Theoretical and computational methods in statistical mechanics

Shmuel Friedland
Univ. Illinois at Chicago

Berkeley, October 26, 2009

Overview

- Motivation: Ising model
- Subshifts of Finite Type
- Pressure P_Γ
- Density points and density entropy
- Convex functions
- P_Γ^* and color density entropy
- First order phase transition
- d-Dimensional Monomer-Dimers
- Friendly colorings
- Computation of pressure

Figure: Uri Natan Peled, Photo - December 2006

Uri N. Peled

Uri was born in Haifa, Israel, in 1944.
Education:
Hebrew University, Mathematics-Physics, B.Sc., 1965.
Weizmann Institute of Science, Physics, M.Sc., 1967
University of Waterloo, Mathematics, Ph.D., 1976
University of Toronto, Postdoc in Mathematics, 1976–78
Appointments:
1978–82, Assistant Professor, Columbia University
1982–91, Associate Professor, University of Illinois at Chicago
1991–2009, Professor, University of Illinois at Chicago
Areas of research: Graphs, combinatorial optimization, boolean functions.
Uri published about 57 paper
Uri died September 6, 2009 after a long battle with brain tumor.

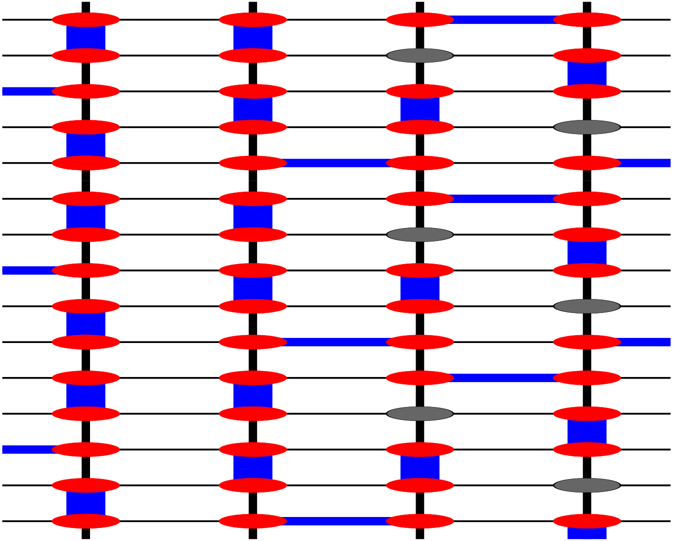


Figure: Matching on the two dimensional grid: Bipartite graph on 60 vertices, 101 edges, 24 dimers, 12 monomers

Motivation: Ising model - 1925

On lattice $\mathbb{Z}^d$ two kinds of particles: spin up 1 and spin down 2. Each neighboring particles located on $(\mathbf{i}, \mathbf{i} + \mathbf{e}_j)$ interact with energy $-J$ if both locations are occupied by the same particles, and with energy J if the two sites are occupied by two different particles. In addition each particle has a magnetization due to the external magnetic field. The energy of the particle of type 1 is H while the energy of the particle of type 2 is $-H$. The energy of $E(\phi)$ of a given finite configuration of particles in $\mathbb{Z}^d$ is the sum of the energies of the above type.

Ferromagnetism $J > 0$: all spins are up or down.

Antiferromagnetism $J < 0$ half spins up and down

(Lowest free energy)

Phase transition:

from one state to another as the temperature varies

Energy: $\frac{k}{T} E(\phi)$

Motivation: Ising model - 1925

On lattice $\mathbb{Z}^d$ two kinds of particles: spin up 1 and spin down 2. Each neighboring particles located on $(\mathbf{i}, \mathbf{i} + \mathbf{e}_j)$ interact with energy $-J$ if both locations are occupied by the same particles, and with energy J if the two sites are occupied by two different particles. In addition each particle has a magnetization due to the external magnetic field. The energy of the particle of type 1 is H while the energy of the particle of type 2 is $-H$. The energy of $E(\phi)$ of a given finite configuration of particles in $\mathbb{Z}^d$ is the sum of the energies of the above type.

Ferromagnetism $J > 0$: all spins are up or down.

Antiferromagnetism $J < 0$ half spins up and down

(Lowest free energy)

Phase transition:

from one state to another as the temperature varies

Energy: $\frac{k}{T} E(\phi)$

Ising: No phase transition in one dimensional Ising model

Onsager 1944: One phase transition point of second order for $d = 2, H = 0$.

$$< n > := \{1, 2, 3, ..., n\}$$

ALPHABET ON n LETTERS - COLORS.

$$\underbrace{\bullet\ \bullet\ \bullet \ldots \bullet\ \bullet\ \bullet}_{m \text{ times}}$$

Coloring of $\mathbb{Z}^d$ in n coloring =
Full $\mathbb{Z}^d$ shift on n symbols
Example of SOFT: $(0-1)$ LIMITED CHANNEL
HARD CORE LATTICE or NEAR NEIGHBOR EXCLUSION
$n = 2$, $< 2 > = \{1, 2\} = \{1, 0\}$ $(2 \equiv 0)$.
NO TWO $1's$ ARE NEIGHBORS.

One dimensional SOFT

$\Gamma \subseteq < n > \times < n >$ directed graph on n vertices $C_\Gamma(< m >)$-all Γ allowable configurations of length m:

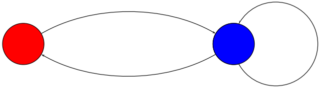
$\{a = a_1 ... a_m = (a_i)_1^m :< m > \to < n > \ (a_i, a_{i+1}) \in \Gamma\}$

$C_\Gamma(\mathbb{Z})$-all Γ allowable configurations (tilings) on $\mathbb{Z}$:

$\{a = (a_i)_{i \in \mathbb{Z}} : \mathbb{Z} \to < n >, (a_i, a_{i+1}) \in \Gamma\}$

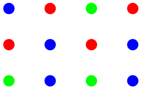
Hard core model:

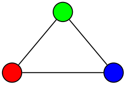
$n = 2, \Gamma = \{\bullet\bullet, \bullet\bullet, \bullet\bullet\}$



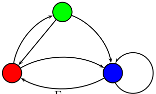
MD SOFT=Potts Models

Dimension $d \geq 2$. For $\mathbf{m} \in \mathbb{N}^d$, $<\mathbf{m}>:=<m_1> \times \ldots \times <m_d>$
$\mathrm{vol}(\mathbf{m}) := |m_1| \times \ldots \times |m_d|$, $\Gamma := (\Gamma_1, \ldots, \Gamma_d), \Gamma_i \subset \langle n \rangle \times \langle n \rangle$
$C_\Gamma(<\mathbf{m}>)$-all Γ allowable configurations of $\mathbf{m}$:
$a = (a_\mathbf{i})_{\mathbf{i} \in \langle \mathbf{m} \rangle} :<\mathbf{m}> \to <n>$ s.t. $(a_\mathbf{i}, a_{\mathbf{i}+\mathbf{e}_j}) \in \Gamma_j$ if $\mathbf{i}, \mathbf{i}+\mathbf{e}_j \in \langle \mathbf{m} \rangle$
$\mathbf{e}_j = (\delta_{1j}, \ldots, \delta_{dj}), j = 1, \ldots, d$.
Example:

$<(4,3)>:=$ 



Γ_1



Γ_2

MD SOFT=Potts Models II

For $\phi \in C_\Gamma(\langle \mathbf{m} \rangle)$ - $\mathbf{c}(\phi) := (c_1(\phi), \ldots, c_n(\phi))$
denotes coloring distribution of configuration ϕ
$c_i(\phi)$-the number of times the particle i appears in ϕ
$\frac{1}{\mathrm{vol}(\mathbf{m})}\mathbf{c}(\phi) \in \Pi_n$ - coloring frequency of ϕ
$\Pi_n(\mathrm{vol}(\mathbf{m}))$ all $\mathbf{c} \in \mathbb{Z}_+^n$ s.t. $\frac{1}{\mathrm{vol}(\mathbf{m})}\mathbf{c} \in \Pi_n$
$C_\Gamma(\langle \mathbf{m} \rangle, \mathbf{c})$ denotes all $\phi \in C_\Gamma(\langle \mathbf{m} \rangle)$ with $\mathbf{c}(\phi) = \mathbf{c}$.
$C_{\Gamma,\mathrm{per}}(\langle \mathbf{m} \rangle) \subseteq C_\Gamma(\langle \mathbf{m} \rangle)$-$\mathbf{m}$-periodic configurations
$C_\Gamma(\mathbb{Z}^d)$-are-Γ allowable configurations of $\mathbb{Z}^d$
Assumption: $C_\Gamma(\mathbb{Z}^d) \neq \emptyset$
$u_i \in \mathbb{R}$ energy of particle $i \in \langle n \rangle$
$\mathbf{u} := (u_1, \ldots, u_n) \in \mathbb{R}^n$ energy vector
$E(\phi) = \mathbf{c}(\phi) \cdot \mathbf{u}$ Energy of configuration ϕ
Near neighbor interaction model, can be fit to the above noninteraction model by considering the coloring of the cube $\langle (3, \ldots, 3) \rangle$ as one particle
Similarly short range interaction model

Pressure

Grand partition function

$Z_\Gamma(\mathbf{m},\mathbf{u}) := \sum_{\phi\in C_\Gamma(\langle\mathbf{m}\rangle)} e^{\mathbf{c}(\phi)\cdot\mathbf{u}}$

$\log Z_\Gamma(\mathbf{m},\mathbf{u})$ subadditive in each component of $\mathbf{m}$ and convex in $\mathbf{u}$

$\frac{1}{\mathrm{vol}(\mathbf{m})}\log Z_\Gamma(\mathbf{m},\mathbf{u})$ - average energy or pressure

$P_\Gamma(\mathbf{u}) := \lim_{\mathbf{m}\to\infty} \frac{1}{\mathrm{vol}(\mathbf{m})}\log Z_\Gamma(\mathbf{m},\mathbf{u})$

Pressure of Γ-SOFT, (Pressure of the Potts model)

$h_\Gamma := P_\Gamma(\mathbf{0})$-ENTROPY of Γ-SOFT

$P_\Gamma(\mathbf{u})$ is a convex Lipschitz function on $\mathbb{R}^n$

$|P_\Gamma(\mathbf{u}) - P_\Gamma(\mathbf{v})| \le ||\mathbf{u}-\mathbf{v}||_\infty := \max|u_i - v_i|$

$P_\Gamma(\mathbf{u} + t\mathbf{1}) = P_\Gamma(\mathbf{u}) + t$

P_Γ has the following properties:

Has subdifferential $\partial P_\Gamma(\mathbf{u})$ for each $\mathbf{u}$

$\partial P_\Gamma(\mathbf{u}) \subseteq \Pi_n$ for each $\mathbf{u}$

Has differentiable $\nabla P_\Gamma(\mathbf{u})$ a.e.

Density points and density entropy

$\mathbf{p} \in \Pi_n$ density point of $C_\Gamma(\mathbb{Z}^d)$ when there exist sequences of boxes $\langle \mathbf{m}_q \rangle \subseteq \mathbb{N}^d$ and color distribution vectors $\mathbf{c}_q \in \Pi_n(\mathrm{vol}(\mathbf{m}_q))$

$\mathbf{m}_q \to \infty, \quad C_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q) \neq \emptyset \ \ \forall q \in \mathbb{N}, \quad$ and $\lim_{q\to\infty} \frac{\mathbf{c}_q}{\mathrm{vol}(\mathbf{m}_q)} = \mathbf{p}$

Π_Γ the set of all density points of $C_\Gamma(\mathbb{Z}^d)$

Π_Γ is a closed set

For $\mathbf{p} \in \Pi_\Gamma$ the color density entropy

$h_\Gamma(\mathbf{p}) := \sup_{\mathbf{m}_q, \mathbf{c}_q} \limsup_{q\to\infty} \frac{\log \# C_\Gamma(\langle \mathbf{m}_q \rangle, \mathbf{c}_q)}{\mathrm{vol}(\mathbf{m}_q)} \geq 0$

where the supremum is taken over all sequences satisfying the above conditions

h_Γ is upper semi-continuous on Π_Γ

Convex functions

$f : \mathbb{R}^n \to [-\infty, \infty]$ convex. $\mathrm{dom}\, f := \{\mathbf{x} \in \mathbb{R}^m : \ f(\mathbf{x}) < \infty\}$
f proper if $f : \mathbb{R}^n \to \bar{\mathbb{R}} := (-\infty, \infty]$ and $f \not\equiv \infty$ f closed if f is lower semi-continuous.
$\mathbf{q}$ subgradient: $f(\mathbf{x}) \geq f(\mathbf{u}) + \mathbf{q}^\top(\mathbf{x} - \mathbf{u})\ \forall \mathbf{x}$ $\partial f(\mathbf{u}) \subset \mathbb{R}^n$ the subset of subgradients of f at $\mathbf{u}$ ASSUMPTION: f is proper and closed
$\partial f(\mathbf{u})$ is a closed nonempty set for each $\mathbf{u} \in \mathrm{ri}\ \mathrm{dom}\, f$ f is differentiable at $\mathbf{u}$ $\iff$ $\partial f(\mathbf{u}) = \{\nabla f(\mathbf{u})\}$ $\mathrm{diff}\, f$ - the set of differentiability points of f
∇f continuous on $\mathrm{diff}\, f$ and $\overline{\mathrm{diff}\, f} \supseteq \mathrm{dom}\, f$
The conjugate, (Legendre transform) f^* defined:
$f^*(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbb{R}^n} \mathbf{x}^\top \mathbf{y} - f(\mathbf{x})$ for each $\mathbf{y} \in \mathbb{R}^m$
f^* is a proper closed function and $f^{**} = f$

P_Γ^* and color density entropy

Thm 1: $h_\Gamma(\mathbf{p}) \leqslant -P_\Gamma^*(\mathbf{p})\ \forall \mathbf{p} \in \Pi_\Gamma$.
$P_\Gamma(\mathbf{u}) = \max_{\mathbf{p}\in\Pi_\Gamma}(\mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})), \mathbf{u} \in \mathbb{R}^n$
$\Pi_\Gamma(\mathbf{u}) := \arg\max_{\mathbf{p}\in\Pi_\Gamma}(\mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})) = \{\mathbf{p} \in \Pi_\Gamma : P_\Gamma(\mathbf{u}) = \mathbf{p}^\top\mathbf{u} + h_\Gamma(\mathbf{p})\}$
For each $\mathbf{p} \in \Pi_\Gamma(\mathbf{u})$, $h_\Gamma(\mathbf{p}) = -P_\Gamma^*(\mathbf{p})$. $\Pi_\Gamma(\mathbf{u}) \subseteq \partial P_\Gamma(\mathbf{u})$.
$\mathbf{u} \in \mathrm{diff}\, P_\Gamma \Rightarrow \Pi_\Gamma(\mathbf{u}) = \{\nabla P_\Gamma(\mathbf{u})\}$.
Therefore $\partial P_\Gamma(\mathrm{diff}\, P_\Gamma) \subseteq \Pi_\Gamma$.
$S(\mathbf{u}), \mathbf{u} \in \mathbb{R}^n \setminus \mathrm{diff}\, P_\Gamma$-
are all the limits of sequences
$\nabla P_\Gamma(\mathbf{u}_i), \mathbf{u}_i \in \mathrm{diff}\, P_\Gamma$ and $\mathbf{u}_i \to \mathbf{u}$.
Then $S(\mathbf{u}) \subseteq \Pi_\Gamma(\mathbf{u})$.
$\mathrm{conv}\ \Pi_\Gamma(\mathbf{u}) = \mathrm{conv}\ S(\mathbf{u}) = \partial P_\Gamma(\mathbf{u})$.
$\partial P_\Gamma(\mathbb{R}^n) \subseteq \mathrm{conv}\ \Pi_\Gamma \subseteq \Pi_n$.
$\mathrm{conv}\ \Pi_\Gamma = \mathrm{dom}\ P_\Gamma^*$.

First order phase transition

Claim: For $\mathbf{u} \in \mathbb{R}^n$ each $\mathbf{p} \in \Pi_\Gamma(\mathbf{u})$ is the set of possible density of n colors in an allowable configurations from $C_\Gamma(\mathbb{Z}^d)$ with the potential $\mathbf{u}$.
For $\mathbf{u} \in \mathrm{diff}\, P_\Gamma$ $\mathbf{p}(\mathbf{u}) = \nabla P_\Gamma(\mathbf{u})$ is a unique density.
Claim: Any point of nondifferentiabity of P_Γ is a point of the phase transition.
Proof Let $\mathbf{u} \in \mathbb{R}^n \setminus \mathrm{diff}\, P_\Gamma$ Then ∂P_Γ consists of more than one point.
Thm 1 yields that $\partial P_\Gamma(\mathbf{u}) = \mathrm{conv}\, S(\mathbf{u}) \subseteq \Pi_\Gamma(\mathbf{u})$. $S(\mathbf{u})$ consists of more than one point. Hence $\Pi_\Gamma(\mathbf{u})$ consists of more than one density for $\mathbf{u}$.
$\mathbf{u} \in \mathbb{R}^n \setminus \mathrm{diff}\, P_\Gamma$ is called a point of phase transition, or a phase transition point of the first order.

d-Dimensional Monomer-Dimers

Dimer: $(\mathbf{i},\mathbf{j}),\ \mathbf{j} = \mathbf{i} + \mathbf{e}_k \in \mathbb{Z}^d$.
any partition of $\mathbb{Z}^d$ to dimers (1-factor).
Monomer: occupies $\mathbf{i} \in \mathbb{Z}^d$.
any partition of $\mathbb{Z}^d$ to monomer-dimers
is 1-factor of a subset of $\mathbb{Z}^d$.

Dimer and Monomer-Dimer are SOFT

$$0 = \tilde{h}_1 \le \tilde{h}_2 \le \ldots \le \tilde{h}_d \le \ldots (\mathrm{dimers})$$

$$\log \frac{1+\sqrt{5}}{2} = h_1 \le h_2 \le \ldots \le h_d \le \ldots$$
$$(\mathrm{monomer-dimer})$$

Fisher, Kasteleyn and Tempreley 61

$$\tilde{h}_2 = \frac{1}{\pi} \sum_{i=0}^{\infty} \frac{(-1)^i}{(2i+1)^2} = 0.29156090\ldots$$

Hammersley's results

Hammersley in 60's studied extensively the monomer-dimer model. He showed $\Pi_\Gamma = \Pi_{d+1}$ for d-dimensional model $\mathbf{p} = (p_1, \ldots, p_d, p_{d+1})$
p_i-the dimer density in $\mathbf{e}_i$-direction $i = 1, \ldots, d$ p_{d+1}-the monomer density Hammersley studied $p := p_1 + \ldots + p_d$-the total dimer density
$h_d(p)$-the p-dimer density in $\mathbb{Z}^d$, $p \in [0, 1]$
He showed $h_d(p)$-concave continuous function on $[0, 1]$
Heilman and Lieb 72: $h_d(p)$ analytic on $(0, 1)$
No phase transition in parameter $p \in [0, 1)$
Au-Yang - Perk 84: Phase transition at dimers $p = 1$

The Graphs for $h_2(p)$

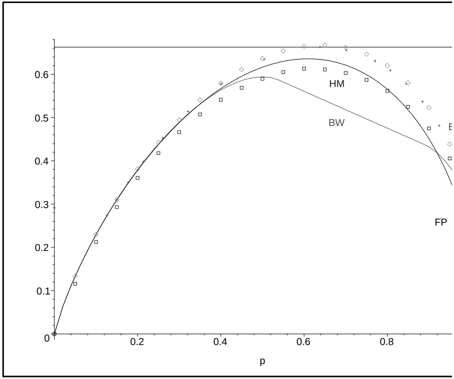


Figure 1: HM is the lower bound of Hammersley-Menon, BW is the lower bound of Bondy-Welsh, FP is the lower bound of Friedland-Peled, MC is the Monte Carlo estimate of Hammersley-Menon, B are Baxter's estimates, and h2 is the true value of $h_2 = \max h_2(p)$.

Graph estimates for $h_2(p)$

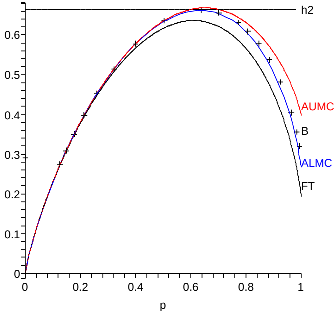


Figure 1: Monomer-dimer tiling of the 2-dimensional grid: entropy as a function of dimer density. FT is the Friedland-Tverberg lower bound, h2 is the true monomer-dimer entropy. B are Baxter's computed values. ALMC is the Asymptotic Lower Matching Conjecture. AUMC is the entropy of a countable union of $K_{4,4}$, conjectured to be an upper bound by the Asymptotic Upper Matching Conjecture.

Friendly colorings

Thm 1 implies:
For any Potts model $h_\Gamma(\cdot) : \Pi_\Gamma \to \mathbb{R}_+$ is concave on every convex subset of $\Pi_\Gamma(\mathbb{R}^n)$.
To get the exact analog of Hammersley's result
$\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ on $\langle n \rangle$ $\mathcal{F} = \cup_{\mathbf{m} \in \mathbb{N}^d} \widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$, where $\widetilde{C}_\Gamma(\langle \mathbf{m} \rangle) \subseteq C_\Gamma(\langle \mathbf{m} \rangle)$ for each $\mathbf{m} \in \mathbb{N}^d$, friendly: if whenever a box $\langle \mathbf{m} \rangle$ is cut in two and each part is colored by a coloring in $\mathcal{F}$, the combined coloring is in $\mathcal{F}$.
Γ friendly if there exist a friendly set $\mathcal{F} = \cup_{\mathbf{m} \in \mathbb{N}^d} \widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$ and a constant vector $\mathbf{b} \in \mathbb{N}^d$ such that if any box $\langle \mathbf{m} \rangle$ is padded with an envelope of width b_i in the direction of $\mathbf{e}_i$, then each Γ-allowed coloring of $\langle \mathbf{m} \rangle$ can be extended in the padded part to a coloring in $\mathcal{F}$.

Examples of friendly colorings

Γ has a friendly color $f \in \langle n \rangle$, i.e., for each $i \in \langle d \rangle$ $(f, j), (j, f) \in \Gamma_i$ for all $j \in \langle n \rangle$

Then $\widetilde{C}_\Gamma(\mathbf{m})$ are Γ-allowed colorings of $\langle \mathbf{m} \rangle$ whose boundary points are colored with f

Hard-core model: $\Gamma_i = \{(1,1), (1,2), (2,1)\}$, has friendly color $f = 1$.

Γ associated with the monomer-dimer covering

$\widetilde{C}_\Gamma(\langle \mathbf{m} \rangle)$ the set of tilings of $\langle \mathbf{m} \rangle$ by monomers and dimers, i.e., the coverings in which no dimer protrudes out of $\langle \mathbf{m} \rangle$, as in Hammersley

P^*_Γ for friendly colorings

Thm 2: Let $\Gamma = (\Gamma_1, \ldots, \Gamma_d)$ be a friendly coloring digraph. Then

1. Π_Γ is convex. Hence $\Pi_\Gamma = \mathrm{dom}\, P^*_\Gamma$.
2. $h_\Gamma(\cdot) : \Pi_\Gamma \to \mathbb{R}_+$ is concave.
3. For each $\mathbf{u} \in \mathbb{R}^n$, $\Pi_\Gamma(\mathbf{u}) = \partial P_\Gamma(\mathbf{u})$.
4. For each $\mathbf{u} \in \mathbb{R}^n$, $h_\Gamma(\cdot)$ is an affine function on $\partial P_\Gamma(\mathbf{u})$.
5. $h_\Gamma(\mathbf{p}) = -P^*_\Gamma(\mathbf{p})$ for each $\mathbf{p} \in \Pi_\Gamma$.

Reduction of one parameter

$P_\Gamma(\mathbf{u}) = t + P_\Gamma(\mathbf{u} - t\mathbf{1}) \Rightarrow \partial P_\Gamma(\mathbf{u}) \in \Pi_n$ It is enough to compute $\hat{P}_\Gamma(\hat{\mathbf{u}}) := P_\Gamma(\hat{\mathbf{u}}), \hat{\mathbf{u}} = (u_1, \ldots, u_{n-1}, 0)$ Hard core model: $\hat{P}_\Gamma(t)$ depends on the energy $t \in \mathbb{R}$.

(It is known that for $d \geq 2$ hard core model has phase transition)

For the dimer problem the pressure $P_d(\mathbf{v})$ depends on $\mathbf{v} = (v_1, \ldots, v_d)$, where v_i is the energy of the dimer in the direction $\mathbf{e}_i, i = 1, \ldots$.

(Non-isotropic model)

Dimer isotropic model in $\mathbb{Z}^d$: pressure $P_d(v)$, where v is the energy of the dimer in any direction.

(Standard model-No phase transition for $v \in \mathbb{R}$)

Computation of pressure

Using the scaled transfer matrices on the torus $T(\mathbf{m}'), \mathbf{m}' = (m_1, \ldots, m_{d-1})$ as in Friedland-Peled 2005 [3].
Assume for simplicity $d = 2, \Gamma = (\Gamma_1, \Gamma_2)$, where Γ_1 symmetric digraph.
Let Δ transfer digraph induced by Γ_2 between the allowable Γ_1 coloring of the circle $T(m)$. Then $V := \mathcal{C}_{\Gamma_1,\mathrm{per}}(m)$ are the set of vertices of $\Delta(m)$. For $\alpha, \beta \in \mathcal{C}_{\Gamma_1,\mathrm{per}}(m)$ the directed edge (α, β) is in $\Delta(m)$ iff the configuration $[(\alpha, \beta)]$ is an allowable configuration on $\mathcal{C}_\Gamma((m, 2))$.
Adjacency matrix $D(\Delta(m)) = (d_{\alpha\beta})_{\alpha,\beta\in\mathcal{C}_{\Gamma_1,\mathrm{per}}(m)}$ is $N \times N$ matrix, where $N := \#\mathcal{C}_{\Gamma_1,\mathrm{per}}(m)$. One dimensional SOFT is $\mathcal{C}_\Gamma(T(m) \times \mathbb{Z})$: all Γ allowable coloring of the infinite torus in the direction $\mathbf{e}_2$ with the basis $T(m)$. The pressure corresponding to this one dimensional SOFT is denoted by $\tilde{P}_{\Delta(m)}(\mathbf{u})$. Its formula:

Computation of pressure II

Let $\tilde{D}(\Delta(m), \mathbf{u}) = (\tilde{d}_{\alpha\beta}(\mathbf{u}))_{\alpha,\beta \in C_{\Gamma_1,\mathrm{per}}(m)}$ $\tilde{d}_{\alpha\beta}(\mathbf{u}) = d_{\alpha\beta} e^{\frac{1}{2}(\mathbf{c}(\alpha)+\mathbf{c}(\beta))^\top \mathbf{u}}$

Then $\tilde{P}_\Delta(\mathbf{u}) := \frac{\theta(\mathbf{u},m)}{m}, \theta(\mathbf{u}, m) := \log \rho(\tilde{D}(\Delta(m), \mathbf{u}))$ (We divide $\log \rho(\tilde{D}(\Delta, \mathbf{u}))$ by m, to have $\tilde{P}_\Delta(\mathbf{u} + t\mathbf{1}) = \tilde{P}_\Delta(\mathbf{u}) + t \quad \text{for any } t \in \mathbb{R}$

Main inequalities

$\frac{1}{p}(\theta(\mathbf{u}, p + 2q) - \theta(\mathbf{u}, 2q)) \leq P_\Gamma(\mathbf{u}) \leq \frac{1}{2m}(\theta(\mathbf{u}, 2m))$ for any $m, p \geq 1$ and $q \geq 0$.

Automorphism Subgroups

$A = (a_{ij})_1^N$ nonnegative matrix
$\mathcal{A}(A) := \{\pi \in S_N : a_{\pi(i)\pi(j)} = a_{ij}, i, j \in < N >\}$ Let
$G \leq \mathcal{A}(A), \mathcal{O}(G) :=< N > /G, M = \#\mathcal{O}(G)$
$\hat{A} = (\hat{a}_{\alpha\beta})_{\alpha,\beta\in\mathcal{O}(G)}, \hat{a}_{\alpha\beta} =: \sum_{j\in\beta} a_{ij},\ i \in \alpha, \rho(A) = \rho(\hat{A})$, If $A = A^T$ then
$\hat{A}$ symmetric for $< \mathbf{x}, \mathbf{y} >= \sum_{\alpha\in\mathcal{O}(G)}(\#\alpha)x_\alpha y_\alpha$.
$M \geq N/\#G$,
In our computations $M \sim N/\#G$
Using these tools we confirmed Baxter's computations with nine digits of precision of $P_2(v)$ and of $h_2(p)$.
We also computed the non-isotropic $P_2((v_1, v_2))$.

Graphs of two dimensional pressure for MD

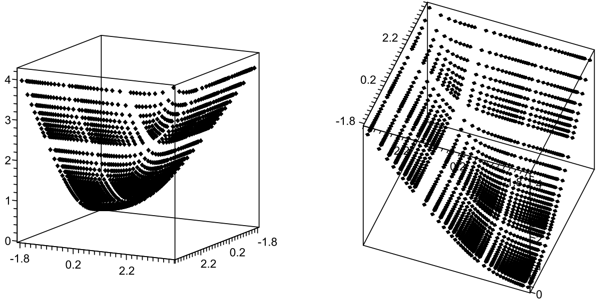


Figure 1: The graph of $\frac{\bar{P}_1(12,(v_1,v_2))}{12}$ for angles $\theta = 28^o, \varphi = 78^o$ and $\theta = -159^o, \varphi = 42^0$

Graphs of two dimensional density entropy for MD

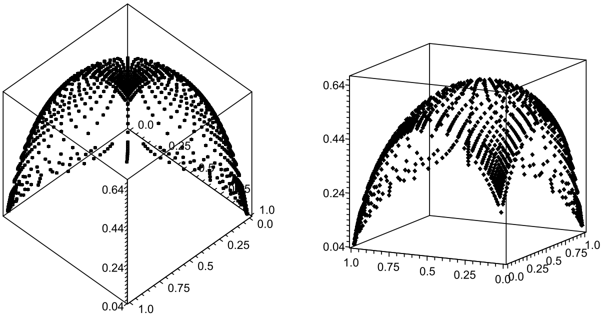


Figure 1: The graph of an approximation of $\bar{h}_2((p_1, p_2)$ for angles $\theta = 45^o, \varphi = 45^o$ and $\theta = -153^o, \varphi = 78^o$

Bibliography I

- H. Au-Yang and J.H.H. Perk, *Phys. Lett. A* 104 (1984), 131–134.
- R. J. Baxter, Dimers on a rectangular lattice, *J. Math. Phys.* 9 (1968), 650–654.
- S. Friedland, On the entropy of Z-d subshifts of finite type, *Linear Algebra Appl.* 252 (1997), 199–220.
- S. Friedland, Multi-dimensional capacity, pressure and Hausdorff dimension, in *Mathematical System Theory in Biology, Communication, Computation and Finance*, edited by J. Rosenthal and D. Gilliam, IMA Vol. Ser. 134, Springer, New York, 2003, 183–222.
- S. Friedland, E. Krop and K. Markstrom,On the number of matchings in regular graphs, *The Electronic Journal of Combinatorics*, 15 (2008), #R110, 1-28.

Bibliography II

- S. Friedland, E. Krop, P.H. Lundow and K. Markström, On the validations of the asymptotic matching conjectures, *Journal of Statistical Physics*, 133 (2008), 513-533.
- S. Friedland, P.H. Lundow and K. Markstrom The 1-vertex transfer matrix and accurate estimation of channel capacity, to appear in *IEEE Transactions on Information Theory*, arXiv:math-ph/0603001v2, arXiv:math-ph/0603001v2, http://arxiv.org/abs/math-ph/0603001v2.
- S. Friedland and U. N. Peled, Theory of Computation of Multidimensional Entropy with an Application to the Monomer-Dimer Problem, *Advances of Applied Math.* 34(2005), 486-522.
- The pressure, densities and first order phase transitions associated with multidimensional SOFT, jointly with U.N. Peled, arXiv:0906.5176v3, http://arxiv.org/abs/0906.5176v3, *submitted*.

Bibliography III

- F. R. Gantmacher, *The Theory of Matrices*, Vol. II, Chelsea Publ. Co., New York 1959.
- J. M. Hammersley, Existence theorems and Monte Carlo methods for the monomer-dimer problem, in *Reseach papers in statistics: Festschrift for J. Neyman*, edited by F.N. David, Wiley, London, 1966, 125–146.
- O. J. Heilman and E. H. Lieb, Theory of monomer-dimer systems, *Comm. Math. Phys.* 25 (1972), 190–232; Errata 27 (1972), 166.
- E. Ising, Beitrag zur Theory des Ferromagnetismus, *Z. Physik* 31 (1925), 253–258.
- P.W. Kasteleyn, The statistics of dimers on a lattice, *Physica* 27 (1961), 1209–1225.
- J. F. C. Kingman, A convexity property of positive matrices. *Quart. J. Math. Oxford Ser.* (2) 12 (1961), 283–284.

Bibliography IV

- L. Onsager, Cristal statistics, I. A two-dimensional model with an order-disorder transition, *Phys. Review* 65 (1944), 117–?
- R.B. Potts, Some generalized order-disorder transformations, *Math. Proc. Cambridge Philos. Soc.* 48 (1952), 106–109.
- R. T. Rockafeller, *Convex Analysis*, Princeton Univ. Press 1970.
- H. N. V. Temperley and M. E. Fisher, Dimer probelm in statistical mechanics—an exact result, *Phil. Mag. (8)* 6 (1961), 1061–1063.
- C. J. Thompson, *Mathematical Statisitical Mechanics*, Princeton Univ. Press, 1972.