On the eigenvalues of graphs: results and conjectures

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Overview

- Maximal spectral radius of directed graphs
- A lower bound for spectral radius of directed Eulerian graphs
- Maximal spectral radius of undirected graphs with *e* edges
- The bipartite case
- The Grone-Merris conjecture

Figure: Uri Natan Peled, Photo - December 2006

Uri N. Peled

Uri was born in Haifa, Israel, in 1944.
Education:
Hebrew University, Mathematics-Physics, B.Sc., 1965.
Weizmann Institute of Science, Physics, M.Sc., 1967
University of Waterloo, Mathematics, Ph.D., 1976
University of Toronto, Postdoc in Mathematics, 1976–78
Appointments:
1978–82, Assistant Professor, Columbia University
1982–91, Associate Professor, University of Illinois at Chicago
1991–2009, Professor, University of Illinois at Chicago
Areas of research: Graphs, combinatorial optimization, boolean functions.
Uri published about 57 paper
Uri died September 6, 2009 after a long battle with brain tumor.

Maximal spectral radius of directed graphs

Maximal spectral radius of directed graphs

$DG = (V, E), V = \{v_1, \dots, v_m\}$ directed graph
$\deg_+(v), \deg_(v)$ - out and in degree of $v \in V$
$e = \sum_{v\in V} \deg_+(v) = \sum_{v \in V} \deg_(v)$- the number of edges in DG
$D_+(G) = \{d_{1,+}(G) \geqslant d_{2,+}(G) \geqslant \cdots \geqslant d_{m,+}(G)\}$
rearranged set of degrees $\deg_+ v_1, \dots, \deg_+ v_m$

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DG represented by adjacency matrix $A = A(G) = [a_{ij}] \in \{0, 1\}^{m \times m}$
a_{ij} number of directed arcs from v_i to v_j
$\rho(G) := \rho(A(G))$ spectral radius of G

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$\mathcal{DG}_D$ set of directed graphs DG with $D(DG) = D$.

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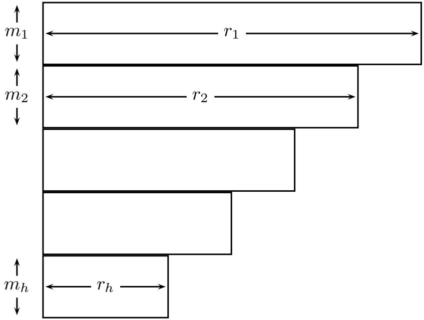
B. Schwarz 1964: $\max_{DG \in \mathcal{DG}_D} \rho(DG)$
achieved at an isomorphic graphs to directed "chain graph" DG_{chain}:
from v_i outgoing arcs to $v_1, \ldots, v_{d_i}$ for $i = 1, \ldots, m$

The adjacency matrix of DG_{chain}

Figure 1: The notation for the row sums of $A(DG)$.



Outline of proof

Fix $\varepsilon > 0$, $J_m = \mathbf{1}_m \mathbf{1}_m^\top$ consider

$$\max_{DG \in \mathcal{DG}_D} \rho(A(DG) + \varepsilon J_m) = \rho(A(DG^\star) + \varepsilon J_m) = \mu$$

choose representation $(A(DG^\star) + \varepsilon J_m)\mathbf{x} = \mu \mathbf{x}$

$$\mathbf{x} = (x_1, \ldots, x_m)^\top, \; x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots \geqslant x_m > 0$$

$A(DG')$ obtained by moving all ones to the left:

$$(A(DG' + \varepsilon J_m)\mathbf{x} \geq \mu \mathbf{x}$$

Wieland characterization:
$\rho(A(DG' + \varepsilon J_m) \geq \mu$
maximality of μ: $(A(DG' + \varepsilon J_m)\mathbf{x} = \mu \mathbf{x}$
since $x_1 \geqslant \ldots \geqslant x_m > 0$ $A(DG')$ of the form in previous slide

Lower bound for spec. radius of Eulerian graphs

DG Eulerian: $\deg_+(v_i) = \deg_-(v_i) := \deg(v_i)$ for $i = 1, \ldots, m$

Friedland 1993: $\rho(DG) \geq \prod_{v \in V}^{m} deg(v)^{\frac{\deg(v)}{e}}$ for Eulerian graphs (1)

Friedland-Karlin 1975: $B \in \mathbb{R}_+^{m\times}$ - irreducible

$B\mathbf{x} = \rho(B)\mathbf{x}, B^\top \mathbf{y} = \rho(B)\mathbf{y}, \mathbf{x}, \mathbf{y} > \mathbf{0}, \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^m x_i y_i = 1$

$\rho(\mathrm{diag}(t_1, \ldots, t_m)B) \geq \rho(B) \prod_{i=1}^m t_i^{x_i y_i}$ where $t_1, \ldots, t_m > 0$

Proof of (1): $E = \mathrm{diag}(\deg(v_1), \ldots, \deg(v_m))$

$B = E^{-1}A(DG)$ - stochastic

$B\mathbf{1} = \mathbf{1}, B^\top(D\mathbf{1}_m) = (D\mathbf{1}_m)$

$\mathbf{1}_i(D\mathbf{1})_i = \deg(v_i), i = 1, \ldots, m, \mathbf{1}^\top(E\mathbf{1}) = e$

$$\rho(A(DG)) = \rho(EB) \geq \prod_{i=1} d(v_i)^{\frac{d(v_i)}{e}}$$

Maximal spec. radius of undir. graphs on e edges

$\mathcal{G}_e$ collection of simple undirected with graphs $G = (V, E)$
with no isolated vertices
for $G = (V, E) \in \mathcal{G}_e$: $\rho(G) \ge \frac{2e}{\#V} = \frac{\mathbf{1}^\top A(G)\mathbf{1}}{\mathbf{1}^\top \mathbf{1}}$
To maximize the lower bound pack edges tightest possible
Characterize the graph which maximize $\rho(G)$ in $\mathcal{G}_e$
Brualdi-Hoffman 1985:
for $e = \frac{n(n-1)}{2}$ the maximal graph K_n-complete graph
Conjecture B-H 1985 for $e = \frac{n(n-1)}{2} + s, s \in [1, n-1]$
a maximal graph is K_n + one vertex
for a maximal graph $A(G)$ if $a_{ij} = 1, i < j$
then $a_{pq} = 1, p < q$, for $p \le i, j \le q$
Friedland 1985:
$s = n - 1$
s fixed and $n > N(s)$
Stanley 1987 $\rho(G) \le \frac{-1+\sqrt{1+8e}}{2}$ sharp for $e = \frac{n(n-1)}{2}$
Rowlinson 1988 proved BH conjecture

The bipartite case

$BG = (V \cup W, E)$, $V = \{v_1, \dots, v_m\}$, $W = \{w_1, \dots, w_n\}$
$R(BG) = [b_{ij}] \in \{0,1\}^{m\times n}$ representation matrix of BG
b_{ij} the number of edges connecting v_i and w_j
$\sigma_1(BG) := \sigma_1(R(BG) \geqslant \sigma_2(BG) := \sigma_2(R(BG)) \geqslant \dots \geqslant 0$

$$A(BG) = \begin{pmatrix} 0 & R(BG) \\ R(BG)^\top & 0 \end{pmatrix} \in \{0,1\}^{(m+n)\times(m+n)}$$

$\lambda_1(BG) = \sigma_1(BG) \geqslant \lambda_2(BG) = \sigma_2(BG) \geqslant \dots$
$\geqslant \lambda_{m+n-1}(BG) = -\sigma_2(BG) \geqslant \lambda_{m+n}(BG) = -\sigma_1(BG)$

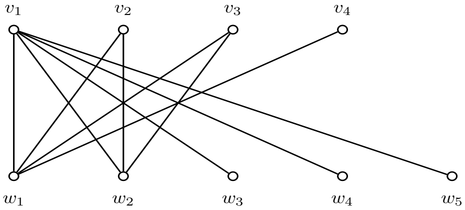
$2\sum_{i=1}^m \sigma_i(BG)^2 = \sum_{k=1}^{m+n} \lambda_k(BG) =$
$\operatorname{tr} A(BG)^2 = 2\operatorname{tr}(R(BG)^\top R(BG) = 2e$
Cor: $\rho(BG) \leq \sqrt{e}$
Equality iff $BG = K_{p,q}$ plus isolated vertices

Conjecture: maximal $\rho(BG)$ for bipartite graphs with no isolated vertices and not a complete bipartite graph achieved for a complete bipartite graph plus one vertex

The Chain graph

$D(BG) = \{d_1(BG) \geqslant d_2(BG) \geqslant \cdots \geqslant d_m(BG)\}$
rearranged set of the degrees $\deg v_1, \ldots, \deg v_m$
Chain graph G_D

Figure 1: The chain graph G_D for $D = \{5, 2, 2, 1\}$.



The Chain graph II

$\mathcal{B}_D$ class of bipartite graphs BG
with no isolated vertices, where $D(G) = D$

B-F-P 2008: $\max_{BG \in \mathcal{B}_D} \rho(BG) = \rho(G_D)$ unique up to isomorphism

Prf: $\sigma_1(BG) = \max_{\mathbf{y} \in \mathbb{R}_+^n, \mathbf{y}^\top \mathbf{y}=1} \|R(BG)\mathbf{y}\|$
$BG^\star$ maximal and $y_1 \geqslant \ldots \geqslant y_n \geq 0$
BG' obtained from $R(BG^\star)$ by moving all ones to the left:
$\sigma_1(BG') \geq \|R(BG')\mathbf{y}\| \geq \|R(BG)\mathbf{y}\| = \sigma(BG^\star)$

Lower estimates for $\sigma_1(G_D)\sigma_2(G_D)$

second compound matrix $\Lambda_2 R$ for $R = [r_{ij}] \in \mathbb{R}^{m\times n}$:
$\binom{m}{2} \times \binom{n}{2}$ rows indexed by (i_1, i_2), $1 \leqslant i_1 < i_2 \leqslant m$
columns indexed by (j_1, j_2), $1 \leqslant j_1 < j_2 \leqslant n$.
entry in row (i_1, i_2) and column (j_1, j_2) of $\Lambda_2 R$

$$\Lambda_2 R_{(i_1,i_2)(j_1,j_2)} = \det \begin{pmatrix} r_{i_1,j_1} & r_{i_1,j_2} \\ r_{i_2,j_1} & r_{i_2,j_2} \end{pmatrix}$$

$$\sigma_1(\Lambda_2 R) = \sigma_1(R)\sigma_2(R)$$

Fact: $-\Lambda_2(R(G_D)) \in \{0,1\}^{\binom{m}{2}\times\binom{n}{2}}$

$\mathbf{w} = (w_{(1,2)}, \ldots, w_{(n-1,n)})^\top \in \{0,1\}^N$, $N := \binom{n}{2}$

$w_{(i,j)} = 1$ iff the column of $\Lambda_2(R(G_D))$ is nonzero

$$\|w\|^2 = \sum_{k=1}^{h-1} r_{k+1}(r_k - r_{k+1})$$

$$\|(\Lambda_2 R(G_D))\mathbf{w}\|^2 = \sum_{1\leqslant k<l\leqslant h} m_k m_l [r_l(r_k - r_l)]^2$$

$$\sigma_1(G_D)^2\sigma_2(G_D)^2 \geqslant \omega \equiv \frac{\sum_{1\leqslant k<l\leqslant h} m_k m_l [r_l(r_k-r_l)]^2}{\sum_{k=1}^{h-1} r_{k+1}(r_k-r_{k+1})}$$

$$\sigma_1(G_D)^2\sigma_2(G_D)^2 \geqslant \omega' \equiv \frac{\sum_{1\leqslant k<l\leqslant h} m'_k m'_l [r'_l(r'_k-r'_l)]^2}{\sum_{k=1}^{h-1} r'_{k+1}(r'_k-r'_{k+1})}$$

The representation matrix of G_D

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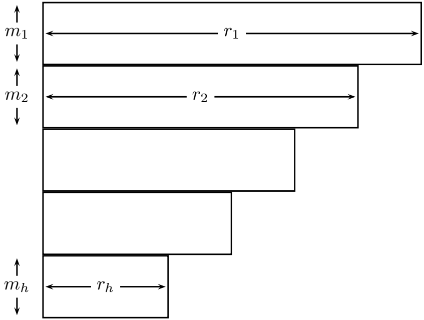

Upper estimates of $\rho(G_D)$

$\sigma_1(G_D)^2\sigma_2(G_D)^2 \geqslant \omega^* \equiv \max(\omega, \omega')$ (1)
$\rho(G_D)^2 \leqslant \frac{e(G_D)+\sqrt{e(G_D)^2-4\omega^*(G_D)}}{2}$
Prf: maximize $\rho(G_D)^2 = \lambda_1(G_D)^2$ under the constraints $\sum_{i=1}^m \sigma_i(G_D)^2 = e(G_D)$ and (1)

$\mathcal{K}(p, q, e)$ the family of subgraphs $\mathcal{K}_{p,q}$ with e edges, no isolated vertices and which are not complete bipartite graphs
$\max_{BG\in\mathcal{K}(p,q,e)} \rho(BG) = \rho(G_{D^\star})$

Weak conjecture: $R(G_{D^\star})$ has rank 2 $\iff h = 2$

Strong conjecture: $h = 2$ and $\min(m_2, m_2') = 1$

Upper estimates of $\rho(G_D)$, rank $R(G_D) = 2$

$h = 2$, $n_1 := m_1' = r_2, n_2 := m_2' = r_1 - r_2, \omega^* = m_1 m_2 n_1 m_2$
$e = m_1 n_1 + m_1 n_2 + n_1 m_2$ (1)
$\min m_1 m_2 n_1 n_2$ subject (1) and
$\max(m_1 + m_2, n_1 + n_2) \leq \max(p, q), \min(m_1 + m_2, n_1 + n_2) \leq \min(p, q)$
and m_1, m_2, n_1, n_2 are positive integers

Thm: for $e = 3k + 1, k \in \mathbb{N}$ $\min m_1 m_2 n_1 n_2$, where $m_1, m_2, n_1, n_2 \in \mathbb{N}$
satisfy (1) and $m_1 + m_2 \geq 3$, $n_1 + n_2 \geq 3$
achieved in the two cases
$(m_1, m_2) = (1, 2), \ (n_1, n_2) = (k, 1)$ or
$(m_1, m_2) = (k, 1), \ (n_1, n_2) = (1, 2)$
(Here $\max(p, q) < \frac{e-1}{2}$)

Upper estim.s of $\rho(G_D)$, rank $R(G_D) = 2$: spec. case

Thm: Let $m_1, m_2, n_1, n_2 \in [1, \infty)$ satisfying
$m_1 + m_2 \geq r, n_1 + n_2 \geq r, m_1 n_1 + m_1 n_2 + m_2 n_1 = e \geq r^2 + 1$
$e = lr + r - 1,\ r \leq p \leq q \leq l + 1 + \frac{l}{r-1}$ where $2 \leq r, l \in \mathbb{N}$ (1)
then the minimum of $m_1 m_2 n_1 n_2$ is $\frac{(r-1)(e-r+1)}{r}$, achieved only
a. $(m_1, m_2) = (r - 1, 1),\ (n_1, n_2) = (\frac{(e-r+1)}{r}, 1)$
b. $(m_1, m_2) = (\frac{e-r+1}{r}, 1),\ (n_1, n_2) = (r - 1, 1)$

Cor: if either $r = 2$, $3 \leq e$ odd and $2 \leq p \leq q, l = \frac{e-1}{2} < q$
or $3 \leq r \in \mathbb{N}$ and (1) holds, then
$\min \rho(G_D), \mathrm{rank}\ R(G_D) = 2$ is achieved only for $G_{D^\star}$ isomorphic to the graph obtained from $K_{r-1,l+1}$ by adding one vertex to the group of $r-1$ vertices and connecting it to l vertices in the group of $l+1$ vertices

Thm: Under the above conditions $\min_{BG \in \mathcal{K}(p,q,e)} = \rho(G_{D^\star})$

C-matrices

$\mathbf{c} = (c_1, \ldots, c_m), c_1 \geqslant c_2 \geqslant \ldots \geqslant c_m \geq 0$
$M(\mathbf{c}) = [c_{\max(i,j)}]_{i,j=1}^m$
Example: $R(G_D)R(G_D)^\top = M(\mathbf{d}), \mathbf{d} := (d_1, \ldots, d_m))$
Fact: $M(\mathbf{c})$ is symmetric totally nonnegative matrix:
$\lambda_1(\mathbf{c}) \geqslant \lambda_2(\mathbf{c}) \geqslant \ldots \geqslant \lambda_m(\mathbf{c}) \geq 0$
$\mathrm{rank}\ M(\mathbf{d}) = h$ - number of distinct degrees in D
$(\max_{BG \in \mathcal{K}(p,q,e)} \rho(BG))^2 = \max \lambda_1(\mathbf{d})$
on all allowable degree sequences $\mathbf{d}$

A generalized maximal problem

As $\lambda_1(S)$ is a convex function on $\mathrm{S}(n,\mathbb{R})$ -
($n \times n$ real symmetric matrices)
$\max_{S\in\mathcal{S}} \lambda_1(S) = \max_{\mathcal{E}(\mathcal{S})} \lambda_1(S)$
$\mathcal{S}$ compact closed subset of $\mathrm{S}(n,\mathbb{R})$, $\mathcal{E}(\mathcal{S})$ - extreme points of $\mathcal{S}$
$\mathcal{F} = \{\mathbf{f} \in \mathbb{R}^m_{+,\searrow}, 1 \le f_m, f_1 \le \max(p,q), \sum_{i=1}^m f_i = e\}$
Extreme points of $\mathcal{F}$ are sequences $f_1 = \ldots = f_t > f_{t+1} = \ldots = f_m$
$(\max_{BG\in\mathcal{K}(p,q,e)})^2 \le \max_{\mathbf{f}\in\mathcal{F}} \lambda_1(M(\mathbf{f})) = \max_{\mathbf{f}\in\mathcal{E}(\mathcal{F})} \lambda_1(M(\mathbf{f}))$
Each $\mathbf{f} \in \mathcal{E}(\mathcal{F})$ induces $m_1 m_2 n_1 n_2$
and the maximal $\lambda_1(\mathbf{f}^\star)$ corresponds to the minimum of $m_1 m_2 n_1 n_2$
For $e = lr + r - 1,\ r \le l + 1 + \frac{l}{r-1}$ where $2 \le r, l \in \mathbb{N}$
the minimum achieved at
a. $(m_1, m_2) = (r-1, 1),\ (n_1, n_2) = (\frac{(e-r+1)}{r}, 1)$
b. $(m_1, m_2) = (\frac{e-r+1}{r}, 1),\ (n_1, n_2) = (r-1, 1)$
which corresponds to $\mathbf{f}_1, \mathbf{f}_2$ and give rise to two isomorphic graphs obtained from $K_{r-1,l+1}$ by adding one vertex to the group of $r-1$ vertices and connecting it to l vertices in the group of $l+1$ vertices

The Grone-Merris conjecture

Laplacian of $G = (V, E), V = \{v_1, \dots, v_n\}$ non-bipartite:
$L(G) = \mathrm{diag}(\mathbf{d}) - A(G)$
Fact $L(G) \sqsupseteq 0$: is singular nonnegative definite $L(G)\mathbf{1} = \mathbf{0}$
$\boldsymbol{\alpha}(G) = \{\alpha_1(G) \geqslant \dots \geqslant \dots \geqslant \alpha_{n-1}(G) \geqslant \alpha_n(G) = 0\}$
For complementary graph G^c: $\alpha_i(G) + \alpha_{n-i-1}(G^c) = n$ for
$i = 1, \dots, n-1$
$d_i'(G)$ -number of degrees of G which greater or equal to i for
$i = 1, \dots, n$
$d_1'(G) \geqslant d_2'(G) \geqslant \dots d_{n-1}'(G) \geqslant d_n'(G) = 0$, $\mathbf{d}'(G) = (d_1'(G), \dots, d_n'(G))$
$\mathbf{d}'$-dual sequence to $\mathbf{d}$ (The column sums in Ferre diagram)

Grone-Merris conjecture 1994:
$\sum_{i=1}^k \alpha_i(G) \leq \sum_{i=1}^k d_i'(G)$ for $k = 1, \dots, n$
$\sum_{i=1}^n \alpha_i(G) = \mathrm{tr}\, L(G) = \sum_{i=1}^n d_i(G) = \sum_{i=1}^n d_i'(G)$
Grone-Merris: Conjecture holds for threshold graphs

Cases $k = 1, 2$

$k = 1$: $\alpha_1(G) = n - \alpha_{n-1}(G^c) \leq n = d'_1(G)$
Equality holds if and only if G has no isolated vertices and G^c is disconnected
If $d_i(G) \geq j \geq 2$ for all i then $\sum_{i=1}^{k} \alpha_i(G) \leq kn = \sum_{i=1}^{k} d'_i(G)$
for $k = 1, \ldots, j$
for $k = j$ equality holds iff G^c has at least $j + 1$ connected components

It is enough to consider the case where G connected and $k \geq 2$
Duval-Reiner 2002, Katz (2007?)
If $d'_2 = n - l, n > l \geq 1$ then GM conjecture holds for $k = 2$
Equality holds if and only if G threshold graph obtained from K_{n-l-1} first adding l isolated vertices and then one vertex connected to all $n - l - 1$.
Observation: it is enough to consider the case where all $n - l$ vertices form a clique:
$L(G) + L(H) \sqsupseteq L(G)$ where H a graph on n vertices with one edge

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