Some open problems in matchings in graphs

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Overview

- Matchings in graphs
- Number of k-matchings in bipartite graphs as permanents and haffnians
- Upper bounds on permanents and haffnians: results and conjectures.
- Lower bounds on permanents and haffnians: results and conjectures.
- Probabilistic methods

Matchings

- $G = (V, E)$ undirected graph with vertices V, edges E.
- matching in G: $M \subseteq E$
  no two edges in M share a common endpoint.
- $e = (u, v) \in M$ is dimer
- v not covered by M is monomer.
- M called monomer-dimer cover of G.
- M is perfect matching $\iff$ no monomers.
- M is k-matching $\iff$ $\#M = k$.

Generating matching polynomial

- $\phi(k,G)$ number of k-matchings in G, $\phi(0,G) := 1$
- $\Phi_G(x) := \sum_k \phi(k,G)x^k$ matching generating polyn.
- roots of $\Phi_G(x)$ are real nonpositive Heilmann-Lieb 1972.
  Newton inequalities hold
- $\Phi_{G_1 \cup G_2}(x) = \Phi_{G_1}(x)\Phi_{G_2}(x)$

Examples:

$$\Phi_{K_{2r}}(x) = \sum_{k=0}^{r} \binom{2r}{2k} \frac{\prod_{j=0}^{k-1} \binom{2k-2j}{2}}{k!} x^k = \sum_{k=0}^{r} \frac{(2r)!}{(2r-2k)!2^k k!} x^k$$

$$\Phi_{K_{r,r}}(x) = \sum_{k=0}^{r} \binom{r}{k}^2 k! x^k$$

$\mathcal{G}(r,2n) \supset \mathcal{GB}(r,2n)$ set of r-regular and regular bipartite graphs on $2n$ vertices, respectively

$qK_{r,r} \in \mathcal{GB}(r,2rq)$ a union of q copies of $K_{r,r}$.
$\Phi_{qK_{r,r}} = \Phi^q_{K_{r,r}}$

Formulas for k-matchings in bipartite graphs

$G=(V,E)$ bipartite $V=V_1\cup V_2, E\subset V_1\times V_2$,
represented by bipartite adjacency matrix
$B(G)=B=[b_{ij}]_{i,j=1}^{m\times n}\in\{0,1\}^{m\times n}$, $\#V_1=m, V_2=n$.

Example: Any subgraph of $\mathbb{Z}^d$ is bipartite

CLAIM: $\phi(k,G)=\mathrm{perm}_k(B(G))$.

Prf: Suppose $n=\#V_1=\#V_2$.
Then permutation $\sigma:\langle n\rangle\to\langle n\rangle$ is a perfect match iff $\prod_{i=1}^n b_{i\sigma(i)}=1$.
The number of perfect matchings in G is $\phi(n,G)=\mathrm{perm}\,B(G)$. □

Computing $\phi(n,G)$ is $\#P$-complete problem Valiant 1979

For $G=(\langle 2n\rangle,E)$ bipartite $G\in\mathcal{GB}(r,2n)\iff\frac{1}{r}B(G)\in\Omega_n\iff$
G is a disjoint (edge) union of r perfect matchings

Matching on nonbipartite graphs

$G = (V, E), |V| = 2n$,
$A(G) = [a_{ij}] \in \mathrm{S}_0(2n, \{0, 1\})$ - adjacency matrix of G

$\phi(n, G) = \mathrm{haf}(A(G)) = \sum_{M \in \mathcal{M}(K_{2n})} \prod_{(i,j) \in M} a_{ij}$
$\mathcal{M}(K_{2n})$ the set of perfect matchings in K_{2n}

$\phi(k, G) = \mathrm{haf}_k(A(G)) = \sum_{M \in \mathcal{M}_k(K_{2n})} \prod_{(i,j) \in M} a_{ij}$
$\mathcal{M}_k(K_{2n})$ the set of k matchings in K_{2n}

Claim $\mathrm{perm}(A(G)) \geq \mathrm{haf}(A(G))^2$. Equality holds if G is bipartite.

Main problems

Find good estimates on

$$s_n(k,r) := \min_{G\in\mathcal{G}(r,2n)} \phi(k,G) \le t_n(k,r) := \min_{G\in\mathcal{GB}(r,2n)} \phi(k,G)$$

$$S_n(k.r) := \max_{G\in\mathcal{G}(r,2n)}\phi(k,G) \ge T_n(k,r) := \max_{G\in\mathcal{GB}(r,2n)} \phi(k,G)$$

Completely solved case $r = 2$ [7]

$S_n(k,2) = T_n(k,2)$ achieved only for $G = nK_{2,2}$ or $G = mK_{2,2} \cup C_6$.

$t_n(k,2)$ achieved only for C_{2n}

$s_n(k,2)$ achieved only for mC_3, $mC_3 \cup C_4$ or $mC_3 \cup C_5$.

The upper bound conjecture

$S_{qr}(k,r) = T_{qr}(k,r) = \phi(k, qK_{r,r})$

$k = qr$ Follows from Bregman's inequality (see also [3])
$\mathrm{perm}\, A \le \prod_{i=1}^{n} (r_i!)^{\frac{1}{r_i}}$

$A = [a_{ij}] \in \{0,1\}^{n\times n}$ $r_i = \sum_{j=1}^{n}, i = 1, \ldots, n$

Egorichev-Alon-Friedland for $G = (V,E), |V| = 2n$
$\phi(n,G) \le \prod_{v\in V} (\deg(v)!)^{\frac{1}{2\deg(v)}}$

Equality holds iff G a union of complete bipartite graphs

$S_n(k,r) \le \binom{2n}{2k}(r!)^{\frac{k}{r}}$
$T_n(k,r) \le \min(\binom{n}{k}^2 (r!)^{\frac{k}{r}}, \binom{n}{k} r^k)$

Friedland-Krop-Lundow-Markström [6]

Asymptotic versions

$$Sa(p,r) = \limsup_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log S_{n_j}(k_j,r)}{2n_j}$$

$$Ta(p,r) = \limsup_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log T_{n_j}(k_j,r)}{2n_j}$$

$$sa(p,r) = \liminf_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log s_{n_j}(k_j,r)}{2n_j}$$

$$ta(p,r) = \liminf_{n_j\to\infty, \frac{k_j}{n_j}\to p\in[0,1]} \frac{\log t_{n_j}(k_j,r)}{2n_j}$$

Next slide gives the graphs of AUMC and the upper bounds for $Ta(p,4)$.

$r = 4$ upper bounds

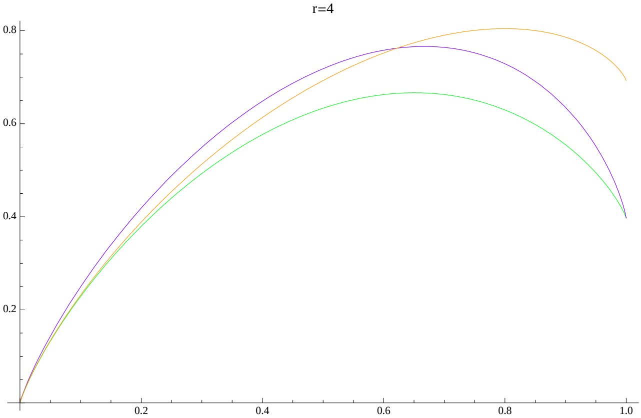

Figure: $h_{K(4)}$-green, $\mathrm{upp}_{4,1}$-blue, $\mathrm{upp}_{4,2}$-orange

The lower bounds: Bipartite case

$r^k \min_{C\in\Omega_n} \operatorname{perm}_k C \le \phi(k,G)$ for any $G \in \mathcal{GB}(r,2n)$

$J_n = B(K_{n,n}) = [1]$ the incidence matrix of the complete bipartite graph $K_{n,n}$ on $2n$ vertices

van der Waerden permanent conjecture 1926:

$$\min_{C\in\Omega_n} \operatorname{perm} C = \operatorname{perm} \frac{1}{n} J_n \ \left(= \frac{n!}{n^n} \approx \sqrt{2\pi n}\, e^{-n} \right)$$

Tverberg permanent conjecture 1963:

$$\min_{C\in\Omega_n} \operatorname{perm}_k C = \operatorname{perm}_k \frac{1}{n} J_n \ \left(= \binom{n}{k}^2 \frac{k!}{n^k} \right)$$

for all $k = 1, \ldots, n$.

History

- In 1979 Friedland showed the lower bound $\operatorname{perm} C \geq e^{-n}$ for any $C \in \Omega_n$ following T. Bang's announcement 1976.
  This settled the conjecture of Erdös-Rényi on the exponential growth of the number of perfect matchings in $d \geq 3$-regular bipartite graphs 1968, Voorhoeve 1979.
- van der Waerden permanent conjecture was proved by Egorichev and Falikman 1981.
- Tverberg conjecture was proved by Friedland 1982
- 79 proof is tour de force according to Bang
- 81 proofs involve directly (Egorichev) and indirectly (Falikman) use of Alexandroff mixed volume inequalities with the conditions for the extremal matrix
- 82 proof uses methods of 81 proofs with extra ingredients
- There are new simple proofs using nonnegative hyperbolic polynomials e.g. Friedland-Gurvits 2008

Lower matching bounds for bipartite graphs

Voorhoeve-1979 ($r = 3$) Schrijver-1998

$$\phi(n, G) \geq \left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n \quad \textit{for} \quad G \in \mathcal{GB}(r, 2n)$$

Gurvits 2006: $A \in \Omega_n$, each column has at most r nonzero entries:

$$\operatorname{perm} A \geq \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{r-1}{r}\right)^{(r-1)n}.$$

$$\textit{Cor}: \ \phi(n, G) \geq \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$$

$$\textit{Con FKM } 2006: \ \phi(k, G) \geq \binom{n}{k}^2 \left(\frac{nr-k}{nr}\right)^{nr-k}\left(\frac{kr}{n}\right)^k, \ G \in \mathcal{GB}(r, 2n)$$

F-G 2008 showed weaker inequalities

Thm: $r \geq 3, s \geq 1$ integers,
$B_n \in \Omega_n, n = 1, 2, \ldots$ each column of B_n has at most r-nonzero entries.
$k_n \in [0, n] \cap \mathbb{N}, n = 1, 2, \ldots, lim_{n\to\infty} \frac{k_n}{n} = p \in (0, 1]$ then

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{k_n} B_n}{2n} \geqslant \frac{1}{2}(-p\log p - 2(1-p)\log(1-p)) + \frac{1}{2}(r+s-1)\log(1-\frac{1}{r+s}) - \frac{1}{2}(s-1+p)\log(1-\frac{1-p}{s})$$

Prf combines properties positive hyperbolic polynomials, capacity and the measure on $\mathcal{G}(r, 2n)$

- Cor: r-ALMC holds for $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$
- Con: under Thm assumptions

$$\liminf_{n\to\infty} \frac{\log \mathrm{perm}_{k_n} B_n}{2n} \geqslant f(r, p) - \frac{p}{2}\log r$$

- For $p_s = \frac{r}{r+s}, s = 0, 1, \ldots,$ conjecture holds

Lower bounds for matchings in regular non-bipartite graphs

Petersen's THM: A bridgeless cubic graph has a perfect match

Problem: Find the minimum of the biggest match in $\mathcal{G}(r, 2n)$ for $r > 2$.

Does every $G \in \mathcal{G}(r, 2n)$ has a match of size $\lfloor \frac{2n}{3} \rfloor$? (True for $r = 2$.)

Esperet-Kardos-King-Král-Norine:
Every cubic bridgeless graph has at least $2^{\frac{|V|}{3656}}$ perfect matchings

An analog the van der Waerden conjecture

THM Edmonds 1965: A symmetric doubly stochastic matrix with zero diagonal of even order $A = [a_{ij}]_{i,j=1}^{2n}$ is a convex combination of symmetric permutation matrices with zero diagonal if and only if $\sum_{i,j\in S} a_{ij} \le |S| - 1$ for any odd subset $S \subset \{1, \ldots, 2n\}$ (*)

Denote by Ψ_{2n} the subset of all symmetric doubly stochastic matrices of the above form

Problem: Find $\min \mathrm{haf}(A), A \in \Psi_{2n}$

CONJECTURE: The minimum is achieved only for the matrix $\frac{1}{2n-1}A(K_{2n})$

$$\mathrm{haf}(\tfrac{1}{2n-1}A(K_{2n})) \approx e^{-n}\sqrt{2e} < \mathrm{haf}(\tfrac{1}{n}A(K_{n,n})) \approx e^{-n}\sqrt{2\pi n}$$

An analog the Tverberg conjecture

$\min_{A\in\Psi_{2n}} \mathrm{haf}_k(A) = \mathrm{haf}_k(\frac{1}{2n-1}A(K_{2n})) =$

$\binom{2n}{2k}\frac{1}{(2n-1)^k}\frac{(2k)!}{2^k k!}$

Note $\frac{1}{3}A(G)\in\Psi_{2n}, G\in\mathcal{G}(3,2n)$ iff there are at least 3 edges coming out of any the set of odd number of vertices (Improves significantly the lower bound [5] if gen. v.d. Waerden conj. true)

Hyperbolic polynomials

THM: Good lower bounds hold for $\mathrm{haf}_k(A)$ if $A \in \Psi_{2n}$ n-1 $n-1$ eigenvalues of A are nonpositive

Outline of proof: Fact $\mathbf{x}^\top A\mathbf{x}$ is a hyperbolic polynomial for a nonnegative symmetric matrix iff A has all but one nonpositive eigenvalues [4]

$$\mathrm{haf}_k A = (2^k k!)^{-1} \sum_{1\le i_1<\ldots<i_{2k}\le 2n} \frac{\partial^{2k}}{\partial x_{i_1}\ldots\partial x_{i_{2k}}} (\mathbf{x}^\top A\mathbf{x})^k$$

Use the arguments of [1] to show

$$\mathrm{haf}\ A \ge \left(\frac{n-1}{n}\right)^{(n-1)n} \ge e^{-n}$$

$A = [a_{ij}] \in \mathbb{R}_+^{n\times n}$, $X(A) := [\sqrt{a_{ij}} x_{ij}]$,
x_{ij} independent random variables $E(x_{ij}) = 0, E(x_{ij}^2) = 1$
$E((\det X(A))^2) = \operatorname{perm} A$. Godsil-Gutman 1981

Concentration results
A. Barvinok 1999 -
1. x_{ij} real Gaussian $\Rightarrow \det X(A)^2$ with high probability $\in [c^n \operatorname{perm} A, \operatorname{perm} A]$ $c \approx 0.28$
2. x_{ij} complex Gaussian $E(|x_{ij}|^2) = 1 \Rightarrow |\det X(A)|^2$ with high probability $\in [c^n \operatorname{perm} A, \operatorname{perm} A]$ $c \approx 0.56$
3. x_{ij} quaternion Gaussian $E(|x_{ij}|^2) = 1 \Rightarrow |\det X(A)|^2$ with high probability $\in [c^n \operatorname{perm} A, \operatorname{perm} A]$ $c \approx 0.76$

Friedland-Rider-Zeitouni 2004:
$0 < a \le a_{ij} \le b$, x_{ij} real Gaussian $\Rightarrow \det X(A)^2$ with high probability $\in [(1-\varepsilon_n) \operatorname{perm} A, \operatorname{perm} A]$ $\varepsilon_n \to 0$

FRZ results use concentration for $\log_\varepsilon \det Z(A) = \mathrm{tr}\, f(Z(A))$,
$Z(A) = X(A)^\top X(A) \succeq 0, f = \log_\varepsilon x = \log \max(x, \varepsilon)$.
or $\log_\varepsilon \det Y(A)$, $Y(A) = \begin{bmatrix} 0 & X(A) \\ -X(A)^\top & 0 \end{bmatrix}$

$E(\det(\sqrt{t}I + Y(A)) = \Phi_{G_w}(t), t \geq 0$
matching polynomial for weighted graph induced by A

Thm: Concentration of $\log\det(\sqrt{t}I + Y(A))$ around expected value
$\log \tilde{\Phi}_{G_w}(t), t > 0$ which less $\log \Phi_{G_w}(t)$
$\frac{1}{n} \log \tilde{\Phi}(t, G_\omega) \leq \frac{1}{n} \log \Phi(t, G_\omega) \leq \frac{1}{n} \log \tilde{\Phi}(t, G_\omega) + \min(\frac{\max_{i,j} |a_{ij}|}{2t}, 1.271)$

Meaning of $\tilde{\Phi}_{G_w}(t)$?

Prob. Methods III-Matching in nonbipartite graphs

Make each undirected edge (i,j) with weight $a_{ij} = a_{ji} \geq 0$
to two opposite directed edges with weights $\pm a_{ij}$ to obtain a skew symmetric matrix
$B = [b_{ij}] \in \mathbb{R}^{(2n)\times(2n)}$, $b_{ii} = 0$

$Y(B) = [\mathrm{sign(b_{ij})}\ \sqrt{|\mathrm{b_{ij}}|}\mathrm{x_{ij}}]$, $x_{ij} = x_{ji}, x_{12}, \ldots, x_{(2n-1),(2n)}$ i.r.v
$E(x_{ij}) = 0, E(x_{ij}^2) = 1$
$E(\det Y(B)) = \mathrm{haf} A$ -
total weight of weighted matchings in induced graph by A

$E(\det(\sqrt{t}I + Y(B)) = \Phi_{G_w}(t)$ - the weighted matching polynomial of $G(A)$.

All the results for bipartite graphs carry over to nonbipatite graphs

Jerrum-Sinclair-Vigoda 2004: fully polynomial randomized approximation scheme (fpras) to compute $\mathrm{perm}\, A$
A variation of MCMC method using rapidly mixed Markov chains converging to equilibrium point

The proofs do not carry over for nonbipartite graphs

Any $\#P$ complete problem has fpras?

Expected values of k-matchings for bipartite graphs

- Permutation $\sigma : \langle nr \rangle \to \langle nr \rangle$ induces $G(\sigma) \in \mathcal{GB}_{\mathrm{mult}}(r, 2n)$ and vice versa
  $G(\sigma) = \{(i, \lceil \frac{\sigma((i-1)r+j)}{r} \rceil),\ j = 1, \ldots, r,\ i = 1, \ldots, n\} \subset \langle n \rangle \times \langle n \rangle$
  number of different σ inducing the same simple G is $(r!)^n$
- μ probability measure on $\mathcal{GB}_{\mathrm{mult}}(r, 2n)$:
  $\mu(G(\sigma)) = ((nr)!)^{-1}$
- FKM 06:
  $E(k, n, r) := \mathrm{E}(\phi(k, G)) = \binom{n}{k}^2 r^{2k} k!(nr-k)!)(nr)!)^{-1}$,
  $k = 1, \ldots, n$
- $1 \le k_l \le n_l, l = 1, ...,$ increasing sequences of integers s.t.
  $\lim_{l\to\infty} \frac{k_l}{n_l} = p \in [0, 1]$. Then

$$\lim_{l\to\infty} \frac{\log E(k_l, n_l, r)}{2n_k} = f(p, r)$$

$$f(p, r) := \frac{1}{2}(p \log r - p \log p - 2(1-p) \log(1-p) + (r-p) \log(1 - \frac{p}{r}))$$

Similar results for non-bipartite graphs?

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