A proof of the set-theoretic version of the salmon conjecture

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April 12, 2011

Abstract

We show that the irreducible variety of $4 \times 4 \times 4$ complex valued tensors of border rank at most 4 is the zero set of polynomial equations of degree 5 (the Strassen commutative conditions), of degree 6 (the Landsberg-Manivel polynomials), and of degree 9 (the symmetrization conditions).

Key words: rank of tensors, border rank of tensors, the salmon conjecture.
2010 Mathematics Subject Classification. 14A25, 15A69.

1 Introduction

Let $V_r(m, n, l) \subseteq \mathbb{C}^m \otimes \mathbb{C}^n \otimes \mathbb{C}^l$ be the variety of tensors of border rank at most r. The projectivization of $V_r(m, n, l)$ is the rth secant variety of $\mathbb{P}^{m-1} \times \mathbb{P}^{n-1} \times \mathbb{P}^{l-1}$. In 2007, Elizabeth Allman posed the problem of determining the ideal $I_4(4, 4, 4)$ generated by all polynomials vanishing on $V_4(4, 4, 4)$, this has been coined the salmon problem [2]. The *salmon conjecture* [7, Conjecture 3.24] states that $I_4(4, 4, 4)$ is generated by polynomials of degree 5 and 9. A first nontrivial step in characterizing $V_4(4, 4, 4)$ is to characterize $V_4(3, 3, 4)$. In [5], Landsberg and Manivel show that $V_4(3, 3, 4)$ satisfies a specific set of polynomial equations of degree 6. (See also [6, Remark 5.7] and [3].) Hence the revised version of the salmon conjecture states that $I_4(4, 4, 4)$ is generated by polynomials of degree 5, 6 and 9 [8, §2]. This in particular implies the set-theoretic version of the salmon conjecture: $V_4(4, 4, 4)$ is the zero set of certain homogeneous polynomials of degree $5, 6, 9$.

It is shown theoretically in [4] that $V_4(4, 4, 4)$ is cut out by polynomials of degree $5, 9, 16$. A main element toward this result is the characterization of the variety of tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^4$ of border rank at most 4. It is shown in [4, Theorem 4.5] that $V_4(3, 3, 4)$ is cut out by polynomials of degrees 9 and 16 equations. The degree 9 equations follow from the observation in [4] that the four frontal slices of $\mathcal{X} \in V_4(3, 3, 4)$, which consists of four 3×3 matrices, are symmetrizable by multiplication on the left and by multiplication on the right by a nonzero matrix $L, R \in \mathbb{C}^{3 \times 3}$ respectively. The existence of nonzero matrices L and R is equivalent to the vanishing of all 9×9 minors of two corresponding 12×9 matrices whose entries

are linear in the entries of $\mathcal{X}$. We call this set of polynomials the symmetrization conditions.

Meanwhile, the entries of L and R are polynomials of degree 8 in the entries of $\mathcal{X}$. The degree 16 equations in [4] are a result of the condition

$$LR^\top = R^\top L = \frac{\operatorname{tr}(LR^\top)}{3} I_3. \tag{1.1}$$

The degree 16 equations are used only in the case A.I.3 of the proof of Theorem 4.5 of [4].

In [5], Landsberg and Manivel exhibited ten linearly independent polynomials of degree 6, referred here as the LM-polynomials, that vanish on $V_4(3, 3, 4)$. Using methods from numerical algebraic geometry, Bates and Oeding give numerical confirmation that $V_4(3, 3, 4)$ is the zero set of specific set of polynomials of degree $6, 9$ [3], where the polynomials of degree 6 are the polynomials in [5].

The aim of this paper to show that $V_4(3, 3, 4)$ is cut out by polynomials of degree 6 and 9. This is done by showing that in Case A.I.3 of [4, Proof of Theorem 4.5] the use of polynomials of degree 16 can be eliminated by use of the LM-polynomials. More precisely we show that any $3 \times 3 \times 4$ tensor $\mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3 \times 3 \times 4}$ whose four frontal are of the form

$$X_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} & 0 \\ x_{2,1,k} & x_{2,2,k} & 0 \\ 0 & 0 & x_{3,3,k} \end{bmatrix}, \quad k = 1, 2, 3, 4, \tag{1.2}$$

has a border rank four at most if and only the ten LM-polynomials vanish on $\mathcal{X}$.

It is not too hard to show that a tensor $\mathcal{X} \in \mathbb{C}^{3 \times 3 \times 4}$ of the form (1.2) has border rank four at most if and only if either the four matrices $\begin{bmatrix} x_{1,1,k} & x_{1,2,k} \\ x_{2,1,k} & x_{2,2,k} \end{bmatrix}, k = 1, 2, 3, 4$ are linearly dependent or $x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. Note that the condition that the above four 2×2 matrices are linearly dependent is equivalent to the vanishing of the polynomial

$$f(\mathcal{X}) = \det \begin{bmatrix} x_{1,1,1} & x_{1,2,1} & x_{2,1,1} & x_{2,2,1} \\ x_{1,1,2} & x_{1,2,2} & x_{2,1,2} & x_{2,2,2} \\ x_{1,1,3} & x_{1,2,3} & x_{2,1,3} & x_{2,2,3} \\ x_{1,1,4} & x_{1,2,4} & x_{2,1,4} & x_{2,2,4} \end{bmatrix}. \tag{1.3}$$

It turns out that the restrictions of ten LM-polynomials to $\mathcal{X}$ of the form (1.2) are the polynomials

$$x_{3,3,k} x_{3,3,l} f(\mathcal{X}) \text{ for } 1 \leq k \leq l \leq 4. \tag{1.4}$$

Hence $\mathcal{X}$ has a border rank four at most if and only the ten LM-polynomial vanish on $\mathcal{X}$. Combining this with the results in [4] we deduce the the set-theoretic version of the salmon conjecture.

We summarize briefly the content of the paper. In §2 we restate the characterization of $V_4(3, 3, 4)$ given in [4, Theorem 4.5]. In §3 we show that the use of polynomials of degree 16 in the proof of [4, Theorem 4.5] can replaced by the use of LM-polynomials. In §4 we summarize briefly the characterization of $V_4(4, 4, 4)$ as the zero set of polynomials of degree $5, 6, 9$.

2 A characterization of $V_4(3, 3, 4)$

We now state [4, Theorem 4.5] which characterizes $V_4(3, 3, 4)$. Let $\mathcal{X} = [x_{i,j,k}]_{i=j=k}^{3,3,4} \in \mathbb{C}^{3\times3\times4}$. The four frontal slices of $\mathcal{X}$ are denoted as the matrices $X_k = [x_{i,j,k}]_{i=j=1}^{3} \in \mathbb{C}^{3\times3}, k = 1, 2, 3, 4$. Assume that $\mathcal{X} \in V_4(3, 3, 4)$. A special case of [4, Lemma 4.3] claims that there exist nontrivial matrices $L, R \in \mathbb{C}^{3\times3}\backslash\{0\}$ satisfying the conditions

$$LX_k - X_k^\top L^\top = 0, \ k = 1, \ldots, 4, L \in \mathbb{C}^{3\times3}, \tag{2.1}$$
$$X_k R - R^\top X_k^\top = 0, \ k = 1, \ldots, 4, R \in \mathbb{C}^{3\times3}. \tag{2.2}$$

These are the symmetrization conditions.

If the entries of R and L are viewed as the entries of two vectors with 9 coordinates each, then the systems (2.1) and (2.2) are linear homogeneous equations with coefficient matrices $C_L(\mathcal{X}), C_R(\mathcal{X}) \in \mathbb{C}^{12\times9}$ respectively. (Observe that for any $A \in \mathbb{C}^{3\times3}$ the matrix $A - A^\top$ is skew symmetric, which has, in general, 3 free parameters.) The entries of $C_L(\mathcal{X}), C_R(\mathcal{X})$ are linear functions in the entries of $\mathcal{X}$. For a generic $\mathcal{X} \in V_4(3, 3, 4)$, rank $C_L(\mathcal{X}) = $ rank $C_R(\mathcal{X}) = 8$. Hence we can express the entries of L and R in terms of corresponding 8×8 minors of $C_L(\mathcal{X}), C_R(\mathcal{X})$ respectively. There is a finite number of ways to express L and R in this way, and some of these expression may be zero matrices [4]. Thus the entries of L and R are polynomials of degree 8 in the entries of in the entries of $\mathcal{X}$. If rank $C_L(\mathcal{X}) < 8$ then each of this expression of L is a zero matrix, and similar statement holds for R. Hence if rank $C_L(\mathcal{X}) = $ rank $C_R(\mathcal{X}) = 8$ then for each expression of L and R the condition (1.1) hold. The characterization of $V_4(3, 3, 4)$ is given by [4, Theorem 4.5].

Theorem 2.1 $\mathcal{X} = [x_{i,j,k}]_{i=j=k=1}^{3,3,4} \in \mathbb{C}^{3\times3\times4}$ *has a border rank 4 at most if and only the following conditions holds.*

1. *Let $X_k := [x_{i,j,k}]_{i=j=1}^{3} \in \mathbb{C}^{3\times3}, k = 1, \ldots, 4$ be the four frontal slices of $\mathcal{X}$. Then the ranks of $C_L(\mathcal{X}), C_R(\mathcal{X})$ are less than 9. (Those are $9 - th$ degree equations.)*

2. *Let R, L be solutions of (2.1) and (2.2) respectively be given by 8×8 minors of $C_L(\mathcal{X}), C_R(\mathcal{X})$. Then (1.1) holds. (Those are $16 - th$ degree equations.)*

The proof of Theorem 2.1 in [4] consists of discussing a number of cases. The 16 degrees polynomial conditions (1.1) are used only in the case A.I.3. In the next section we show how to prove the theorem in the case A.I.3 using only the ten LM-polynomials of degree 6.

3 The case A.I.3 of [4, Theorem 4.5]

Suppose $\mathcal{X} \in \mathbb{C}^{3\times3\times4}$ and there exist two nonzero matrices $L, R \in \mathbb{C}^{3\times3}$ such that (2.1)–(2.2) hold. The case A.I.3 assumes that L and R are rank one matrices. The degree 16 equations yield that $LR^\top = R^\top L = 0$, thus, the remainder of the proof of [4, Theorem 4.5] in the case A.I.3 resolves the case where $LR^\top = R^\top L = 0$. Therefore, to eliminate the use of polynomial conditions of degree 16 we need to show the following.

Claim 3.1 *Let $\mathcal{X} \in \mathbb{C}^{3\times3\times4}$. Let $R, L \in \mathbb{C}^{3\times3}$ be rank one matrices satisfying the conditions (2.1)–(2.2) respectively. Suppose furthermore that either $LR^\top \neq 0$ or $R^\top L \neq 0$. If the ten LM-polynomials vanish on $\mathcal{X}$ then $\mathcal{X} \in V_4(3,3,4)$.*

In the rest of this section we prove Claim 3.1. Assume that $L = \mathbf{u}\mathbf{v}^\top, R = \mathbf{x}\mathbf{y}^\top$. The following claim is straightforward.

$$\mathbf{u}\mathbf{v}^\top A \text{ is symmetric if and only if } \mathbf{v}^\top A = b\mathbf{u}^\top \text{ for some } b \in \mathbb{C}, \tag{3.1}$$

$$A\mathbf{x}\mathbf{y}^\top \text{ is symmetric if and only if } A\mathbf{x} = c\mathbf{y} \text{ for some } c \in \mathbb{C}. \tag{3.2}$$

By changing bases in two copies of $\mathbb{C}^3$ we can assume that $\mathbf{u} = \mathbf{v} = \mathbf{e}_3 = (0,0,1)^\top$. Let $P, Q \in \mathbf{GL}(3, \mathbb{C})$ such that

$$P^\top \mathbf{e}_3, Q^\top \mathbf{e}_3 \in \mathrm{span}(\mathbf{e}_3). \tag{3.3}$$

Then if $A \in \mathbb{C}^3 \times \mathbb{C}^3$ such that (3.1) and (3.2) hold, $\mathbf{e}_3\mathbf{e}_3^\top(PAQ)$ is symmetric. Observe next that $PAQ(Q^{-1}\mathbf{x})(P\mathbf{y})^\top$ is also symmetric. Thus we need to analyze what kind of vectors can be obtained from two nonzero vectors $\mathbf{x}, \mathbf{y}$ by applying $Q^{-1}\mathbf{x}, P\mathbf{y}$, where P, Q satisfy (3.3). By letting $Q_1 := Q^{-1}$ we see that Q_1 satisfies the same conditions Q in (3.3). Hence Q_1, P have the zero pattern

$$\begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}. \tag{3.4}$$

Lemma 3.2 *Let $\mathbf{y} \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$. If $\mathbf{e}_3^\top \mathbf{y} \neq 0$ then there exists $P \in \mathbf{GL}(3, \mathbb{C})$ of the form (3.4) such that $P\mathbf{y} = \mathbf{e}_3$. If $\mathbf{e}_3^\top \mathbf{y} = 0$ then there exists $P \in \mathbf{GL}(3, \mathbb{C})$ of the form (3.4) such that $P\mathbf{y} = \mathbf{e}_2$.*

Proof. Assume first that $\mathbf{e}_3^\top \mathbf{y} \neq 0$. Let $\mathbf{f} = (f_1, 0, f_3)^\top, \mathbf{g} = (0, g_2, g_3)^\top \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ such that $\mathbf{f}^\top \mathbf{y} = \mathbf{g}^\top \mathbf{y} = 0$. Then $f_1 g_2 \neq 0$. Hence there exists $P \in \mathbf{GL}(3, \mathbb{C})$ of the form (3.4), whose first and the second rows are $\mathbf{f}^\top, \mathbf{g}^\top$ respectively, such that $P\mathbf{y} = \mathbf{e}_3$.

Suppose now that $\mathbf{e}_3^\top \mathbf{y} = 0$. Hence there exists $P = P_1 \oplus [1], P_1 \in \mathbf{GL}(2, \mathbb{C})$ such that $P\mathbf{y} = \mathbf{e}_2$. $\qquad\square$

Corollary 3.3 *Let $A \in \mathbb{C}^{3\times3}$ and assume that LA and AR are symmetric matrices for some rank one matrices $L, R \in \mathbb{C}^{3\times3}$. Then there exists $P, Q \in \mathbf{GL}(3, \mathbb{C})$ such that by replacing A, L, R by $A_1 := PAQ, L_1 := Q^\top LP^{-1}, R_1 = Q^{-1}RP^\top$ we can assume $L_1 = \mathbf{e}_3\mathbf{e}_3^\top$ and R_1 has one of the following 4 forms*

$$\mathbf{e}_3\mathbf{e}_3^\top, \quad \mathbf{e}_3\mathbf{e}_2^\top, \quad \mathbf{e}_2\mathbf{e}_3^\top, \quad \mathbf{e}_2\mathbf{e}_2^\top. \tag{3.5}$$

To prove Claim 3.1 we need to consider the first three choices of R_1 in (3.5). Note that by changing the first two indices in $\mathcal{X} \in \mathbb{C}^{3\times3\times4}$ we need to consider only the first two choices of R_1 in (3.5).

3.1 The case $L = R = \mathbf{e}_3\mathbf{e}_3^\top$

Let $X_1, X_2, X_3, X_4 \in \mathbb{C}^{3\times 3}$ be the four frontal sections of $\mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3\times 3\times 4}$. Assume that (2.1)–(2.2) hold. Then each X_k has the form of (1.2). (This is the case discussed in [4, (4.7)].)

Using Mathematica, we took the 10 LM-polynomials available in the ancillary material of [3, deg_6_salmon.txt] and let $x_{1,3,k} = 0$, $x_{2,3,k} = 0$, $x_{3,1,k} = 0$, $x_{3,2,k} = 0$ for $k = 1, 2, 3, 4$. The resulting polynomials had 24 terms. We then factored $f(\mathcal{X})$ from these restricted polynomials. This symbolic computations shows that that the restriction of the ten LM-polynomials to $\mathcal{X}$ satisfying (2.1)–(2.2) are the polynomials given in (1.4). Therefore, by the result of Landsberg-Manivel [5], if $\mathcal{X} \in V_4(3,3,4)$ then all polynomials in (1.4) vanish on $\mathcal{X}$.

Vice versa, suppose that all polynomials in (1.4) vanish on $\mathcal{X}$. Assume first that the polynomial $f(\mathcal{X})$ given by (1.3) vanishes in $\mathcal{X}$. Let

$$Y_k = \begin{bmatrix} x_{1,1,k} & x_{1,2,k} \\ x_{2,1,k} & x_{2,2,k} \end{bmatrix}, \quad k = 1, 2, 3, 4, \tag{3.6}$$

be the projection of the four frontal sections of $\mathcal{X}$ given by (1.2) on $\mathbb{C}^{2\times 2}$. Then $f(\mathcal{X}) = 0$ if and only if Y_1, Y_2, Y_3, Y_4 are linearly dependent. Assuming the generic case that X_1, X_2, X_3, X_4 are linearly independent we can choose a new basis in $\mathrm{span}(X_1, X_2, X_3, X_4)$ of the form

$$Z_k = \begin{bmatrix} z_{1,1,k} & z_{1,2,k} & 0 \\ z_{2,1,k} & z_{2,2,k} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k = 1, 2, 3, \quad Z_4 = \mathbf{e}_3\mathbf{e}_3^\top. \tag{3.7}$$

The tensor $\mathcal{Z} = [z_{i,j,k}] \in \mathbb{C}^{3\times 3}$, whose three frontal sections are Z_1, Z_2, Z_3 is essentially $2 \times 2 \times 3$ tensor. Hence its border rank is at most 3. (Since Z_1, Z_2, Z_3 are linearly independent the rank of $\mathcal{Z}$ is 3.) Since rank $Z_4 = 1$ we deduce that the border rank of $\mathcal{X}$ is 4 at most. Therefore any $\mathcal{X}$ of the form (1.2) which satisfies $f(\mathcal{X}) = 0$ has border rank 4 at most.

Assume now that $f(\mathcal{X}) \neq 0$. Since the ten polynomials in (1.4) vanish on $\mathcal{X}$ it follows that $x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. In this case $\mathcal{X}$ is essentially a $2 \times 2 \times 4$ tensor. Hence its rank is 4 at most.

3.2 The case $L = \mathbf{e}_3\mathbf{e}_3^\top$, $R = \mathbf{e}_3\mathbf{e}_2^\top$

Let $X_1, X_2, X_3, X_4 \in \mathbb{C}^{3\times 3}$ be the four frontal sections of $\mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3\times 3\times 4}$. Assume that (2.1)–(2.2) hold. This means that our tensor $\mathcal{X} = [x_{i,j,k}] \in \mathbb{C}^{3\times 3\times 4}$ has the following zero entries $x_{1,3,k} = x_{3,1,k} = x_{3,2,k} = x_{3,3,k} = 0$ for $k = 1, 2, 3, 4$. So our tensor is essentially a $2 \times 3 \times 4$ and hence its border rank is 4 at most [4].

4 The defining polynomials of $V_4(4,4,4)$

In this section we state for reader's convenience the defining equations of $V_4(4,4,4)$. We briefly repeat the arguments in [4] by replacing the degree 16 polynomial conditions with the degree 6 polynomial conditions. Let $\mathcal{X} = [x_{i_1,i_2,i_3}] \in \mathbb{C}^{4\times 4\times 4}$. For each $l \in \{1, 2, 3\}$ we fix i_l while we let $i_p, i_q = 1, 2, 3, 4$ where $\{p, q\} = \{1, 2, 3\} \setminus \{l\}$.

In this way we obtain four l-sections $X_{1,l}, \ldots, X_{4,l} \in \mathbb{C}^{4 \times 4}$. (Note that $X_{k,3} = [x_{i,j,k}]_{i=j=1}^4, k = 1, 2, 3, 4$ are the four frontal sections of $\mathcal{X}$.) Denote by $\mathbf{X}_l = \mathrm{span}(X_{1,l}, \ldots, X_{4,l}) \subset \mathbb{C}^{4 \times 4}$ the l-section subspace corresponding to $\mathcal{X}$. For each $l \in \{1, 2, 3\}$ we define the following linear subspaces of polynomials of degrees $5, 6, 9$ respectively in the entries of $\mathcal{X}$. The defining polynomials could be any basis in each these linear subspaces.

We first describe the Strassen commutative conditions. (These conditions where rediscovered independently in [1].) Take $U_1, U_2, U_3 \in \mathbf{X}_l$. View $U_i = \sum_{j=1}^4 u_{j,i} X_{j,l}$ for $i = 1, 2, 3$. So the entries of each $X_{j,l}$ are fixed scalars and $u_{j,i}, i = 1, 2, 3, j = 1, 2, 3, 4$ are viewed as variables. Let adj U_2 be the adjoint matrix of U_2. Then the Strassen commutative conditions are

$$U_1(\mathrm{adj}\ U_2)U_3 - U_3(\mathrm{adj}\ U_2)U_1 = 0$$

Since the values of $u_{j,i}, i = 1, 2, 3, j = 1, 2, 3, 4$ are arbitrary, we regroup the above condition for each entry as a polynomial in $u_{j,i}$. The coefficient of each monomial in $u_{j,i}$ variables is a polynomial of degree 5 in the entries of $\mathcal{X}$ and must be equal to zero. The set of all such polynomial of degree 5 span a linear subspace, and we can choose any basis in this subspace.

The degree 6 and 9 polynomial conditions are obtained in a a slightly different way. Let $P = [p_{ij}], Q = [q_{ij}] \in \mathbb{C}^{4 \times 4}$ be matrices with entries viewed as variables. View $P X_{k,l} Q, k = 1, 2, 3, 4$ as the four frontal section of $4 \times 4 \times 4$ tensor $\mathcal{X}(P, Q, l) = [x_{i,j,k}(P, Q, l)]_{i,j,k=1}^4$.

Let $\mathcal{Y} = [x_{i,j,k}(P, Q, l)]_{i,j,k=1}^{3,3,4}$. Now $\mathcal{Y}$ must satisfy the 6 degree polynomial conditions of Landsberg-Manivel and the 9 degree symmetrization conditions. Since the entries of P, Q are variables, this means that the coefficients of the monomials in the variables $v_{ij}, w_{ij}, i, j = 1, 2, 3, 4$ must vanish identically. This procedure gives rise to a finite number of polynomial conditions of degree 6 and 9 respectively. Again choose a finite number of linear independent conditions of degree 6 and 9 respectively.

The zero set of the above polynomials of degrees $5, 6, 9$ defines $V_4(4, 4, 4)$.

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