ENTROPY OF RATIONAL SELFMAPS OF PROJECTIVE VARIETIES

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December 4, 1990

§0. Introduction

Let $X \subset \mathbf{CP^N}$ be an irreducible projective variety. Assume that $F : X \to X$ is a rational continuous map. Denote by $h(F)$ the entropy of F. In [**Fri**] we showed that $h(F) = log\rho(F)$ if X is smooth. Here $\rho(F)$ is the spectral radius of the induced linear map on the homology groups of X over the rationals. In the first part of this paper (§1) we show that this result is valid for any irreducible normal projective variety X. More general, $h(F) = log\rho(F)$ for a regular selfmap F of an irreducible projective variety X. We conjecture that the regularity assumption of F can be replaced by the continuity assumption.

The second part of this paper (§2-3) deals with the case where $F : X \to X$ is a rational but not a continuous map. One can extend naturally F to the restriction of the standard shift map to the space $\hat{\Omega}(F)$ which is the closure of the orbit space of F [**Fri**]. Using this extension we define the entropy $h(F)$ as in [**Fri**]. On the other hand one can define $H(F)$ - the volume growth of algebraic subvarieties on X. As in the case of smooth X discussed in [**Fri**] our arguments show that

$$h(F) = H(F) \tag{0.1}$$

for a rational regular map $F : X \to X$. We conjecture that this equality holds for a discontinuous rational map. Consult [**Fri**] for examples where this conjecture holds. In §2 using Gromov's results [**Gro1**] we prove that $h(F) \le M(F)$. Here $M(F) \ge H(F)$ is a natural extension of $H(F)$. In §3 we discuss some extensions, examples and conjectures.

§1. Regular rational maps

Let n be the complex dimension of X and denote by $H_{2k,a}(X)$ the subgroup of $H_{2k}(X, \mathbf{Q})$ generated by all irreducible algebraic varieties in X of complex dimension k. In this section we shall assume that F is rational and continuous. It then follows that

$$F_* : H_{2k}(X, \mathbf{Q}) \to H_{2k}(X, \mathbf{Q}), \quad F_* : H_{2k,a}(X) \to H_{2k,a}(X). \tag{1.1}$$

Denote by $\rho_k(F)$ and $\rho_{k,a}(F)$ the spectral radii of the above linear maps. Set

$$\rho(F) = \max_{0 \le k \le n} \rho_k(F), \quad \rho_a(F) = \max_{0 \le k \le n} \rho_{k,a}(F). \tag{1.2}$$

Let ω be the the standard $(1, 1)$ form on $\mathbf{CP}^N$. Assume that $Y \subset X$ is an irreducible variety of complex dimension k. Then the volume of Y is given by the Wirtinger formula:

$$vol(Y) = \frac{1}{k!} \int_Y \omega^k.$$

Thus we can view the restriction of ω^k to X as a linear functional on $H_{2k,a}(X)$. Let $\mathcal{A}(X)$ be the set of all irreducible varieties $Y \subset X, 0 \le dim(X) \le n$. We then let

$$H(F) = \sup_{Y \in \mathcal{A}(X)} \limsup_{m \to \infty} log \frac{vol(F^{\circ m}(Y))}{m}. \tag{1.3}$$

Remark. If $dim(F^{\circ m}(Y)) = dim(Y)$ then $vol(F^{\circ m}(Y))$ is the standard volume of the irreducible variety $F^{\circ m}(Y)$ mulitplied by the degree of the branched covering map covering map $F : Y \to F^{\circ m}(Y)$. If $dim(F^{\circ m}(Y)) < dim(Y)$ then $vol(F^{\circ m}(Y)) = 0$. Equivalently:

$$vol(F^{\circ m}(Y)) = \frac{1}{k!}\int_Y ((F^{\circ m})^*\omega)^k,\ k = dim(Y).$$

Thus $H(F)$ is the volume growth of algebraic varieties on X. As $\omega^k : H_{2k,a}(X) \to \mathbf{C}$ it follows:

$$H(F) \le log\rho_a(F). \tag{1.4}$$

In [**Fri**] we showed that if X is smooth then

$$H(F) = log\rho_a(F) = log\rho(F). \tag{1.5}$$

We will prove this equality for an irreducible projective variety X and a rational regular map $F : X \to X$. We now recall some basic facts about varieties and rational maps needed here. Most of them can be found in [**Sha**].

For $a \in X$ we let $T_a(X)$ to be the tangent space of X at a. Intrinsically, $T_a(X)$ can be identified with the linear space of all derivations of the local ring $\mathcal{O}_a(X)$ at a. We prefer here a coordinate dependent definition which will be needed for our purposes. Let $COT_a(X)$ be the cotangent space of X at a given as follows. Assume that the affine piece of X - $\tilde{X} = X \cap \mathbf{C}^N$ which contains the point a is the zero set the k polynomials $p_1, ..., p_k \in \mathbf{C}[\mathbf{C}^N]$. We then let

$$COT_a(X) = span\{(\frac{\partial p_j}{\partial x_i}(a))_{i=1}^{i=N},\ j = 1, ..., k\} \subset \mathbf{C}^N,\ T_a(X) = COT_a(X)^{\perp}\} \tag{1.6}$$

Note that $dim(T_a(X)) \ge dim(X)$ and the equality holds iff a is a smooth point of X. Recall that a is called a normal point of X if the local ring $\mathcal{O}_a(X)$ is integrally closed. X is called a normal variety if it is normal at all of its points. Assume that $X \subset \mathbf{CP}^N$ is an irreducible projective variety and $F : X \to X$ is a rational continuous map. Let $a \in X,\ b = F(a)$. F is called regular at a if there exists an affine piece $\tilde{X} = X \cap \mathbf{C}^N$ so that $a, b \in \tilde{X}$ and the following condition is satisfied:

$$\begin{aligned} &F(x) = (\frac{f_1(x)}{g_1(x)}, ..., \frac{f_N(x)}{g_N(x)}),\ x \in \mathbf{C}^N, \\ &f_i, g_i \in \mathbf{C}[\tilde{X}],\ g_i(a) \ne 0,\ i = 1, ..., N \end{aligned} \tag{1.7}$$

F is called regular if F is regular at any $a \in X$. Assume that X is a normal variety and $F : X \to X$ is a rational continuous map. It is quite straightforward to show that F is a regular mapping.

In what follows we will assume that $F : X \to X$ is a rational regular mapping unless stated otherwise. Let $a \in X$. Then there exists an open ball $B(a, r(a)) \subset \mathbf{C}^N$ so that the polynomials $g_i,\ i = 1, ..., N$ appearing do not vanish at any point of $Closure(B(a, r(a))$. Let $U_a = B(a, r(a))$. Then $\mathcal{U} = \cup_{a\in X} U_a$ is an open cover of X. Let $\mathcal{U}_f = U_{a_1} \cup ... \cup U_{a_t}$ a finite cover of X. Then on each U_{a_i} the map F has the form (1.7). Thus, F can be considered locally as a restriction of a holomorphic map $\bar{F} : U_a \to \mathbf{C}^N$ to X. Let $D(\bar{F})(a) \in M_N(\mathbf{C})$ be the full differential matrix of $\bar{F}$ at a. From the definition of the finite cover $\mathcal{U}_f$ it follows that there exists $K > 0$ so that $\|D(\bar{F})(a)\| \le K,\ \forall a \in X$. It is straightforward to show that $COT_b(X)D(\bar{F})(a) \subset COT_a(X),\ b = F(a)$. Thus, if $D(F)(a)$ denotes the restriction of $D(\bar{F})(a)$ to the tangent space $T_a(X)$ we get the expected relation $D(F)(a) : T_a(X) \to T_{F(a)}(X)$. As usual, let $Sing(X) \subset X$ be the variety of the singular points of X. Observe that

$$F : X_r \to X_r,\ F^{\circ -1}(Y) \subset Y,\ X_r = X\backslash Y,\ Y = \cup_0^{\infty} F^{\circ -m}(Sing(X)). \tag{1.8}$$

If $Y \ne X$ it then follows that X_r is a complex manifold of complex dimension n. We view X_r as a Riemannian manifold with the Riemannian metric obtained by the restriction of Fubini-Study metric in $\mathbf{CP}^N \supset X$. We thus showed:

Lemma 1.9. *Let $X \subset \mathbf{CP}^N$ be an irreducible complex projective variety. Assume that $F : X \to X$ is a rational regular map. Suppose furthermore that $Y \neq X$. Then the norms $\|D(F(a))\|$, $a \in X_r$ are uniformly bounded.*

Theorem 1.10. *Let X be an irreducible projective variety. Assume that $F : X \to X$ is a rational regular map. Then*

$$h(F) = H(F) = log\rho_a(F) = log\rho(F) \tag{1.11}$$

Proof. We now modify the arguments of [**New**] to show the inequality

$$h(F) \leq H(F). \tag{1.12}$$

Assume first that F is not dominating. Then $W_1 = F(X)$ is an irreducible variety, $dim(W_1) < dim(X)$, $F : W_1 \to W_1$ is a rational regular map and $h(F) = h(F, W_1)$. Continue this process until we get an irreducible subvariety $W_k \subset X$, $F : W_k \to W_k$ is a rational regular dominating map and $h(F) = h(F, W_k)$. Thus, w.l.o.g. we may assume that F is dominating. Hence $Y \neq X$ where Y is defined by (1.8). Recall that $h(F)$ is the supremum of all measure theoretic entropies $h_\mu(F)$ where μ is an F invariant ergodic measure. See for example [**Wal**, Ch. 8]. Let μ be an F invariant ergodic measure. Thus, either $\mu(Y) = 0$ or $\mu(Y) = 1$.

Assume first that $\mu(Y) = 0$. In view of Lemma 1.9 we can define the Lyapunov exponents for the map $F|_{X_r}$ with respect to μ. Using the fact that $X_r \subset X$ where X is compact and the observation that F is a (local) restriction of a homolomorphic map we can combine the arguments of [**New**] and [**Fri**] to deduce $h_\mu(F) \leq H(F)$.

Assume now that $\mu(Y) = 1$. Let $Z \subset X$ be an irreducible variety. Since F is a rational regular map it follows that $F(Z)$ is an irreducible variety. Furthermore $dim(F(Z)) \leq dim(Z)$. Let $Sing(X) = \cup_1^t Z_j$ where each Z_j is an irreducible variety. Set $Y_j = \cup_0^\infty F^{\circ -i}(Z_j) \subset Y$. Clearly, $F^{\circ -1}(Y_j) \subset Y_j$. The ergodicity of μ implies that $\mu(Y_j)$ is either 0 or 1. As $Y = \cup_1^t Y_j$ w.l.o.g. we may assume that $\mu(Y_1) = 1$. As $\mu(Y_1) = 1$ and μ is an F invariant measure it follows that $\mu(\cup_{i \geq k} F^{\circ i}(Z_1)) = 1,\ k = 0, 1, ...$ Let $V = \cup_0^\infty F^{\circ -i}(Sing(Z_1))$. Then $F^{\circ -1}(V) \subset V$. Thus, either $\mu(V) = 1$ or $\mu(V) = 0$. In the first case we can repeat our arguments by replacing $Sing(X)$ with $Sing(Z_1)$. Thus, it is enough consider the case $\mu(V) = 0$. In particular $\mu(Sing(Z_1)) = 0$. Replace Z_1 by $F^{\circ i}(Z_1)$ to deduce that it suffices to consider the case where $\mu(Sing(F^{\circ i}(Z_1))) = 0,\ i = 0, 1, ...$ That is

$$\mu(W) = 1,\ W = \{x,\ x \in Z_1,\ F^{\circ i}(x) \notin Sing(F^{\circ i}(Z_1)),\ i = 0, 1, ...\}. \tag{1.13}$$

In that case we can define the Lyapunov exponents of F on W with respect to μ. The arguments of [**New**] and [**Fri**] yield the inequality $h_\mu(F) \leq H(F)$. The maximal characterization of $h(F)$ coupled with the above inequality yields (1.12).

We now use the arguments of [**Yom**] as given by [**Gro2**] to deduce the inequality $H(F) \leq h(F)$. Let $Y \subset X$ be an irreducible subvariety. As F is a restriction of a (locally) holomorphic map the arguments in [**Gro2**] yield directly that

$$\limsup_{m \to \infty} log \frac{vol(F^{\circ m}(Y))}{m} \leq h(F).$$

Hence $H(F) \leq h(F)$. Combine this inequality with (1.12) to deduce that $h(F) = H(F)$. Using Yomdin's arguments and (1.8) we deduce that $h(F) \geq log\rho(F)$. Combine this inequality with the previous equality and (1.4) to deduce the theorem. ⋄

§2. Discontinuous rational maps

Assume that X is an irreducible projective variety of complex dimension n and $F : X \to X$ is rational map. Denote by $Sing(F) \subset X$ the set of points where F is discontinuous. A standard argument shows that $Sing(F)$ is a quasi subvariety. Thus, $X\backslash(Sing(X)\cup Sing(F))$ is a connected manifold and $Z = Closure(F(X\backslash(Sing(X)\cup Sing(F))))$ is an irreducible variety. If $Z = X$ then F is called dominating. Otherwise, from the dynamics point of view it is enough to study the map $F : Z \to Z$. Continuing the above process it is enough to consider dominating rational maps. In what follows we shall assume that $F : X \to X$ is a dominating rational discontinuous (at least at one point) map. Furthermore, we shall assume that X is a smooth variety. This is not a serious restriction. Indeed, according to Hironaka [**Hir**] it is possible to blow up the ambient space $\mathbf{CP}^N \supset X$ to obtain a smooth projective variety Y which is a resolution of X. It then follows that F lifts to a rational dominating map $G : Y \to Y$. Set

$$\begin{aligned}&V_0 = Sing(F),\ V_i = F(X\backslash V_0\cup ...\cup V_{i-1})\cap V_{i-1} =\\&\{x, F^{\circ j}(x)\notin Sing(F),\ j=0,...,i-1,\ F^{\circ i}(x)\in Sing(F)\},\ i=1,...,\ V=\cup_0^\infty V_i.\end{aligned} \tag{2.1}$$

Hence, each V_i is a quasi subvariety of X. In particular $\mu(V_i) = 0$ where μ is measure with respect to volume form ω^n. Thus, $\mu(V) = 0$. It is natural to consider the orbit space $\Omega(F) \subset X^\infty$ on which the action of the standard shift is equivalent to the map F:

$$\begin{aligned}&X^\infty = \prod_1^\infty X_i,\ X_i = X,\ i=1,...,\\&X^\infty \supset \Omega(F) = \{(F^i(x))_0^\infty,\ x\in X,\ F\ is\ holomorphic\ at\ F^i(x),\ i=0,1,...\}.\end{aligned} \tag{2.2}$$

Let $d : X\times X \to \mathbf{R}_+$ be the metric induced by the Fubini-Study metric on $X \subset \mathbf{CP}^N$. Clearly, X has a finite diameter: $d(x,y)\le D,\ \forall x,y\in X$. It then follows that X^∞ is a compact metric space with respect to the metric:

$$\delta((x_i)_1^\infty,(y_i)_1^\infty) = \max_{1\le i}\frac{d(x_i,y_i)}{2^{i-1}},\ (x_i)_1^\infty,(y_i)_1^\infty \in X^\infty.$$

Let $\pi_m : X^\infty \to X^m = \prod_1^m X_i$ be the projection on the first m components. Recall that the shift map $\sigma : X^\infty \to X^\infty$ is a continuous map given by $\sigma((x_i)_1^\infty) = (x_i)_2^\infty$. It is easy to see that $\sigma : \Omega(F) \to \Omega(F)$. Moreover, the map $F : \pi_1(\Omega(F)) \to \pi_1(\Omega(F))$ is equivalent to the restriction of σ to $\Omega(F)$. Set $\hat\Omega(F) = Closure(\Omega(F))$. Thus, $\hat\Omega(F)$ is a compact set which is mapped into itself by σ. Let

$$\begin{aligned}&\Gamma_i(F) = \pi_i(\hat\Omega(F)),\ i=1,...,\ \Gamma(F) = \Gamma_2(F),\\&\sigma_i : \Gamma_i(F) \to \Gamma_{i-1}(F),\ (x_j)_1^i \mapsto (x_j)_2^i,\ i=1,...\end{aligned} \tag{2.3}$$

Note that $\Gamma_1(F) = X$, $\Gamma(F) \subset X\times X$ is the standard graph of F and $\Gamma_i(F)$ is an irreducible variety of dimension $dim(X)$. The map

$$\sigma_2 : \Gamma(F) \to X \tag{2.4}$$

can be viewed as a regular resolution of the rational map F.

As in [**Fri**] we define the entropy $h(F)$ by

$$h(F) = limsup_{m\to\infty}\frac{h(\sigma,\hat\Omega(F^{\circ m}))}{m}. \tag{2.5}$$

This definition yields straightforward the inequality:

$$h(F^{\circ m}) \le mh(F). \tag{2.6}$$

Of course, if F is regular (continuous) then the equality sign hold in (2.6). In [**Fri**] we conjectured that $h(F) = H(F)$ where X is smooth. In what follows we define the quantity $H(F)$ - the volume growth of algebraic subvarieties on X by the iterates of F in a slightly different way then in (1.3). The arguments of [**Fri**] imply that these two definitions are the same if F is a holomorphic rational map. We view X a smooth projective variety in the ambient projective space $\mathbf{CP}^N$. A hyperplane S of complex dimension $N - dim(X) + k$ is called in general position if the following condition hold:

$$\begin{aligned} &S \cap X = \cup_1^m Z_i,\ Z_i\ irreducible,\ dim(Z_i) = k,\ Z_i \not\subset V, \\ &dim(Closure(F^{\circ j}(Z_i \backslash V))) = k,\ j = 1, ..., i = 1, ..., m, \\ &Closure(F^{\circ j}(Z_i \backslash V)) \neq Closure(F^{\circ j}(Z_l \backslash V))),\ for\ i \neq l. \end{aligned} \tag{2.7}$$

Since $F : X \to X$ is a dominating map the standard arguments of algebraic geometry yield that "most" of $N - dim(X) + k$ dimensional hyperplanes of $\mathbf{CP}^N$ (with respect to the appropriate measure) are generic. Denote by $\mathcal{A}_k(X, F),\ k = 1, ..., dim(X)$ the set of all k dimensional algebraic subvarieties of X of the form $S \cap X$ where S is an $N - dim(X) + k$ dimensional hyperplane in general position. Let $Y \subset \mathbf{CP}^N$ be an irreducible algebraic variety of complex dimension k. Denote by $deg(Y)$ the degree of Y. That is $deg(Y)$ is the number of the intersection points (counted with multiplicities) with any $N - dim(Y)$ dimensional hyperplane S so that $Y \cap S$ consists of a finite number of points. Equivalently, $deg(Y) = vol(Y)$. Set

$$\begin{aligned} &\alpha_{j,k} = \sup_{Y \in \mathcal{A}_k(X,F)} vol(Closure(F^{\circ j}(Y \backslash V))),\ j = 0, 1, ..., \\ &\beta_k = \limsup_{j \to \infty} \frac{log \alpha_{j,k}}{j},\ k = 1, ..., dim(X), \\ &H(F) = \max_{1 \leq k \leq dim(X)} \beta_k. \end{aligned} \tag{2.8}$$

We conjecture:

Conjecture 2.9. *Let $X \subset \mathbf{CP}^N$ be an irreducible smooth projective variety. Assume that $F : X \to X$ is a dominating rational map. Let $h(F)$ and $H(F)$ be as defined above. Then $h(F) \leq H(F)$.*

To support this conjecture we will recall some results of [**Gro1**]. Let X be a compact Riemann manifold. Assume that $\Gamma \subset X \times X$ is an arbitrary closed set. Set

$$\Gamma^\infty = \{\xi,\ \xi = (x_i)_1^\infty \in X^\infty,\ (x_i, x_{i+1}) \in \Gamma,\ i = 1, ...\},\ \Gamma_m = \pi_m(\Gamma^\infty),\ m = 1, ... \tag{2.10}$$

It then follows that Γ^∞ is a compact set in X^∞ such that $\sigma : \Gamma^\infty \to \Gamma^\infty$. Let $h(\Gamma) = h(\sigma, \Gamma^\infty)$. We view X^k as a Riemannian manifold endowed with the Riemannian product metric. Assume that the Hausdorff dimension of $\Gamma \subset X^2$ is a positive integer n. Let $vol(\Gamma^k) \leq \infty$ be the n dimensional volume of Γ^k. We shall assume:

$$vol(\Gamma^k) < \infty,\ k = 2, ... \tag{2.11}$$

Let $B_k(a, r) \subset X^k$ be an open ball of radius r centered at a with respect to the induced metric on X^k by X:

$$B_k(a, r) = \{x,\ x = (x_i)_1^k, a = (a_i)_1^k \in X^k,\ \sum_1^k d(x_i, a_i)^2 < r^2.\}$$

Set

$$\begin{aligned} &lov(\Gamma) = \limsup_{k \to \infty} \frac{log\ vol(\Gamma^k)}{k}, \\ &Dens_\epsilon(\Gamma_k) = \inf_{a \in \Gamma^k} vol(\Gamma^k \cap B_k(a, \epsilon)), \\ &lodn_\epsilon(\Gamma) = \liminf_{k \to \infty} \frac{log\ Dens_\epsilon(\Gamma^k)}{k}, \\ &lodn(\Gamma) = \lim_{\epsilon \to 0} lodn_\epsilon(\Gamma). \end{aligned} \tag{2.12}$$

Lemma 2.13 (Gromov) *Let X be a compact Riemannian manifold, $\Gamma \subset X \times X$ a closed set of integer Hausdorff dimension n satisfying condition* (2.11). *Then*

$$h(\Gamma) \leq lov(\Gamma) - lodn(\Gamma). \tag{2.14}$$

Proof. Let

$$\begin{aligned}\delta_j(\xi,\eta) = \max_{0\le l\le j-1} \delta(\sigma^{\circ l}(\xi),\sigma^{\circ l}(\eta)) = \\ \max_{1\le i} \frac{d(x_i,y_i)}{2^{(i-j)^+}},\ \xi=(x_i)_1^\infty, \eta=(y_i)_1^\infty \in X^\infty, j=1,\dots\ .\end{aligned} \tag{2.15}$$

Here, $a^+ = max(a,0),\ a\in\mathbf{R}$. Fix $\epsilon>0$. Let $L(k,\epsilon,\Gamma^\infty)$ be the maximal size of (k,ϵ) separated set in Γ^∞. That is for any finite set $E\subset\Gamma^\infty$ with the property $\xi,\eta\in E,\ \xi\ne\eta\Rightarrow\delta_k(\xi,\eta)>\epsilon$ we have the inequality $Card(E)\le L(k,\epsilon,\Gamma^\infty)$. Furthermore, the equality sign holds for at least one such a set E. The standard definition of $h(\sigma,\Gamma)$ is [**Wal**, Ch.7]:

$$h(\sigma,\Gamma) = \lim_{\epsilon\to 0}\limsup_{k\to\infty}\frac{log\ L(k,\epsilon,\Gamma^\infty)}{k}. \tag{2.16}$$

Let $E(k,\epsilon,\Gamma^\infty)$ be a (k,ϵ) separated set of cardinality $L(k,\epsilon,\Gamma^\infty)$. It then follows that

$$\max_{1\le i\le k+K(\epsilon)} d(x_i,y_i)>\epsilon,\ \xi=(x_i)_1^\infty\ne\eta=(y_i)_1^\infty\in E(k,\epsilon,\Gamma^\infty),\ K(\epsilon)=\lceil log_2 D - log_2\epsilon\rceil.$$

Here D is the diameter of X. In particular the $L(k,\epsilon,\Gamma^\infty)$ balls

$$B_{k+K(\epsilon)}(\pi_{k+K(\epsilon)}(\xi),\frac{\epsilon}{2}),\ \xi\in E(k,\epsilon,\Gamma^\infty)$$

are disjoint. Hence:

$$vol(\Gamma^{k+K(\epsilon)}) \ge L(k,\epsilon,\Gamma^\infty) Dens_{\frac{\epsilon}{2}}(\Gamma^{k+K(\epsilon)}).$$

Take the logarithm of this inequality, divide by $k+K(\epsilon)$, take limsup of the both sides of this inequality and let ϵ tend to zero to deduce the lemma. ⋄

Theorem 2.17. (Gromov) *Let X be a compact complex Kähler manifold. Assume that $\Gamma\subset X\times X$ is a closed irreducible analytic subvariety. Then $lodn(\Gamma)\ge 0$. Hence*

$$h(\Gamma)\le lov(\Gamma). \tag{2.18}$$

Proof. Let n be the complex dimension of Γ. According to [**Fed**, Sec. 5.4.19] the irreducible analytic subvariety $\Gamma^k\subset X^k$ is minimal. Thus

$$vol(\Gamma^k\cap B_k(a,\epsilon))\ge C(n)\epsilon^{2n}.$$

Here $C(n)$ depends on the space X and the dimension n but not on $k,\ a,\ \epsilon$. (Consult [**Gro1**] for a detailed proof.) Thus, $lodn(\Gamma)\ge 0$ and (2.14) yields (2.18). ⋄

Let $F: X\to X$ be a dominating rational map. Assume that ω is the restriction of the Fubini-Study (1,1) form to $X\subset\mathbf{CP}^N$. Set

$$\begin{aligned}&\phi_m = (F^{\circ m})^*\omega,\ m=0,1,\dots\ ,\\ &\phi_{m_1,\dots,m_k} = \phi_{m_1}\dots\phi_{m_k},\ 0\le m_i,\ i=1,\dots,k,\ 1\le k\le dim(X).\end{aligned} \tag{2.19}$$

Here ϕ_m is the pull back of ω by the $F^{\circ m}$ and $\phi_{m_1,\dots,m_k}$ the exterior products of $\phi_{m_1},\dots,\phi_{m_k}$. Note that $\phi_{m_1,\dots,m_k}$ is a rational (k,k) form. As the singularities of this form are "mild" (on subvariety of codimension 2 at least) the (k,k) form $\phi_{m_1,\dots,m_k}$ is a linear functional on $H_{2k,a}(X)$. Thus, we can replace this rational form by a regular (k,k) form $\psi_{m_1,\dots,m_k}\in H^{2k}$. If F is not holomorphic we usually would not have the functorial equality:

$$\psi_{m_1,\dots,m_j}\psi_{m_{j+1},\dots,m_k} = \psi_{m_1,\dots,m_k}.$$

The arguments in [**Fri**] would yield the inequality:

$$\psi_{m_1,\dots,m_j}\psi_{m_{j+1},\dots,m_k} \ge \psi_{m_1,\dots,m_k}. \tag{2.20}$$

as linear functionals on $H_{2k,a}(X)$ with respect to the cone generated by the k complex dimensional analytic cycles in X. Let $\|\phi_{m_1,...,m_k}\|$ be the norm of the linear functional $\phi_{m_1,...,m_k} : H_{2k,a} \to \mathbf{R}$. Note

$$\|\phi_{m_1,...,m_n}\| = |\phi_{m_1,...,m_n}([X])|.$$

Here $[X]$ is the fundamental class of X, i.e. the generator of the one dimensional free group $H_{2n}(X, \mathbf{Z})$. A straightforward argument shows that

$$\beta_k = \limsup_{m_1=...=m_k=j-1\to\infty} \frac{log\|\phi_{m_1,...,m_k}\|}{j} = \limsup_{m_1=...=m_k=j-1\to\infty} \frac{log|\phi_{m_1,...,m_k,0,...,0}([X])|}{j},\ k = 1, ..., n. \tag{2.21}$$

We now define another invariant $M(F)$ of F

$$M(F) = \limsup_{j\to\infty} \frac{log(\max_{0\le m_i<j,\ i=1,...,n} |\phi_{m_1,...,m_n}([X])|)}{j}. \tag{2.22}$$

The definitions (2.8), (2.22) and the equalities (2.21) yield the inequality $H(F) \le M(F)$. We shall show that for holomorphic F we have the equality $H(F) = M(F)$.

Theorem 2.23. *Let $X \subset \mathbf{CP}^N$ be an irreducible projective variety of complex dimension n and assume that $F : X \to X$ is a dominating rational map. Denote by $\Gamma(F) \subset X \times X$ the graph of F as given by (2.3). Then*

$$lov(\Gamma(F)) = \limsup_{j\to\infty} \frac{log|(\sum_{i=0}^{i=j-1} \phi_i)^n([X])|}{j}. \tag{2.24}$$

Hence

$$h(F) \le h(\Gamma(F)) \le M(F). \tag{2.25}$$

Proof. The points of $\pi_k(\Omega(F)) \subset \Gamma_k(F)$ are of the form $(x, F(x), ..., F^{\circ k-1}(x)),\ x \in X\backslash V$. Hence, in terms of the variable x, the restriction of the $(1,1)$ form ω on the $j-th$ coordinate of $\pi_j(\Omega(F))$ is ϕ_{j-1} - the pull back of ϕ_0 by $F^{\circ j-1}$. Thus, the restriction of the standard $(1,1)$ form ω in $(\mathbf{CP}^N)^k$ to $X\backslash V$ is

$$\theta_k(x) = \sum_{j=0}^{k-1} \phi_j(x),\ x \in X\backslash V,\ k = 0, 1, ... \tag{2.26}$$

So $vol(\Gamma_k(F)) = \frac{1}{n!}\theta_k^n([X])$ and (2.24) follows. Clearly

$$\theta_k^n([X]) \le k^n \max_{0\le m_i<k, i=1,...,n} |\phi_{m_1,...,m_n}|.$$

Use Theorem 2.17 ($\Gamma(F) \subset \mathbf{CP}^N \times \mathbf{CP}^N$), equality (2.24), definition (2.22) and the above inequality to deduce that $h(\sigma, \Gamma^\infty) = h(\Gamma(F)) \le M(F)$. Recall that $h(\sigma^{\circ m}, \Gamma(F)^\infty) = mh(\sigma, \Gamma(F)^\infty)$. Next note that the action of $\sigma^{\circ m}$ on $\Gamma(F)^\infty$ decomposes in an obvious way to m subshifts. One of this subshifts is $\sigma : \hat\Omega(F^{\circ m}) \to \hat\Omega(F^{\circ m})$. Therefore

$$h(\sigma, \hat\Omega(F^{\circ m})) \le h(\sigma^{\circ m}, \hat\Omega(F)) = mh(\Gamma(F)^\infty) \le mM(F).$$

Combine this inequality with the definition (2.5) to deduce the theorem. $\diamond$

Theorem 2.27. *Let X be a compact Kähler manifold. Assume that $F : X \to X$ is holomorphic. Then*

$$h(F) = lov(\Gamma(F)^\infty) = M(F) = log\rho(F). \tag{2.28}$$

Proof. Let μ be the dimension of $H^2(X)$. Assume that $A \in \mathbf{R}^{\mu,\mu}$ is a matrix representation of the linear operator $F^* : H^2(X) \to H^2(X)$. Suppose first that A is similar to a diagonal matrix over the complex numbers (A is semi simple). Let $\lambda_1, ..., \lambda_\mu$ and $u_1, ..., u_\mu \in \mathbf{C}^\mu \equiv H^2(X, \mathbf{C})$ be the μ complex eigenvalues and the μ corresponding eigenvectors of A. Then

$$\phi_k = \sum_{j=1}^{j=\mu} a_j \lambda_j^k u_j, \ a_j \in \mathbf{C}, \ j = 1, ..., \mu.$$

It then follows that

$$\phi_{m_1,...,m_n}([X]) = \sum_{1 \le j_1,...,j_n \le \mu} b_{j_1,...,j_n} \lambda_{j_1}^{m_1} ... \lambda_{j_n}^{m_n}. \tag{2.29}$$

Here, the tensor $b_{j_1,...,j_n}$ is a symmetric tensor. W.l.o.g. we may assume that $|\lambda_1| \ge |\lambda_2| \ge ... \ge |\lambda_\mu|$. A straightforward argument shows that

$$\begin{aligned} &M(F) = log|\lambda_{i_1}\lambda_{i_2}...\lambda_{i_k}|, \ 1 \le i_1 ... \le i_k, \ 1 \le k \le n, \\ &M(F) \le log\rho_k(F) \le log\rho(F). \end{aligned} \tag{2.30}$$

For a general matrix A one has to use a corresponding modification of (2.29) to deduce (2.30). Next note that (2.25) is valid in this case. Combine the above inequalities with (2.25) and Yomdin's inequality to deduce (2.28). $\diamond$

We now note that the arguments of the proof of Theorem 2.27 yield the validity of (2.30) for a projective variety X and a rational continuous map F. (However, we can not apply Yomdin's inequality.) Use (2.25) to deduce:

Theorem 2.31. *Let $X \subset \mathbf{CP}^N$ be an irreducible complex projective variety. Assume that $F : X \to X$ is a rational continuous map. Then $h(F) \le log\rho(F)$.*

§3. Extensions, examples and conjectures.

Let $X \subset \mathbf{CP}^N$ be a smooth projective variety of complex dimension n. Assume that $\Gamma \subset X \times X$ is an irreducible variety. We call Γ regular if $dim(\Gamma) = dim(X) = n$ and the projection of Γ on each X component of $X \times X$ is X. Assume that $\Gamma \subset X^2$ is a regular irreducible variety It then follows that there exists a subvariety $U \subset X$ so that for each $x \in X \backslash U$ we have

$$x \times X \cap \Gamma = \cup_1^v (x, y_i(x)), \ y_i(x) \ne y_j(x) \ for \ i \ne j. \tag{3.1}$$

We remark that there exists many regular irreducible graphs Γ. Indeed, let us view $X \times X$ a subset of $\mathbf{CP}^N \times \mathbf{CP}^N$. In $\mathbf{CP}^N$ choose an affine chart $\mathbf{C}^N$ so that $X_a = X \cap \mathbf{C}^N \subset X$ is an irreducible affine variety of dimension n. Intersect $X_a \times X_a \subset \mathbf{C}^{2N}$ with a hyperplane L of codimension n in general position. It then follows that each irreducible component of $Closure((X_a \times X_a) \cap L) \subset X \times X$ is a regular irreducible graph.

Let $Sym(X^k)$ be the symmetric k product of X. That is, $Sym(X^k)$ be the space of $k-th$ unordered pairs $\{x_1, ..., x_k\}, \ x_i \in X, \ i = 1, ..., k$. Thus, a regular irreducible Γ induces a rational map

$$F : X \to Sym(X^v), \ x \mapsto \{y_1(x), ..., y_v(x)\}. \tag{3.2}$$

We identify F with the v valent map $F : X \to X$ and no ambiguity will arise. Let $Sing(F)$ be the set of points where the map (3.2) is discontinous. Thus, $F(x) = \{F_1(x), ..., F_v(x)\}, \ x \notin Sing(F)$, where each $F_i(x)$ appears according to its multiplicity. ($F_i(x) \ne F_j(x) \ for \ i \ne j \ and \ x \notin U$.). We now show that most of the results of the previous section apply to the v valent map F.

A standard argument yields that $Sing(F)$ is a quasi-variety of codimension 2 at least, e.g. [**G-H**, p'491]. Let $V_i,\ i = 0, 1, ...$ and V be defined as in (2.1). We then define $\Omega(F) \subset \Gamma^\infty$ as (2.2). Thus $\Gamma_k(F) \equiv \Gamma^k$ and $\hat\Omega(F) = \Gamma^\infty$. We can define the quanties $\alpha_{j,k},\ \beta_k$ and $H(F)$ as in (2.8). We let $F^{\circ m} : X \to X$ be the v^m valent map obtained by the composition of F m times. Equivalently, $F^{\circ m}$ can be defined in terms of the graph Γ^{m+1}:

$$\begin{aligned} &F^{\circ m} : X \to Sym(X^{v^m}),\ x \mapsto \{z_1(x), ..., z_{v^m}(x)\} \\ &, x \in V_m,\ x \times X^m \cap \Gamma^{m+1} = \cup_1^{v^m}(x, ..., z_i(x)). \end{aligned} \tag{3.3}$$

We now let ϕ_m and $\phi_{m_1,...,m_k}$ be defined as in (2.19), where ω is the $(1,1)$ form on $Sym(X^{v^m})$ induced by the standard $(1,1)$ form on X^{v^m}. Then the formulas $(2.20) - (2.22)$ hold. We define $h(F)$ by (2.5). The arguments of the proof of Theorem 2.23 yield:

Theorem 3.4. *Let $X \subset \mathbf{CP}^N$ be a smooth projective variety of dimension n. Assume that $\Gamma \subset X \times X$ is a regular irreducible variety. Then*

$$lov(\Gamma) = \limsup_{j\to\infty} \frac{log|(\sum_{i=0}^{i=j-1} \phi_i)^n([X])|}{j}. \tag{3.5}$$

Hence

$$h(F) \le h(\Gamma) \le M(F). \tag{3.6}$$

Conjecture 3.7 *Let the assumptions of Theorem 3.4 hold. Then*

$$h(F) = h(\Gamma) = lov(\Gamma) = M(F) = H(F). \tag{3.8}$$

Let $F : \mathbf{C}^N \to \mathbf{C}^N$ be a polynomial map. We will express in relatively simple terms the quantity $M(F)$. Assume first that F is dominating. Let $L \subset \mathbf{C}^N$ be a hyperplane of codimension 1 in general position. (L is the dual of ω.) Denote by $Q_m = F^{\circ -m}(L) \subset \mathbf{C}^N$ the hypersurface of codimension 1 obtained by the pull back of L. Set $n = N$. It then follows

$$M(F) = \limsup_{j\to\infty} \frac{log(\max_{0 \le m_i < j} t_{m_1,...,m_n})}{j},\ t_{m_1,...,m_n} = Card(\cap_{i=1}^{i=n} Q_{m_i}). \tag{3.9}$$

Suppose now that F is not dominating. In §1 we showed that there exists an irreducible variety $X \subset \mathbf{C}^N$ of complex dimension n so that $F : X \to X$ is a dominating map and the dynamics of of F is reduced to the restriction of F to X. We then let $Q_m = F^{\circ -m}(L) \cap X$ and the equality (3.9) applies.

We now study the structure of the set Γ^∞ where $\Gamma = \Gamma(F),\ F : X \to X$ where X be a compact smooth projective surface and F is a dominating rational map which satisfies the following condition. It is well known that F is holomorphic except at a finite number of points $\zeta_i \in X,\ i = 1, ..., k$. We shall assume that

$$F : \tilde X \to \tilde X,\ \tilde X = X \backslash \{\zeta_1, ..., \zeta_k\}. \tag{3.10}$$

The above assumption simplifies enormously the dynamics of $\sigma : \hat\Omega(F) \to \hat\Omega(F)$. More precisely, we have

Lemma 3.11. Let X be a smooth projective surface and $F : X \to X$ is a rational dominating map which is not holomorphic exactly at the points $\{\zeta_1, ..., \zeta_k\}$. Assume the condition (3.10) holds. Then

$$\begin{aligned} &\hat\Omega(F) = Y \cup Z,\ Y \cap Z = \emptyset,\ Closure(Y) = Y \cup Z, \\ &Y = \{(x_j)_1^\infty,\ \exists j\ x_j \notin \{\zeta_1, ..., \zeta_k\}\},\ \sigma(Y) \subset Y, \\ &Z \subset \mathcal{U} = \prod_1^\infty U_i,\ U_i = \{\zeta_1, ..., \zeta_k\},\ i = 1, ... \end{aligned} \tag{3.12}$$

Furthermore, Y is an open complex (algebraic) space (in the sense of Grauert-Remmert).

Proof. Let $(x_j)_1^\infty \in Y$. Assume that $x_i \notin \{\zeta_1, ..., \zeta_k\}$. Then $x_j = F^{j-i}(x_i),\ j > i$. It then follows that there exists a neighborhood $U \subset \hat{\Omega}(F)$ of x so it is homeomorphic to a corresponding neighborhood $V \subset \Gamma_i(F)$ of the point $\xi = (x_j)_1^i \in \Gamma_i(F)$. As $\Gamma_i(F)$ is an irreducible algebraic variety we deduce that Y is a complex (algebraic) space in a natural way. More rigorously, a neighborhood $U \subset \hat{\Omega}(F)$ of a point $(x_j)_1^\infty$ consists of a finite product of the affine neighborhoods $U_j \subset X$ of the points x_j for $j = 1, ..., m$ so that

$$x_m \notin \{\zeta_1, ..., \zeta_k\},\ x_j \notin \{\zeta_1, ..., \zeta_k\} \Rightarrow U_j \cap \{\zeta_1, ..., \zeta_k\} = \emptyset.$$

Furthermore, the ring of analytic function $\mathcal{O}_Y(U)$ coincide with $\mathcal{O}_{\pi_m(\hat{\Omega}(F))}(U)$. $\diamond$

We may view Z as the boundary of Y. Note that $\mathcal{U}$ can be naturally identified with the unit circle S^1. It then follows that either $\mu(Z) = 1$ or $\mu(Z) = 0$ where μ is the σ invariant probability measure induced by the uniform probability measure on $\mathcal{U}$. (μ is the Haar measure on S^1.)

We now bring an example of a map satisfying condition (3.10). Let $F = (p, q) : \mathbf{C}^2 \to \mathbf{C}^2$ be a polynomial map. Suppose that

$$p = \sum_0^m p_i(x, y), \quad q = \sum_0^m q_i(x, y)$$

are the homogeneous expansions of p and q, where each $p_i,\ q_i$ is either a zero polynomial or a homogeneous polynomial of degree i. Assume furthermore that at least one of the polynomials $p_m,\ q_m$ is not a zero polynomial. Suppose furthermore that p_m and q_m are linearly dependent. (This would be the case if $deg(p) \neq deg(q)$.) It then follows that F extends to a rational map $F : \mathbf{CP}^2 \to \mathbf{CP}^2$ so that the line at infinity L is mapped to one point $\zeta = (u, v, 0) \in L$. The condition (3.10) in this case is equivalent ot the condition $|p_m(u, v)| + |q_m(u, v)| > 0$. Thus, there are many dominating polynomial maps which satisfy these conditions.

We now discuss the definition of the entropy $h(F)$ as given by (2.5) for F satisfying conditions of Lemma 3.11. Let μ be an invariant ergodic probability measure under the shift σ on the space $\Gamma(F^{\circ m})^\infty$. Suppose first that μ is supported on Z given in (3.12). Let h_μ be the measure theoretical entropy of σ. The topological entropy of σ on $\mathcal{U}$ is equal to $logk$. The variational characterization of the topological entropy yields the inequality $h_\mu \leq logk$. See for example [**Wal**, Ch.7-8]. Assume that $h(F) > 0$. The variational characterization

$$h(\sigma, \Gamma(F^{\circ m})^\infty) = \sup_\mu h_\mu \tag{3.13}$$

together with the above arguments and the definition (2.5) imply that for m sufficiently large it is enough to consider in (3.13) the invariant ergodic measures so that $\mu(Z) = 0$. It then follows that μ can be considered as an $F^{\circ m}$ invariant ergodic measure on X so that $\mu(\{\zeta_1, ..., \zeta_k\}) = 0$. Thus, we can define the Lyapunov exponents of $F^{\circ m}$ with respect to μ. The arguments of [**Fri**] (using basically inequalities of the type (2.20)) yield that $M(F) < \infty$. The inequality (3.6) suggests that all the Lyapunov exponents are finite. Now the modified arguments of [**New**] should yield $h(F) \leq H(F)$ (Conjecture 2.9).

References.

[**Fed**] H. Federer, *Geometric Measure Theory*, Springer 1969.

[**Fri**] S. Friedland, Entropy of polynomial and rational maps, *Annals Math.* 1990, to appear.

[**G-H**] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley 1978.

[**Gro1**] M. Gromov, On the entropy of holomorphic maps, preprint, 1977.

[**Gro2**] M. Gromov, Entropy, homology and semialgebraic geometry, *Astérisque* 145-146 (1987), 225-240.

[**Hir**] H. Hironaka, On resolution of singularities, *Proc. Int. Congress Math.*, Stockholm 1962, 507-525.

[**New**] S. Newhouse, Entropy and volume, *Ergod.Th. & Dynam. Sys.* 8(1988), 283-299.

[**Sha**] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer 1977.

[**Wal**] P. Walters, *An Introduction to Ergodic Theory*, Springer 1982.

[**Yom**] Y. Yomdin, Volume growth and entropy, *Israel J. Math.* 57(1987), 285-299.