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ON SPACES OF MATRICES CONTAINING A NONZERO MATRIX OF BOUNDED RANK

Dmitry Falikman, Shmuel Friedland and Raphael Loewy

Let $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$ be the spaces of $n \times n$ real matrices and real symmetric matrices respectively. We continue to study $d(n, n-2, \mathbb{R})$: the minimal number ℓ such that every ℓ-dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix of rank $n-2$ or less. We show that $d(4, 2, \mathbb{R}) = 5$ and obtain some upper bounds and monotonicity properties of $d(n, n-2, \mathbb{R})$. We give upper bounds for the dimensions of $n-1$ subspaces (subspaces where every nonzero matrix has rank $n-1$) of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$, which are sharp in many cases. We study the subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$ where each nonzero matrix has rank n or $n-1$. For a fixed integer $q > 1$ we find an infinite sequence of n such that any $\binom{q+1}{2}$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least q.

1. Introduction.

Let $\mathbb{F}$ be a field, $M_{m,n}(\mathbb{F})$ the space of all $m \times n$ matrices over $\mathbb{F}$ and $S_n(\mathbb{F})$ the space of all $n \times n$ symmetric matrices over $\mathbb{F}$. We write $M_n(\mathbb{F})$ for $M_{n,n}(\mathbb{F})$. Let V be either $M_{m,n}(\mathbb{F})$ or $S_n(\mathbb{F})$, and let k be a positive integer. In the last 80 years there has been interest in the following types of subspaces, all related to the *rank* function:

(a) A subspace of V where each matrix has rank bounded above by k.
(b) A subspace of V where each nonzero matrix has rank k. A subspace of this type is said to be a *k-subspace*.
(c) A subspace of V where each nonzero matrix has rank bounded below by k.

See the works [**13**], [**11**], [**2**], [**8**], [**5**], [**3**], [**4**], [**7**] and many others. Roughly speaking these problems are divided into two classes depending on whether $\mathbb{F}$ is algebraically closed or not. The classical case, which goes back to Radon-Hurwitz, discusses the maximal dimension $\rho(n)$ of an n-subspace U of $M_n(\mathbb{R})$ where each $0 \neq A \in U$ is an orthogonal matrix times $r \in \mathbb{R}^*$. Write $n = (2a+1)2^{c+4d}$, where a and d are nonnegative integers, and $c \in$

$\{0, 1, 2, 3\}$. Then the Radon-Hurwitz number $\rho(n)$ is defined by

$$\rho(n) = 2^c + 8d. \tag{1.1}$$

In his famous work Adams [1] gave a nonlinear version of the Radon-Hurwitz number by showing that $\rho(n) - 1$ is the maximal number of linearly independent vector fields on the $n-1$ dimensional sphere S^{n-1}. ($S^{n-1} \subset \mathbb{R}^n$ denotes the Eucledian sphere of radius one centered at the origin.) From this result Adams deduced that the maximal dimension of an n-subspace of $M_n(\mathbb{R})$ is exactly $\rho(n)$. Let $\rho(x) = 0$ if x is not a positive integer. Define now

$$\rho_s(n) = \rho\left(\frac{n}{2}\right) + 1. \tag{1.2}$$

Adams, Lax and Phillips [2] *showed that the maximal dimension of an n-subspace of $S_n(\mathbb{R})$ is exactly $\rho_s(n)$.* Friedland, Robbin and Sylvester [8] and Berger and Friedland [5] gave further nonlinear versions of the above results by considering odd maps ϕ from S^p to matrices of rank n in $M_n(\mathbb{R}), S_n(\mathbb{R})$ and $M_{n,n+1}(\mathbb{R})$ respectively. In this paper (§4) we generalize these results to odd maps from S^p to rank $n-1$ matrices in $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$.

The main motivation of this paper is the following quantity studied in [7]. For an integer k, such that $1 \le k \le n-1$, let $d(n, k, \mathbb{F})$ be the smallest integer ℓ such that every ℓ dimensional subspace of $S_n(\mathbb{F})$ contains a nonzero matrix whose rank is at most k. Note that it is clear that the maximal dimension of a subspace of $S_n(\mathbb{F})$ of type (c) above is exactly $d(n, k-1, \mathbb{F}) - 1$. It is our purpose to continue the study of $d(n, k, \mathbb{R})$ started in [7]. Note that any $d(n, k, \mathbb{R}) - 1$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least $n-k$. (One of the main results of [8] was that any subspace of $S_n(\mathbb{R})$ of dimension $\sigma(n) + 1$, where

$$\sigma(n) = 2 \text{ if } n \not\equiv 0, \pm 1 \pmod 8,$$
$$\sigma(n) = \rho(4b) \text{ if } n = 8b, 8b \pm 1,$$

contains a nonzero matrix with a multiple eigenvalue.) The equality

$$d(n, k, \mathbb{C}) = \binom{n-k+1}{2} + 1, \tag{1.3}$$

established in [7] is derived straightforwardly from the following dimension computations. Let $V_{k,n}(\mathbb{C})$ ($V_{k,n}(\mathbb{R})$) be the variety of all matrices in $S_n(\mathbb{C})$ ($S_n(\mathbb{R})$) of rank k or less. Then in the projective space $\mathbb{P}S_n(\mathbb{C})$ the projective variety $\mathbb{P}V_{k,n}(\mathbb{C})$ is an irreducible variety of codimension $d(n, k, \mathbb{C}) - 1$, which yields (1.3). In particular $d(n, n-1, \mathbb{C}) = 2$. The results of Adams,Lax and Phillips, cited above, yield that $d(n, n-1, \mathbb{R}) = \rho_s(n) + 1$. This shows that in general the computation of $d(n, k, \mathbb{R})$ is much more difficult than the computation of $d(n, k, \mathbb{C})$. In [7] we gave a simple condition on n when $d(n, k, \mathbb{C}) = d(n, k, \mathbb{R})$ for $k \le n-2$, which trivially holds for

$k = n-1$. It was shown by Harris and Tu [10] that the degree of the variety $\mathbb{P}V_{k,n}(\mathbb{C})$ is given by the formula

$$\delta_{k,n} := \deg \mathbb{P}V_{k,n}(\mathbb{C}) = \prod_{j=0}^{n-k-1} \frac{\binom{n+j}{n-k-j}}{\binom{2j+1}{j}}. \tag{1.4}$$

Then $d(n,k,\mathbb{C}) = d(n,k,\mathbb{R})$ if $\delta_{k,n}$ is odd. (In this case the result that any $d(n,k,\mathbb{R})-1$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least $n-k$ is best possible.) We show that $\delta_{n-q,n}$ is odd if $n > q \geq 1$ and

$$n \equiv \pm q \pmod{2^{\lceil \log_2 2q \rceil}}. \tag{1.5}$$

In particular, under the above conditions,

$$d(n, n-q, \mathbb{C}) = d(n, n-q, \mathbb{R}) = \binom{q+1}{2} + 1. \tag{1.6}$$

If $n \geq q$ and n satisfies (1.5) then any $\binom{q+1}{2}$ subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity at least q. This statement for $q = 2$ yields the original Lax's result [12] that any 3 dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix with a multiple eigenvalue for $n \equiv 2 \pmod 4$. (This result and its generalization in [8] is of importance in the study of singularities of hyperbolic systems.)

An important part of the paper is devoted to the study of the numbers $d(n, n-2, \mathbb{R})$. Besides the cases given in (1.5-1.6) for $q = 2$ it is easy to see that $d(3, 1, \mathbb{R}) = 6$ [7]. (Hence the inequality $d(n, n-2, \mathbb{R}) \geq \sigma(n) + 2$ established in [7] is not sharp for some n.) Partial results regarding $d(n, n-2, \mathbb{R})$ were obtained in [7]. In particular, it was shown that for $m \geq 1$

$$d(4m, 4m-2, \mathbb{R}) \leq 4m + 1. \tag{1.7}$$

We obtain here additional results on $d(n, n-2, \mathbb{R})$. In particular, using numerical and symbolic computations we show that $d(4, 2, \mathbb{R}) = 5$, which implies that the upper bound in (1.7) is sharp for $m = 1$.

The paper is organized as follows. In Section 2 we show that $d(4, 2, \mathbb{R}) = 5$. In Section 3 we obtain some upper bounds and some monotonicity properties of $d(n, n-2, \mathbb{R})$. In Section 4 we consider the existence of continuous and smooth odd maps from the sphere S^p to matrices of rank $n-1$ in $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. As a consequence we obtain new upper bounds for the dimension of $(n-1)$-subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. These upper bounds are shown to be sharp in many cases. In Section 5 we discuss the existence of subspaces $U \subset S_{2n-1}(\mathbb{R})$ of dimension $2n$, which do not contain a nonzero matrix of rank less than $2n-2$. The last section is devoted to remarks and conjectures.

2. Computation of $d(4,2,\mathbb{R})$.

As indicated in the introduction, the first unknown number in the sequence $\{d(n,n-2,\mathbb{R})\}_{n=3}^{\infty}$ is $d(4,2,\mathbb{R})$. By (1.7) $d(4,2,\mathbb{R}) \le 5$. It is our purpose to prove:

Theorem 2.1. $d(4,2,\mathbb{R}) = 5$.

Proof. It suffices to exhibit a 4-dimensional subspace L of $S_4(\mathbb{R})$ with the property that every $0 \ne A \in L$ has rank 3 or 4.

Let L be the subspace of $S_4(\mathbb{R})$ spanned by the matrices

$$B_1 = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & -1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix}, \qquad B_4 = \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

It is straightforward to check that dim $L = 4$. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be indeterminates and let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, and $B(\alpha) = \sum_{i=1}^{4} \alpha_i B_i$.

We show that if $0 \ne \alpha \in \mathbb{R}^4$ then rank $B(\alpha) \ge 3$. So let $d(\alpha) = \det B(\alpha)$, and for $1 \le j \le 4$ let $d_j(\alpha)$ denote the determinant of the principal submatrix of $B(\alpha)$ obtained by deleting row and column j. We let Maple compute the common zeros of the 5 polynomials $d(\alpha), d_1(\alpha), d_2(\alpha), d_3(\alpha), d_4(\alpha)$ in $\mathbb{C}^4$.

It turns out that there are no common zeros with $\alpha_1 = 0$ except the trivial solution $\alpha = 0$. Now we can assume $\alpha_1 = 1$. We list the ten solutions found by Maple. They are:

(i) The two solutions given by

$$\alpha = (1, 0, 1 + x, x),$$

where x is a zero of the polynomial $f(z) = z^2 + z + 1$.

(ii) The two solutions given by

$$\alpha = (1, x, 2x - 1, 3x - 2),$$

where x is a zero of the polynomial $f(z) = 7z^2 - 8z + 3$.

(iii) The two solutions given by

$$\alpha = (1, x, -x, -5x - 1),$$

where x is a zero of the polynomial $f(z) = 14z^2 + 4z + 1$.

(iv) The four solutions given by

$$\alpha = \left(1, x, \frac{1}{2}x^2 + x + \frac{1}{2}, -\frac{1}{2}x^2 - x - \frac{1}{2}\right),$$

where x is a zero of the polynomial
$f(z) = z^4 + 4z^3 + 8z^2 + 4z + 3 = (z+1)^4 + 2z^2 + 2$.

It follows that for any $0 \neq \alpha \in \mathbb{R}^4$ either $d(\alpha) \neq 0$ or $\exists\ 1 \leq j \leq 4$ such that $d_j(\alpha) \neq 0$. Thus, the rank of every nonzero matrix in L is at least 3.

We explain now why Maple did indeed give us *all* the common zeros of $d(\alpha), d_1(\alpha), d_2(\alpha), d_3(\alpha), d_4(\alpha)$. Let L_1 be the subspace of $S_4(\mathbb{C})$ spanned by B_1, B_2, B_3, B_4. We have by (1.3)

$$d(4, 2, \mathbb{C}) = 4.$$

Moreover, by (1.4) we get

$$\deg \mathbb{P}(V_{2,4}(\mathbb{C})) = \frac{\binom{4}{2}\binom{5}{1}}{3} = 10. \tag{2.1}$$

We also use the known fact that a rank r symmetric matrix with entries in a field has a nonsingular $r \times r$ principal submatrix. Thus, the ten distinct solutions found by Maple yield ten matrices in L_1 whose rank is at most 2, and no two of those matrices lie on the same line. *In fact, a computation showed that each of those ten matrices has rank* 2.

It remains to explain why Maple did not omit other solutions, assuring us it did not miss any real solutions in particular. It follows from (2.1) that if L' is a four dimensional generic subspace of $S_4(\mathbb{C})$ then $\mathbb{P}(V_{2,4}(\mathbb{C}))$ meets $\mathbb{P}(L')$ in at most ten distinct points. Hence, we have to check that L_1 is indeed generic. It suffices to show that $\mathbb{P}(V_{2,4}(\mathbb{C}))$ and $\mathbb{P}(L_1)$ intersect transversally. So let A denote any of the ten matrices in L_1 of rank 2 found by Maple. Let $x, y \in \mathbb{C}^4$ be linearly independent vectors such that $Ax = Ay = 0$. A computation showed that the rank of

$$\begin{bmatrix} x^tB_2x & x^tB_3x & x^tB_4x \\ y^tB_2y & y^tB_3y & y^tB_4y \\ x^tB_2y & x^tB_3y & x^tB_4y \end{bmatrix} \in M_3(\mathbb{C})$$

is 3. This implies that the linear system of three equations

$$x^tBx = 0$$
$$y^tBy = 0$$
$$x^tBy = 0,$$

has no solution B in L_1 such that A, B are linearly independent. This shows that L_1 is generic.

In addition to the computation using Maple, a numerical procedure (using Matlab) was performed to check that there are no nonzero matrices of rank 1 or 2 in L.

3. Upper bounds and monotonicity properties of $d(n, n-2, \mathbb{R})$.

It follows from (1.5) and (1.6) that if $n \equiv 2(\text{mod } 4)$ then $d(n, n-2, \mathbb{R}) = d(n, n-2, \mathbb{C}) = 4$. It was shown in [7] that

$$d(n, n-2, \mathbb{R}) \le 7 \quad \text{for } n \equiv 4(\text{mod } 8). \tag{3.1}$$

It is our purpose here to get some additional upper bounds and some monotonicity properties of $d(n, n-2, \mathbb{R})$.

Proposition 3.1. *Let $n \equiv 3, 5(\text{mod } 8)$. Then $d(n, n-2, \mathbb{R}) \le 7$.*

Proof. It follows from (1.4) that for every $n \ge 3$

$$\deg \mathbb{P}(V_{n-3,n}(\mathbb{C})) = \frac{\binom{n}{3}\binom{n+1}{2}\binom{n+2}{1}}{1 \cdot 3 \cdot 10} = \frac{(n+2)(n+1)n^2(n-1)(n-2)}{2^3 \cdot 3^2 \cdot 5},$$

and this is odd for $n \equiv 3, 5(\text{mod } 8)$. So, as indicated in the introduction, and using (1.3) we have for $n \equiv 3, 5(\text{mod } 8)$

$$d(n, n-3, \mathbb{R}) = d(n, n-3, \mathbb{C}) = \binom{4}{2} + 1 = 7,$$

and since $d(n, n-2, \mathbb{R}) \le d(n, n-3, \mathbb{R})$, the proposition follows. $\square$

It follows from [7] that $d(n, n-2, \mathbb{R}) \ge 4$ for $n \equiv 3, 4, 5(\text{mod } 8)$. Thus, we have the exact value for $d(n, n-2, \mathbb{R})$ whenever $n \equiv 2, 6(\text{mod } 8)$, and good upper and lower bounds whenever $n \equiv 3, 4, 5(\text{mod } 8)$.

Proposition 3.2. *Let k be a fixed integer such that $k \in \{3, 4, 5, 11, 12, 13\}$. Then the sequence $\{d(16m+k, 16m+k-2, \mathbb{R})\}_{m=0}^{\infty}$ is a (weakly) monotone increasing sequence, bounded above by 7.*

There exists $M = M(k)$ such that

$$d(16m+k, 16m+k-2, \mathbb{R}) = d(16M+k, 16M+k-2, \mathbb{R}) \quad \text{for all } m \ge M.$$

Proof. It suffices to prove the monotonicity of the given sequence, because the required boundedness follows from (3.1) and Proposition 3.1.

Let L be any 8-dimensional 8-subspace of $M_8(\mathbb{R})$. Such a subspace exists because $\rho(8) = 8$ by (1.1). Let

$$L_1 = \left\{ \begin{bmatrix} 0 & A \\ A^t & 0 \end{bmatrix}, \ A \in L \right\}.$$

Then L_1 is an 8-dimensional subspace of $S_{16}(\mathbb{R})$. Note that every nonzero matrix in L_1 is nonsingular. Let $\{B_i\}_{i=1}^8$ be a basis of L_1.

Let $d_{m,k} = d(16m+k, 16m+k-2, \mathbb{R})$ and $d_0 = d_{m,k} - 1$. Let $W_{m,k}$ be any d_0-dimensional subspace of $S_{16m+k}(\mathbb{R})$ such that the rank of any nonzero matrix in $W_{m,k}$ is at least $16m+k-1$. Note that $d_0 \le 7-1=6$. Let $\{C_i\}_{i=1}^{d_0}$ be a basis of $W_{m,k}$. Let $\widetilde{W}_{m,k} = \mathrm{span}\{C_i \oplus B_i : 1 \le i \le d_o\}$. Then $\widetilde{W}_{m,k}$ is a d_0-dimensional subspace of $S_{16(m+1)+k}(\mathbb{R})$, and clearly every nonzero matrix in it has rank $\ge 16+16m+k-1 = 16(m+1)+k-1$. This shows that

$$\begin{aligned} &d(16(m+1)+k, 16(m+1)+k-2, \mathbb{R}) \ge d_0+1 \\ &= d_{m,k} = d(16m+k, 16m+k-2, \mathbb{R}). \end{aligned}$$

□

The following proposition justifies some claims we made in the introduction.

Proposition 3.3. *Let $1 \le k \le n-2$. Then any $d(n,k,\mathbb{R})-1$ dimensional subspace of $S_n(\mathbb{R})$ contains a nonzero matrix which has an eigenvalue of multiplicity at least $n-k$. If $d(n,k,\mathbb{R}) = d(n,k,\mathbb{C})$ then this result is sharp.*

Proof. Let V be a $d(n,k,\mathbb{R})-1$ dimensional subspace of $S_n(\mathbb{R})$. If $I_n \in V$ then 1 is the eigenvalue of I_n of multiplicity $n \ge n-k$. Suppose that $I_n \notin V$. Let $\hat{V} = \mathrm{span}\{V, I_n\}$. Then $\hat{V}$ contains $0 \ne \hat{A}$ such that rank $\hat{A} \le k$. $\hat{A} = A + aI, 0 \ne A \in V$, so $-a$ is an eigenvalue of A of multiplicity at least $n-k$.

Let $\Sigma_{n-k} \subset S_n(\mathbb{R})$ be the variety of all $A \in S_n(\mathbb{R})$ such that A has an eigenvalue of multiplicity at least $n-k$. Clearly

$$\Sigma_{n-k} = \{B \in S_n(\mathbb{R}) : B = A + aI_n \quad \text{for some } A \in V_{k,n}(\mathbb{R}) \text{ and } a \in \mathbb{R}\}.$$

Since $V_{k,n}(\mathbb{R})$ is an irreducible variety of codimension $d(n,k,\mathbb{C})-1$ it follows that Σ_{n-k} is an irreducible variety of codimension $d(n,k,\mathbb{C})-2$. Hence there exists an $d(n,k,\mathbb{C})-2$ dimensional subspace $W \subset S_n(\mathbb{R})$ such that $\Sigma_{n-k} \cap W = \{0\}$.

4. Upper bounds for the dimension of $(n-1)$-subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$.

In this section we obtain upper bounds for the dimension of $(n-1)$-subspaces of $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. This is done by considering certain odd, smooth maps from S^p into $M_n(\mathbb{R})$ and $S_n(\mathbb{R})$. We also need several known results.

Let $M_n^{(k)}(\mathbb{R})$ $\left(S_n^{(k)}(\mathbb{R})\right)$ denote the set of all rank k matrices in $M_n(\mathbb{R})(S_n(\mathbb{R}))$. Given any field $\mathbb{F}$ and positive integers m, n, let $M_{m,n}^0(\mathbb{F})$ denote the set of all matrices in $M_{m,n}(\mathbb{F})$ whose rank is equal to $\min\{m,n\}$.

Lemma 4.1. *Let $C \in M_{n,n+1}^0(\mathbb{F})$. For $j = 1, 2, \ldots, n+1$, let $C^{(j)}$ denote the $n \times n$ matrix obtained from C by deleting its j-th column. Then the* *solution of the homogeneous linear system $Cy = 0$ is a line spanned by*

$$y = \left(-\det C^{(1)}, \det C^{(2)}, -\det C^{(3)}, \ldots, (-1)^{n+1}\det C^{(n+1)}\right)^t . \tag{4.1}$$

Proof. This is an easy consequence of well-known properties of the determinant. □

The next theorem appears as a part of Theorems A and F in [8]. See there how it is related to classical results due to Radon, Hurwitz, Adams and Adams-Lax-Phillips.

Theorem 4.1. *Let $\varphi : S^p \to M_n(\mathbb{R})$ be an odd continuous map, i.e., $\varphi(-\alpha) = -\varphi(\alpha) \quad \forall \alpha \in S^p$.*

(i) *Suppose that $\varphi(S^p) \subset M_n^{(n)}(\mathbb{R}) = GL(n, \mathbb{R})$. Then, $p \leq \rho(n) - 1$.*
(ii) *Suppose that $\varphi(S^p) \subset S_n^{(n)}(\mathbb{R})$. Then, we have $p = 0$ if n is odd, and $p \leq \rho(\frac{n}{2})$ if n is even.*

All the inequalities in (i) *and* (ii) *are sharp.*

The next theorem appears in [5].

Theorem 4.2. *Let $\varphi : S^p \to M^0_{n,n+1}(\mathbb{R})$ be an odd continuous map. Then $p \leq \max\{\rho(n) - 1, \rho(n+1) - 1\}$, and this inequality is sharp.*

Theorem 4.3. *Let $n \geq 2$. Let $\varphi : S^p \to M_n^{(n-1)}(\mathbb{R})$ be n odd smooth map. Then, if $n \neq 3, 5, 9$*

$$p \leq \max\{\rho(n-1) - 1, \rho(n) - 1, \rho(n+1) - 1, 2\}. \tag{4.2}$$

Furthermore, if $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$ then $p = 0$ if n is even, and

$$p \leq \max\left\{\rho\left(\frac{n-1}{2}\right), \rho(n+1) - 1\right\} \quad \text{if } n \text{ is odd.} \tag{4.3}$$

Proof. We prove first (4.2). If $p \leq 1$ (4.2) certainly holds, so we may assume $p \geq 2$. Hence S^p is simply connected. Therefore, we can choose $x(\alpha) \in S^{n-1}$ in a well-defined and continuous way such that

$$x^t(\alpha)\varphi(\alpha) = 0 \quad \forall \alpha \in S^p. \tag{4.4}$$

For $\alpha \in S^p$ let $B(\alpha)$ be the matrix in $M_{n,n+1}(\mathbb{R})$ obtained from $\varphi(\alpha)$ by augmenting it with the column vector $x(\alpha)$, i.e.,

$$B(\alpha) = (\varphi(\alpha), x(\alpha)) \in M_{n,n+1}(\mathbb{R}).$$

Then rank $\varphi(\alpha) = n - 1$ and (4.4) imply that $B(\alpha) \in M^0_{n,n+1}(\mathbb{R})$ for all $\alpha \in S^p$.

Note that the smoothness of φ implies that $x(\alpha)$ is also smooth. It is also clear that for each $\alpha \in S^p$ $x(-\alpha) = \pm x(\alpha)$. Hence, we must have $x(-\alpha) = -x(\alpha)$ for all $\alpha \in S^p$, or $x(-\alpha) = x(\alpha)$ for all $\alpha \in S^p$.

Case 1. Suppose that $x(-\alpha) = -x(\alpha)$ for all $\alpha \in S^p$. Then the map from S^p to $M^0_{n,n+1}(\mathbb{R})$ defined by $\alpha \to B(\alpha)$ is an odd continuous map, so by Theorem 4.2 we have

$$p \leq \max\{\rho(n) - 1, \rho(n+1) - 1\}, \tag{4.5}$$

and (4.2) holds.

Case 2. We now assume $x(-\alpha) = x(\alpha)$ for all $\alpha \in S^p$. We apply Lemma 4.1 to $B(\alpha) \in M^0_{n,n+1}(\mathbb{R})$, for each $\alpha \in S^p$, and denote the normalized solution given by (4.1) $\left(\text{that is, } \frac{1}{\|y\|}y\right)$ by $\psi(\alpha)$. Let

$$\psi(\alpha) = (\psi_1(\alpha), \psi_2(\alpha), \ldots, \psi_n(\alpha), \psi_{n+1}(\alpha)).$$

There are two subcases now.

Case 2a. Suppose that n is even. It follows from (4.1) that

$$\psi(-\alpha) = (-\psi_1(\alpha), -\psi_2(\alpha), \ldots, -\psi_n(\alpha), \psi_{n+1}(\alpha)).$$

We define

$$C(\alpha) = \begin{bmatrix} B(\alpha) \\ \psi(\alpha) \end{bmatrix} \in M_{n+1}(\mathbb{R}).$$

This matrix is nonsingular because $\psi(\alpha)$ is orthogonal to all the rows of $B(\alpha)$. Let $\eta(\alpha) = (\psi_1(\alpha), \psi_2(\alpha), \ldots, \psi_n(\alpha))$ and let

$$D(\alpha) = \begin{bmatrix} \varphi(\alpha) \\ \eta(\alpha) \end{bmatrix} \in M_{n+1,n}(\mathbb{R}),$$

that is, $D(\alpha)$ is the matrix obtained from $C(\alpha)$ by deleting its last column. Its rank is n for each $\alpha \in S^p$. Since $D(-\alpha) = -D(\alpha)$ for all $\alpha \in S^p$ we apply Theorem 4.2 again and conclude that (4.5) holds, so (4.2) holds.

Case 2b. Suppose that n is odd. So $n \geq 3$. Suppose first that $p < n - 1$. Since $x(\alpha)$ is smooth, it follows from Sard's theorem that $\{x(\alpha) : \alpha \in S^p\}$ has measure zero in S^{n-1}. In particular, $\exists\, \xi \in S^{n-1}$ such that $x(\alpha) \neq \pm\xi$ for all $\alpha \in S^p$.

For every $\alpha \in S^p$ let $L_\alpha = \operatorname{span}\{x(\alpha), \xi\}$. Let $Q(\alpha)$ be the unique $n \times n$ orthogonal matrix satisfying: $Q(\alpha)$ is the identity on $L^\perp_\alpha$ and its restriction to L_α is the rotation in that plane by an angle $< \pi$ that sends $x(\alpha)$ to ξ. Since $x(-\alpha) = x(\alpha)$ we have $Q(-\alpha) = Q(\alpha)$. Also, $Q(\alpha)$ is continuous in α (cf. the Proposition in [**8**]). It follows from (4.4) that

$$\xi^t(Q^t(\alpha)\varphi(\alpha)Q(\alpha)) = 0, \quad \text{for all } \alpha \in S^p. \tag{4.6}$$

Let $\varphi_1(\alpha) = Q^t(\alpha)\varphi(\alpha)Q(\alpha)$. Then $\varphi_1(\alpha)$ is an odd continuous function. Without loss of generality we may assume $\xi^t = (0, 0, \ldots, 0, 1)$. It follows from (4.6) that for each $\alpha \in S^p$ the last row of $\varphi_1(\alpha)$ is 0, so deleting it results in an $(n-1)\times n$ matrix of rank $n-1$. Applying Theorem 4.2 again we conclude that

$$p \le \max\{\rho(n-1)-1, \rho(n)-1\}$$

so (4.2) holds.

Now suppose that $p \ge n-1$. We may consider S^{n-2} as contained in S^p, and then consider the restriction of the given $\varphi(\alpha)$ and $x(\alpha)$ to S^{n-2}. We repeat the proof given for $p < n-1$ and conclude in the same way that

$$n-2 \le \max\{\rho(n-1)-1, \rho(n)-1\} = \rho(n-1)-1.$$

Since this cannot happen by (1.1) when $n \ne 3, 5, 9$ we have completed the proof of (4.2).

Suppose now that $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$. Recall that the *inertia* of $A \in S_n(\mathbb{R})$ is the triple (π, ν, δ), where π is the number of positive eigenvalues of A, ν is the number of negative eigenvalues of A and δ is the number of zero eigenvalues of A.

Let n be even and suppose that $p > 0$. Let $\alpha \in S^p$, and consider a path in S^p from α to $-\alpha$. Let $\mathcal{J}$ denote the image of this path under φ. Since every matrix in $\mathcal{J}$ has rank $n-1$, it follows that the inertia of each matrix on this path is equal to inertia $(\varphi(\alpha))$. In particular, inertia $(\varphi(-\alpha)) =$ inertia$(\varphi(\alpha))$, but this is impossible since $\varphi(-\alpha) = -\varphi(\alpha)$. Hence $p = 0$.

Let n be odd. Suppose first that $n = 3$. We show that $p \le 1$ in this case, so (4.3) holds. Suppose to the contrary that $p \ge 2$. It is clear that $\varphi(\alpha)$ has the same inertia for each $\alpha \in S^p$. In particular, given any $\alpha \in S^p$, $\varphi(\alpha)$ and $\varphi(-\alpha) = -\varphi(\alpha)$ have the same inertia. Hence each $\varphi(\alpha)$ has one positive, one negative and one zero eigenvalue. So for each $\alpha \in S^p$, the eigenvalues of $\varphi(\alpha)$ are pairwise distinct, contradicting Theorem B of [**8**].

Note that while we have shown that there is no odd continuous map from S^2 to $S_3^{(2)}(\mathbb{R})$, there is an odd continuous map from S^2 to $M_3^{(2)}(\mathbb{R})$. For example, consider the map that sends $(x, y, z) \in S^2$ to

$$\begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

We assume now $n \ge 5$. Since $\max\{\rho(\frac{n-1}{2}), \rho(n+1)-1\} \ge 2$, (4.3) holds if $p \le 1$. Hence we may assume that $p \ge 2$. We go along the proof of the nonsymmetric part of the theorem. If $x(\alpha)$ is odd, then (4.5) holds, and since $\rho(n) = 1$ for odd n, we get

$$p \le \rho(n+1) - 1 \le \max\left\{\rho\left(\frac{n-1}{2}\right), \rho(n+1)-1\right\},$$

and (4.3) holds.

So we may assume that $x(\alpha)$ is even. Suppose also that $p < n-1$. Then we get (4.6) again, where $\varphi_1(\alpha) = Q^t(\alpha)\varphi(\alpha)Q(\alpha)$ is an odd continuous function, and we may assume $\xi^t = (0, 0, \ldots, 0, 1)$. It follows that for each $\alpha \in S^p$ the matrix obtained from $\varphi_1(\alpha)$ by deleting its last row and column is in $S_{n-1}^{(n-1)}(\mathbb{R})$. Hence, by part (ii) of Theorem 4.1 we get

$$p \leq \rho\left(\frac{n-1}{2}\right),$$

so (4.3) holds.

If $p \geq n-1$ we consider S^{n-2} as contained in S^p, and then consider the restriction of $\varphi(\alpha), x(\alpha)$ to S^{n-2}. A repetition of the argument for $p < n-1$ yields $n-2 \leq \rho(\frac{n-1}{2})$, which is impossible for n odd, $n \geq 5$. This completes the proof. □

Corollary 4.1. *Let $n \geq 2$ and let L be an $(n-1)$-subspace of $M_n(\mathbb{R})$.*

(a) *If $n \neq 3, 5, 9$, then*

$$\dim L \leq \max\{\rho(n-1), \rho(n), \rho(n+1), 3\}. \tag{4.7}$$

(b) *If $L \subset S_n(\mathbb{R})$ (that is, L is a subspace of $S_n(\mathbb{R})$), then*

$$\dim L \leq \begin{cases} 1 & \text{if } n \text{ is even}, \\ \max\left\{\rho\left(\frac{n-1}{2}\right)+1, \rho(n+1)\right\} & \text{if } n \text{ is odd}. \end{cases} \tag{4.8}$$

Proof. (a) Let $d = \dim L$ and let $A_1, A_2, \ldots, A_d$ be a basis of V. For $A \in V$ we write $A = \sum_{i=1}^d \alpha_i A_i$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$. Then we may view the map $\alpha \to A$, restricted to α such that $\|\alpha\| = 1$, as an odd smooth map from S^{d-1} to $M_n^{(n-1)}(\mathbb{R})$. Then $d-1$ is bounded by the right-hand side of (4.2), so (4.7) follows.

(b) The proof is similar to the previous case, using now (4.3) instead of (4.2). □

Consider next the sharpness of the inequalities (4.2), (4.3), (4.7) and (4.8). For that purpose we have the following lemma.

Lemma 4.2.

(i) *There exists a $\rho(n-1)$ dimensional $(n-1)$-subspace V of $M_n(\mathbb{R})$.*

(ii) *There exists an $(n-1)$-subspace V of $S_n(\mathbb{R})$ such that*

$$\dim V = \begin{cases} 1 & \text{if } n \text{ is even}, \\ \rho\left(\frac{n-1}{2}\right)+1, & \text{if } n \text{ is odd}. \end{cases}$$

Proof. (i) As indicated in the introduction, there exists an $\rho(n-1)$ dimensional $(n-1)$-subspace V_1 of $M_{n-1}(\mathbb{R})$. Then we let V be the subspace of $M_n(\mathbb{R})$ obtained from V_1 by appending a row and column of 0's to each matrix in V_1.

(ii) The claim is trivial if n is even, so suppose n is odd. As indicated in the introduction, there exists an $(n-1)$-subspace V_1 of $S_{n-1}(\mathbb{R})$ of dimension $\rho\left(\frac{n-1}{2}\right)+1$. Now V is obtained from V_1 as in Part (i). $\square$

Corollary 4.2. *Let $n \equiv 1 \pmod 4$. Then the inequalities (4.3) and (4.8) are sharp, and the inequalities (4.2) and (4.7) are sharp provided that $n \neq 5, 9$.*

Proof. For $n \equiv 1 \pmod 4$ the maximum of the right-hand sides of (4.2) and (4.7) are $\rho(n-1)-1$ and $\rho(n-1)$, respectively. Let V be the $\rho(n-1)$ dimensional $(n-1)$-subspace defined in the Proof of Lemma 4.2, Part (i). This shows the sharpness of (4.7) provided that $n \neq 5, 9$. The sharpness of (4.2) is obtained by considering an odd smooth map from $S^{\rho(n-1)-1}$ to $M_n^{(n-1)}(\mathbb{R})$ as in the proof of Corollary 4.1 Part (a).

The sharpness of (4.3) and (4.8) is proved similarly, using the $\rho\left(\frac{n-1}{2}\right)+1$ dimensional $(n-1)$-subspace V of $S_n(\mathbb{R})$ in Part (ii) of Lemma 4.2. $\square$

Remark. In Theorem 4.3 it is possible to replace the assumption that φ is an odd smooth map by the assumption that φ is an odd continuous map.

The proof of the remark is achieved as follows. First approximate $\varphi(\alpha)$ arbitrarily by a smooth odd $\widetilde{\varphi}(\alpha)$ (which is in $S_n(\mathbb{R})$ if $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$). We can assume that for each $\alpha \in S^p$, $\widetilde{\varphi}(\alpha)$ has a simple eigenvalue $\lambda(\alpha)$ which is the smallest in absolute value among all eigenvalues of $\widetilde{\varphi}(\alpha)$ (and is also real if $\varphi(S^p) \subset S_n^{(n-1)}(\mathbb{R})$). Clearly $\lambda(-\alpha) = -\lambda(\alpha)\ \forall \alpha \in S^p$. Let $\psi(\alpha)$ be the projection of $\widetilde{\varphi}(\alpha)$ corresponding to $\lambda(\alpha)$. Then $\psi(\alpha)$ is a smooth odd function of rank at most 1, and it follows that $\widetilde{\varphi}(\alpha) - \psi(\alpha)$ satisfies the conditions of Theorem 4.3, so (4.2) and (4.3) hold respectively.

5. Existence of certain $2n$ dimensional subspaces of $S_{2n-1}(\mathbb{R})$.

Let $n \geq 4$ be an even integer. It follows from [7] and this paper that $d(n, n-2, \mathbb{R}) \leq n+1$, with equality for $n = 4$. Now suppose that $n \geq 5$ is an odd integer. We do not know if a similar result holds, although it seems plausible. In this section we show that if there exists an $n+1$ dimensional subspace L of $S_n(\mathbb{R})$ such that each nonzero matrix in L has rank $n-1$ or n, then there is a severe restriction on the possible inertias attained in L. We also derive here additional results on $n+1$ dimensional subspaces of $n \times n$ matrices where each nonzero matrix has rank $n-1$ or n.

Let $\mathbb{F}$ be any field and let V be an $n+1$ dimensional subspace of $M_n(\mathbb{F})$ spanned by $A_1, A_2, \ldots, A_{n+1}$. Sometimes we find it convenient to identify $\mathbb{PF}^n$ with $\mathbb{F}^{n+1}$ and $\mathbb{P}V$ with $\mathbb{PF}^n$. That is, we identify $\widetilde{\alpha} \in \mathbb{PF}^n$ and $\widetilde{A} \in \widetilde{V} := \mathbb{P}V$ with lines spanned by $0 \neq \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})^t$ and $0 \neq A$ in F^{n+1} and V respectively. Let $A(\alpha) = \sum_{i=1}^{n+1} \alpha_i A_i$. We identify $\widetilde{\alpha}$ with $\widetilde{A(\alpha)}$.

Given any $0 \neq y \in \mathbb{F}^n$, let

$$L_y = \{A \in V : Ay = 0\}. \tag{5.1}$$

Since $\dim V \geq n+1$ it is clear that $\dim L_y \geq 1$ for every $y \neq 0$. Let

$$M(y) = [A_1 y, A_2 y, \ldots, A_{n+1} y] \in M_{n,n+1}(\mathbb{F}). \tag{5.2}$$

For $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})^t$ it is clear that $\left(\sum_{i=1}^{n+1} \alpha_i A_i\right) y = 0$ if and only if $M(y)\alpha = 0$. It follows that

$$\dim L_y = 1 \text{ if and only if } \operatorname{rank} M(y) = n.$$

Observe that if rank $M(y) = n$ then, by Lemma 4.1, ker $M(y)$ is spanned by

$$\alpha(y) = \left(-\det M(y)^{(1)}, \det M(y)^{(2)}, \ldots, (-1)^{n+1} \det M(y)^{(n+1)}\right)^t \tag{5.3}$$

Let

$$T = \left\{\widetilde{y} \in \mathbb{PF}^{n-1} : \alpha(y) = 0\right\}. \tag{5.4}$$

Hence, T consists of those $\widetilde{y}$ for which rank $M(y) < n$.

Given $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ we consider the homogeneous polynomial

$$\psi(\alpha) = \det\left(\sum_{i=1}^{n+1} \alpha_i A_i\right),$$

and let

$$\widetilde{Z}(\psi) = \text{ the zero set of } \psi(\alpha), \text{ considered as a subset of } \mathbb{PF}^n. \tag{5.5}$$

Lemma 5.1. *Let V be an $n+1$ dimensional subspace of $M_n(\mathbb{C})$, spanned by $A_1, A_2, \ldots, A_{n+1}$, and such that:*

(i) *$\exists A \in V$ such that $\det A \neq 0$.*
(ii) *$\exists B \in V$ such that $\operatorname{rank} B = n-1$ and $\ker B$ is spanned by a vector $0 \neq u \in \mathbb{C}^n$ with $\dim L_u = 1$.*

Then the following holds:

Let $H \subset \mathbb{PC}^n$ be the irreducible component of the hypersurface $\widetilde{Z}(\psi)$ passing through $\widetilde{B}$. Then there exists a birational map

$$\varphi : H - - \to \mathbb{PC}^{n-1}, \quad \theta = \varphi^{-1} : \mathbb{PC}^{n-1} - - \to H$$

defined as follows:

(a) *For each $\widetilde{\alpha} \in H$ with $\operatorname{rank} C = n-1$, where $C = \sum_{i=1}^{n+1} \alpha_i A_i$, $\varphi(\widetilde{\alpha}) = \widetilde{y}$, where $y \in \mathbb{C}^n$ is the basis of $\ker C$.*
(b) *For each $\widetilde{y} \in \mathbb{PC}^{n-1}$ such that $\dim L_y = 1$, $\theta(\widetilde{y}) = \widetilde{\alpha}$, where $\alpha = \alpha(y)$.*

Proof. The maps φ and θ are rational. For y in the neighborhood of u, $\dim L_y = 1$. Hence θ is holomorphic in the neighborhood of $\widetilde{u}$. Let $B = \sum_{i=1}^{n+1} \beta_i A_i$. The assumptions (ii) imply that φ is holomorphic in the neighborhood of $\widetilde{\beta}$ and $\varphi \cdot \theta$ is the identity map (as a rational map). Hence

$$\theta : \mathbb{PC}^{n-1} - - \to H.$$

□

Remark. As $\theta : \mathbb{PC}^{n-1} - - \to H$ is a rational map, it is not holomorphic on a variety of codimension at least 2 (cf. [**9**, Chapter 4, Section 2]). Hence codim $T \geq 2$.

Lemma 5.2. *Let V be an $n+1$ dimensional subspace of $M_n(\mathbb{R})$, such that each nonzero matrix in V has* $\operatorname{rank} n-1$ *or n. Then:*

(i) $\exists A \in V$ *such that* $\det A \neq 0$.
(ii) $\exists B \in V$ *such that* $\operatorname{rank} B = n-1$ *and* $\ker B$ *is spanned by a vector* $0 \neq y \in \mathbb{R}^n$ *with* $\dim L_y = 1$.

Proof. By Theorem 2 of [**3**], V cannot be an $(n-1)$-subspace or an n-subspace. Hence V contains nonsingular matrices and singular matrices. Let

$$\widetilde{V}_s = \{\widetilde{A} \in \widetilde{V} : \det A = 0\}. \tag{5.6}$$

Let

$$\eta : \widetilde{V}_s \to \mathbb{PR}^{n-1}$$

be defined as follows:

Suppose that $B \in V$ is any singular matrix, and suppose $0 \neq y \in \mathbb{R}^n$ satisfies $By = 0$. Then let $\eta(\widetilde{B}) = \widetilde{y}$.

It is known that the set of singular points of $\widetilde{V}_s$ is a proper subvariety of $\widetilde{V}_s$. As $\eta(\widetilde{V}_s) = \mathbb{PR}^{n-1}$, it follows that there exists a regular point $\widetilde{B}$ of $\widetilde{V}_s$ such that $D\eta(\widetilde{B})$ has the rank $n-1$. Let $y \in S^{n-1}$ be such that $By = 0$. Suppose that $\dim L_y \geq 2$. Then it is clear that $\dim \ker D\eta(\widetilde{B}) \geq 1$.

It follows that the dimension of the tangent space to $\widetilde{B}$ in $\widetilde{V}_s$ has dimension $\geq n$, implying that $\dim \widetilde{V}_s \geq n$, a contradiction. □

Lemma 5.3. *Let n be odd, $n \geq 3$, and let V be an $n+1$ dimensional subspace of $S_n(\mathbb{R})$ such that each nonzero matrix in V has* $\operatorname{rank} n-1$ *or n. Suppose that $V^{\perp} = \{B \in S_n(\mathbb{R}) : \operatorname{tr}(AB) = 0 \ \forall A \in V\}$ does not contain rank one matrices. Let $\widetilde{V}_s$ be as in (5.6). Then, $\widetilde{V}_s$ is a smooth connected variety in $\mathbb{PR}^n$.*

Proof. Let $A_k = \left(a_{ij}^{(k)}\right)_{i,j=1}^{n}$, $k = 1, 2, \ldots, n+1$ be a basis of V. Let $\psi(\alpha) = \mathit{det}\left(\sum_{i=1}^{n+1} \alpha_i A_i\right)$. For $\widetilde{\alpha} \in \widetilde{V}_s$ we have $\psi(\alpha) = 0$, so $A(\alpha) = \sum_{i=1}^{n+1} \alpha_i A_i$ has rank $n-1$. Hence the $(n-1)$th compound of $A(\alpha), C_{n-1}(A(\alpha))$, has rank 1. So there exists $0 \neq x(\alpha) = (x_1(\alpha), x_2(\alpha), \ldots, x_n(\alpha))^t$ and $\epsilon = \pm 1$ such that $C_{n-1}(A(\alpha)) = \epsilon x(\alpha) x^t(\alpha)$. Let $C_{n-1}(\alpha) = (c_{ij}(\alpha))_{i,j=1}^n$. Observe that

$$\frac{\partial \psi(\alpha)}{\partial \alpha_k} = \sum_{i,j=1}^{n} a_{ij}^{(k)} c_{ij}(\alpha) = \epsilon x^t(\alpha) A_k x(\alpha), \;\; k = 1, 2, \ldots, n+1.$$

Since $V^\perp$ does not contain a rank one matrix, we deduce that $\nabla \psi(\alpha) \neq 0$. Thus each $\widetilde{\alpha} \in \widetilde{V}_s$ is a smooth point and the local dimension of $\widetilde{V}_s$ at $\widetilde{\alpha}$ is $n-1$.

Let $V_{\mathbb{C}}$ be the complexification of V. Let $T_{\mathbb{C}}, T_{\mathbb{R}}$ be defined by (5.5) for $\mathbb{F} = \mathbb{C}, \mathbb{R}$, respectively. By the remark following Lemma 5.1, $\mathrm{codim}_{\mathbb{C}} T_{\mathbb{C}} \geq 2$. Since $T_{\mathbb{R}} = T_{\mathbb{C}} \cap \mathbb{PR}^{n-1}$, we have $\mathrm{codim}_{\mathbb{R}} T_{\mathbb{R}} \geq 2$.

Let $Y_r = \mathbb{PR}^{n-1} \backslash T_{\mathbb{R}}$. Then Y_r is connected. Using (ii) of Lemma 5.2 we can define H and θ as in Lemma 5.1. Let $H_{\mathbb{R}} = H \cap \mathbb{PR}^n$. Then $\theta(Y_r)$ is connected in $H_{\mathbb{R}}$, and this implies that $\overline{\theta(Y_r)} = H_{\mathbb{R}}$ is connected.

To finish the proof it suffices to show that $\widetilde{V}_s = H_{\mathbb{R}}$. Since $\overline{Y}_r = \mathbb{PR}^{n-1}$ it follows that for any $\widetilde{y} \in \mathbb{PR}^{n-1}$ there exists $\widetilde{\beta} \in H_{\mathbb{R}}$ such that $A(\beta) y = 0$. Let $\widetilde{\gamma}$ be an arbitrary point in $\widetilde{V}_s$. Then $\exists\ 0 \neq u \in \mathbb{R}^n$ such that $A(\gamma) u = 0$. If $\dim L_u = 1$, then $\widetilde{\gamma} \in H_{\mathbb{R}}$. So suppose $\dim L_u \geq 2$. Then $\exists\ \widetilde{\beta} \in H_{\mathbb{R}}$ such that $A(\beta) \in L_u$. Hence $\widetilde{\beta}, \widetilde{\gamma} \in \widetilde{L}_u$, and clearly $\widetilde{\beta}$ and $\widetilde{\gamma}$ are connected in $\widetilde{L}_u \subset \widetilde{V}_s$. As $\widetilde{V}_s$ is a smooth variety, it follows that $\widetilde{\gamma} \in H_{\mathbb{R}}$. Hence $\widetilde{V}_s = H_{\mathbb{R}}$. □

Recall that the standard inner product in $S_n(\mathbb{R})$ is given by $\langle A, B \rangle = \mathrm{tr}(AB)$, so with respect to this inner product a matrix A is normalized if and only if $\mathrm{tr}(A^2) = 1$.

Corollary 5.1. *Let n and V satisfy the assumptions of Lemma 5.3. Let*

$$V_\nu = \{A \in V : \mathrm{tr}\,(A^2) = 1\}$$

and

$$V_{\nu s} = \{A \in V_\nu : \det A = 0\}.$$

Then $V_{\nu s}$ is a smooth connected hypersurface in V_ν.

Proof. Assume that the basis $A_1, A_2, \ldots, A_{n+1}$ is orthonormal with respect to the standard inner product in $S_n(\mathbb{R})$. Let $\widetilde{V}_s, T_{\mathbb{R}}$ and Y_r be as in Lemma 5.3. Identify $\widetilde{V}_s$ with $V_{\nu s}/\{\pm I\}$. It follows from Lemma 5.3 that $V_{\nu s}$ is smooth. To show that $V_{\nu s}$ is connected, it suffices to show we can connect some $A \in V_{\nu s}$ to $-A$ in $V_{\nu s}$.

Take $y \in S^{n-1}$ such that $\widetilde{y} \in Y_r$. Connect y to $-y$ by a path $\mathcal{J}$(in S^{n-1}) such that $\widetilde{\mathcal{J}} \subset Y_r$. For $z \in S^{n-1}$ such that $\widetilde{z} \notin T_{\mathbb{R}}$ let

$$\hat{\alpha}(z) = \frac{1}{\|\alpha(z)\|_2} \alpha(z),$$

where $\alpha(z)$ is given by (5.3). Since α is odd, $A(\hat{\alpha}(z)), z \in \mathcal{J}$, connects $A(\hat{\alpha}(y))$ to $A(\hat{\alpha}(-y)) = -A(\hat{\alpha}(y))$. $\square$

Theorem 5.1. *Let n be odd, $n \geq 3$, and let V be an $n+1$ dimensional subspace of $S_n(\mathbb{R})$ such that each nonzero matrix in V has* rank $n-1$ *or n. Then each nonzero matrix in V has at least $\frac{n-1}{2}$ positive and negative eigenvalues.*

Proof. Let $A_1, A_2, \ldots, A_{n+1}$ be a basis of V. We distinguish two cases.

Case 1. Suppose that $V^{\perp}$ does not contain a matrix of rank one. Let $V_{\nu s}$ be as in Corollary 5.1. Then, each matrix in $V_{\nu s}$ has rank $n-1$, and by the connectedness of $V_{\nu s}$ it follows that A and $-A$ have the same inertia for each $A \in V_{\nu s}$. It follows that every singular matrix in $V \backslash \{0\}$ has $\frac{n-1}{2}$ positive eigenvalues, $\frac{n-1}{2}$ negative eigenvalues, and 1 zero eigenvalue. Continuity yields immediately that each $0 \neq A \in V$ has at least $\frac{n-1}{2}$ positive and negative eigenvalues.

Case 2. Suppose now that $V^{\perp}$ does contain a rank one matrix. Note that we have $\dim \mathbb{P}(V^{\perp}) = \frac{n(n+1)}{2} - (n+1) - 1$, while $\dim \mathbb{P}V_{1,n}(\mathbb{R}) = n-1$. Hence, for any $\epsilon > 0$ there exist $A_{1,\epsilon}, A_{2,\epsilon}, \ldots, A_{n+1,\epsilon} \in S_n(\mathbb{R})$ which satisfy the following three conditions:

(i) $\|A_{i,\epsilon} - A_i\| < \epsilon, \ i = 1, 2, \ldots, n+1$.
(ii) $V_\epsilon = \mathrm{span}\{A_{1,\epsilon}, A_{2,\epsilon}, \ldots, A_{n+1,\epsilon}\}$ is $n+1$ dimensional.
(iii) $V_\epsilon^{\perp}$ does not contain a rank one matrix.

For $\epsilon > 0$ small enough, we can apply Case 1 and conclude that any nonzero matrix in V_ϵ has at least $\frac{n-1}{2}$ positive and negative eigenvalues. Let $\epsilon \to 0$ and use continuity to finish the proof. $\square$

6. Concluding remarks and conjectures.

Theorem 6.1. *Let $1 \leq q \in \mathbb{Z}$. Then* $\deg \mathbb{P}V_{n-q,n}(\mathbb{C})$ *is odd if $n > q$ and*

$$n \equiv \pm q \left(\mathrm{mod}\ 2^{\lceil \log_2 2q \rceil}\right). \tag{6.1}$$

Proof. Let $0 \neq q \in \mathbb{Z}$. Let $\mu(q) = \lceil \log_2 2|q| \rceil$ and $0 \leq \nu(q) \in \mathbb{Z}$ be the largest integer such that $2^{\nu(q)}$ divides q. Then $\nu(q) + 1 \leq \mu(q)$. Clearly if $\ell \equiv m \not\equiv 0 \pmod{2^\mu}$ for some $1 \leq \mu \in \mathbb{Z}$ then $\nu(\ell) = \nu(m)$. Let $\delta_{k,n} := \deg \mathbb{P}V_{k,n}(\mathbb{C})$. We claim that for n satisfying (6.1)

$$\nu(\delta_{n-q,n}) = \sum_{j=0}^{q-1} \nu\left(\binom{\hat{q}+j}{q-j}\right) - \nu\left(\binom{2j+1}{j}\right), \quad \hat{q} = \pm q. \tag{6.2}$$

Here for any real x and $0 \le k \in \mathbb{Z}$ we let

$$\binom{x}{k} = \frac{x(x-1)\dots(x-k+1)}{k!}.$$

In particular

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}, \quad k = 1, \dots . \tag{6.3}$$

Observe that for $j \le q-1$

$$\binom{n+j}{q-j} = \frac{(n+j)(n+j-1)\dots(n-q+2j+1)}{1 \cdot 2 \dots (q-j)}.$$

For n satisfying (6.1) it follows that $n + \ell \equiv \pm q + \ell \not\equiv 0 \pmod{2^{\mu(q)}}$ if $-q+1 \le \ell \le q-1$. Hence (6.2) holds.

Our theorem is equivalent to the statement that $\nu(\delta_{n-q,n}) = 0$ if (6.1) holds. We first consider the case $n \equiv -q \pmod{2^{\mu(q)}}$. Then

$$\begin{aligned}
\nu(\delta_{n-q,n}) &= \sum_{j=0}^{q-1} \nu\left(\binom{-q+j}{q-j}\right) - \nu\left(\binom{2j+1}{j}\right) \\
&= \sum_{j=0}^{q-1} \nu\left((-1)^{q-j}\binom{2(q-j)-1}{q-j}\right) - \sum_{j=0}^{q-1} \nu\left(\binom{2j+1}{j}\right) \\
&= \sum_{j=0}^{q-1} \nu\left(\binom{2(q-j)-1}{q-j-1}\right) - \sum_{j=0}^{q-1} \nu\left(\binom{2j+1}{j}\right) = 0.
\end{aligned}$$

We now consider the case $n \equiv q \pmod{2^{\mu(q)}}$. In this case it is enough to show the identity

$$\prod_{j=0}^{q-1} \frac{\binom{q+j}{q-j}}{\binom{2j+1}{j}} = 1, \quad q = 1, \dots . \tag{6.4}$$

We prove the above identity by induction on q. For $q = 1$ (6.4) trivially holds. Assume that (6.4) holds for $q = m \ge 1$. Let $q = m+1$. Use the identity

$$\binom{m+1+j}{m+1-j} = \frac{m+1+j}{m+1-j}\binom{m+j}{m-j}, \quad j = 0, 1, \dots, m,$$

to deduce (6.4) for $q = m+1$ from $q = m$. □

Corollary 6.1. *Let $n > q$ and let (6.1) hold. Then*

$$d(n, n-q, \mathbb{R}) = d(n, n-q, \mathbb{C}) = \binom{q+1}{2} + 1.$$

Corollary 6.2. *Let $n \geq q \geq 2$ and assume that (6.1) holds. Then any $\binom{q+1}{2}$ dimensional subspace U of $S_n(\mathbb{R})$ contains a nonzero matrix with an eigenvalue of multiplicity q at least. Furthermore, there exists an $\binom{q+1}{2}-1$ dimensional subspace $V \subset S_n(\mathbb{R})$ such that each eigenvalue of $0 \neq A \in V$ has a multiplicity less than q.*

Proof. For $n = q$ $U = S_n(\mathbb{R})$ and the eigenvalue 1 of I_q has multiplicity q. Assume that $V \subset S_q(\mathbb{R})$ does not contain I_q. Then each eigenvalue of $0 \neq A \in V$ has multiplicity less than q. Assume that $q < n$. Then the claim of the corollary follows from Corollary 6.1 and Proposition 3.3. □

Clearly $\delta_{n-1,n} = n$ is odd if and only if n is odd. As $d(n, n-1, \mathbb{R}) = \rho_s(n) + 1$ it follows that $d(n, n-1, \mathbb{R}) > d(n, n-1, \mathbb{C}) = 2$ if n is even.

Conjecture 6.1. *Let $n > q \geq 2$ and assume that (6.1) does not hold. Then $\delta_{n-q,n}$ is even. Furthermore $d(n, n-q, \mathbb{R}) > d(n, n-q, \mathbb{C})$.*

For $q = 2, 3, 4, 5$ it is straightforward to check that $\delta_{n-q,n}$ is odd if and only if (6.1) holds.

Finally, we return to the sequence $\{d(n, n-2, \mathbb{R})\}_{n=3}^{\infty}$ and state two questions. The first unknown number in this sequence is $d(5, 3, \mathbb{R})$. As we have seen in Section 3, $d(5, 2, \mathbb{R}) = 7$, so $d(5, 3, \mathbb{R}) \leq 7$.

Question 1. Is $d(5, 3, \mathbb{R}) = 6$?

We think that the answer is yes. In [6] it is shown that $d(5, 3, \mathbb{R}) \leq 6$. Let L be the 5 dimensional subspace of $S_5(\mathbb{R})$ spanned by the matrices

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \\ -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \end{bmatrix}.$$

A local minimization procedure using Matlab seems to indicate that every nonzero matrix in L has rank 4 or 5. This has yet to be confirmed by other means, but if it is correct, then $d(5, 3, \mathbb{R}) = 6$.

Note that for every field $\mathbb{F}$ we have

$$d(n,1,\mathbb{F}) \geq d(n,2,\mathbb{F}) \geq \cdots \geq d(n,n-2,\mathbb{F}) \geq d(n,n-1,\mathbb{F}). \tag{6.5}$$

Question 2. Is there strict inequality everywhere in (6.5) if $\mathbb{F} = \mathbb{R}$?

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Department of Industrial Engineering
Technion – Israel Institute of Technology
E-mail address: dmfalikm@ie.technion.ac.il

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
E-mail address: friedlan@uic.edu

Department of Mathematics
Technion – Israel Institute of Technology
E-mail address: loewy@techunix.technion.ac.il

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