bea**]. Thus

$$t_0 \ge dim_H\Lambda(F) > 0.$$

We now prove that $dim_H\Lambda(F) \ge t_0$. Recall that the Borel sigma algebra of $\mathcal{U}(\epsilon)$ is generated by the sets

$$C((a_i)_1^n)\cap\mathcal{U}(\epsilon), \quad (a_i)_1^n \in M(n), \quad n = 1, ..., .$$

Set

$$\Psi((a_i)_1^n) := \cap_{j=1}^n \hat{f}^{-j+1}\Lambda_{a_j}(\epsilon), \quad (a_i)_1^n \in M(n).$$

Note that $\Psi((a_i)_1^n)$ is in the Borel sigma algebra $\mathcal{B}$ of $\hat{\Lambda}(\epsilon)$. Let $\mathcal{B}' \subset \mathcal{B}$ be the sub-sigma algebra generated by $\Psi((a_i)_1^n), (a_i)_1^n \in M(n), n = 1, ...,$ Note that $\cup_{(a_i)_1^n\in M(n)}\Psi((a_i)_1^n), n = 1, ...$ form an increasing sequence of measurable partitions of $\hat{\Lambda}(\epsilon)$, such that

$$\lim_{n\to\infty} \max_{(a_i)_1^n\in M(n)} \text{diam }\Psi((a_i)_1^n) = 0.$$

Therefore $\mathcal{B}' = \mathcal{B}$. Hence any $\mu \in \mathcal{E}(\mathcal{U}(\epsilon))$ induces an $\hat{f}$ ergodic measure $\hat{\mu}$ on $\mathcal{B}$. Clearly $h(\mu) = h(\hat{\mu})$. Then the $\hat{\mu}$-Lyapunov exponent of f is given by the formula

$$\lambda(\hat{\mu}) = \int \log|f'|d\hat{\mu} = \alpha(\mu).$$

According to Young [**You**], $\frac{h(\hat{\mu})}{\lambda(\hat{\mu}}) = dim_H\hat{\mu}$. In particular, $dim_H\hat{\mu} \le dim_H\hat{\Lambda}(\epsilon) = dim_H\Lambda(F)$. Choose μ to be a maximal measure for $P(-t_0 q)$ to deduce

$$t_0 = \frac{h(\hat{\mu}^*)}{\lambda(\hat{\mu}^*)} \le dim_H\Lambda(F).$$

Hence $t_0 = dim_H\Lambda(F)$. $\diamond$.

Theorem 5.8. *Let $F \le PSL(2, \mathbf{C})$ be a nonelementary, torsion free, geometrically finite, purely loxodromic Kleinian group. Let $\Lambda(F)$ be the limit set of F and assume that $\epsilon \in (0, \epsilon_0)$ satisfies the assumptions above. Let $\mathcal{T}(\epsilon) = (V(\epsilon), E(\epsilon))$ be the induced tree by $\mathcal{U}(\epsilon)$. Identify the root $o \in V(\epsilon)$ with a point $o \in H^3$. Set*

$$\phi_m((a_i)_1^m) = d_h(\gamma(a_i)_1^m)o, o), \quad (a_i)_1^m \in V(\epsilon),$$
$$\psi_m((a_i)_1^\infty) = d_h(\gamma(a_i)_1^m)o, o), \quad m = 1, ..., .$$

Then

$$dim_H \Lambda(F) = \hat{\delta}(\phi).$$

Proof. Without loss of generality we assume that $o \in D(F)$. The arguments of proofs of Theorem 4.4 yield (4.8), where $f := \hat{f}, \Theta := \iota$. Theorem 5.3 yields that $dim_H \Lambda(F) = \hat{\delta}(\phi)$. $\diamond$

Assume that the conditions of Theorem 5.8 hold. Let $\mu \in \mathcal{E}(\mathcal{U}(\epsilon))$. Then Corollary 2.6 implies a lower bound

$$\delta(\phi, \mu) \ge \frac{h(\mu)}{\alpha_m(\mu)}, \quad m = 1, ..., .$$

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