Tensors

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Foreword

In the past ten years, tensors again became a hot topic of research in pure and applied mathematics. In applied mathematics it is driven by data which has a few parameters. In pure math. it is quantum information theory, and multilinear algebra. There are many interesting numerical and theoretical problems that need to be resolved. Tensors are related to matrices one one hand and on the other hand are related to polynomial maps.

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To paraphrase Max Noether:
Matrices were created by God and tensors by Devil.

Overview

Ranks of 3-tensors

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Characterization of tensor in $\mathbb{C}^{4\times4\times4}$ of border rank 4

Basic notions

scalar $a \in \mathbb{F}$, vector $\mathbf{x} = (x_1, \ldots, x_n)^\top \in \mathbb{F}^n$, matrix $A = [a_{ij}] \in \mathbb{F}^{m \times n}$,
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COR $\mathrm{rank}\,\mathcal{T} \leq \min(mn, ml, nl)$

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PRF: 3-sat with n variables m clauses
satisfiable iff $\operatorname{rank} \mathcal{T} = 4n + 2m, \mathcal{T} \in \mathbb{F}^{(2n+3m)\times(3n)\times(3n+m)})$
otherwise rank is larger

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In all the examples we know $\mathrm{mtrank}(m,n,l) \le \mathrm{grank}(m,n,l) + 1$

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$f_r:(\mathbb{C}^m\times\mathbb{C}^n\times\mathbb{C}^l)^r\to\mathbb{C}^{m\times n\times l}$, $\mathbf{x}\otimes\mathbf{y}\otimes\mathbf{z}=(a\mathbf{x})\otimes(b\mathbf{y})\otimes((ab)^{-1}\mathbf{z})$
$\mathrm{grank}_{\mathbb{C}}(m,n,l)(m+n+l-2)\ge mnl\Rightarrow\mathrm{grank}_{\mathbb{C}}(m,n,l)\ge\lceil\frac{mnl}{(m+n+l-2)}\rceil$

Generic rank of $\mathbb{C}^{m\times n\times l}$

THM: $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \min(l, mn)$ for $(m-1)(n-1)+1 \le l$.

Reason: For $l = (m-1)(n-1)+1$ a generic subspace of matrices of dimension l in $\mathbb{C}^{m\times n}$ intersect the variety of rank one matrices in $\mathbb{C}^{m\times n}$ at least at l lines which contain l linearly independent matrices

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Conjecture $\mathrm{grank}_{\mathbb{C}}(m,n,l) = \lceil \frac{mnl}{(m+n+l-2)} \rceil$

for $2 \le m \le n \le l < (m-1)(n-1)$ and $(3,n,l) \ne (3, 2p+1, 2p+1)$

Generic rank of $\mathbb{C}^{m\times n\times l}$

THM: $\mathrm{grank}_{\mathbb{C}}(m,n,l)=\min(l,mn)$ for $(m-1)(n-1)+1\le l$.

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Fact: $\mathrm{grank}_{\mathbb{C}}(3,2p+1,2p+1)=\lceil\frac{3(2p+1)^2}{4p+3}\rceil+1$

Bilinear maps and product of matrices

bilinear map: $\phi : \mathbf{U} \times \mathbf{V} \to \mathbf{W}$

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$M = N = L = 2$, $\mathrm{grank}_{\mathbb{C}}(4,4,4) = \lceil \frac{4 \times 4 \times 4}{4+4+4-2} \rceil = \lceil 6.4 \rceil = 7$

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Product of two 2×2 matrices is done by 7 multiplications

Known cases of rank conjecture

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$\mathbf{w}_r \in (\mathbb{Z}^m \times \mathbb{Z}^n \times \mathbb{Z}^l)^r$ find $\mathrm{rank}\, J(f_r)(\mathbf{w}_r)$ exact arithmetic

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I checked the conjecture up to $m, n, l \leq 14$

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For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

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$\mathrm{mtrank}(2,n,l) = \mathrm{grank}(2,n,l) = \min(l,2n)$ if $2 \le n < l$ - one typical rank
$\mathrm{mtrank}(2,n,n) = \mathrm{grank}(2,n,n) + 1 = n+1$ if $2 \le n$ - two typical ranks

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For $l = (m-1)(n-1) + 1$ $\exists m, n$:
$c(m, n, l) > 1, \mathrm{mtrank}(m, n, l) \geq \mathrm{grank}(m, n, l) + 1$

Generic rank III - the real case

For $mn \leq l$ $\mathrm{mtrank}(m,n,l) = \mathrm{grank}(m,n,l) = mn$.

For $2 \leq m \leq n \leq l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise distinct open connected semi-algebraic sets s.t.

Closure$(\cup_{i=1}^{c(m,n,l)}) = \mathbb{R}^{m \times n \times l}$
$\mathrm{rank}\ \mathcal{T} = \mathrm{grank}(m,n,l)$ for each $\mathcal{T} \in V_1$
$\mathrm{rank}\ \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$
$\{\rho_1, \ldots, \rho_{c(m,n,l)}\} = \{grank(m,n,l), \ldots, \mathrm{mtrank}(m,n,l)\}$

$\mathrm{mtrank}(2,n,l) = \mathrm{grank}(2,n,l) = \min(l, 2n)$ if $2 \leq n < l$ - one typical rank
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Examples [3]

Rank one approximations

Rank one approximations

$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$

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How many distinct singular values are for a generic tensor?

ℓ_p maximal problem and Perron-Frobenius

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$\|(x_1, \ldots, x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

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Assume that $\mathcal{T} \geq 0$. Then $\mathbf{x}, \mathbf{y}, \mathbf{z} \geq 0$

For which values of p we have an analog of Perron-Frobenius theorem?

Yes, for $p \geq 3$, No, for $p < 3$,
Friedland-Gauber-Han [1]

(R_1, R_2, R_3)-rank approximation of 3-tensors

(R_1, R_2, R_3)-rank approximation of 3-tensors

Fundamental problem in applications:
Approximate well and fast $\mathcal{T} \in \mathbb{R}^{m_1 \times m_2 \times m_3}$ by rank (R_1, R_2, R_3) 3-tensor.

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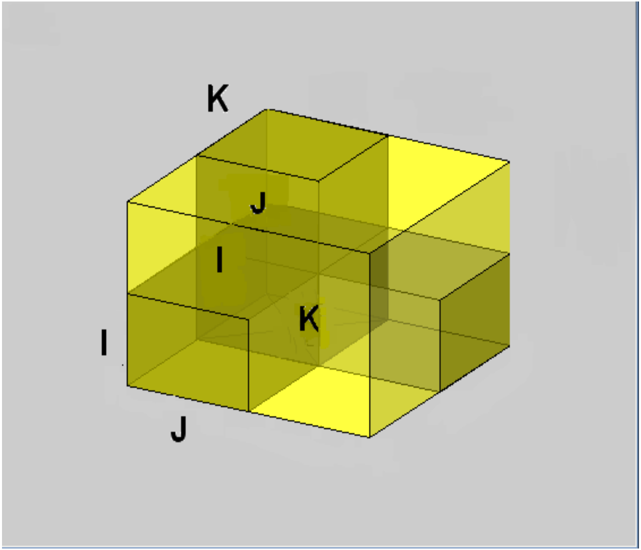
Use Newton method on Grassmannians - Eldén-Savas 2009 [1]

Fast low rank approximation I:



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CUR approximation of $\mathcal{A}$ obtained by choosing E, F, G submatrices of unfolded $\mathcal{A}$ in the mode $1, 2, 3$.

List of applications

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Face recognition

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Factor analysis

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$f_{\mathcal{T}}$ is strictly convex implies $\mathcal{T}$ is not decomposable: $\mathcal{T} \ne \mathcal{T}_1 \oplus \mathcal{T}_2$.

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Characterization of tensor in $\mathbb{C}^{4\times4\times4}$ of border rank 4

Major problem in algebraic statistics:
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Strassen's condition hold for any $3 \times 3 \times 3$ subtensor of $\mathcal{T}$:
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Friedland [5] one needs a equations of degree 16

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