Maximizing Sum Rates in Gaussian Interference-limited Channels

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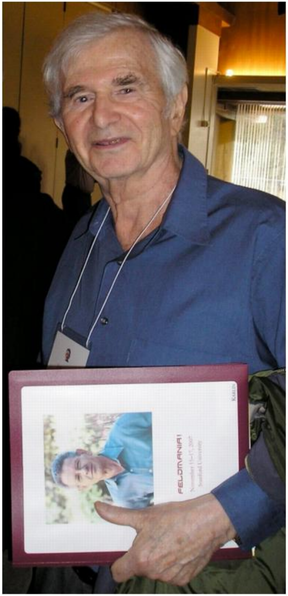

Figure: Karlin

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He died Dec. 18, 2007 at Stanford Hospital after a massive heart

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Every critical point in interior Π_n is local minimum (M-matrices!)
Hence the critical point $\mathbf{x}(A) \in \Pi_n$ is global minimum

Lower bound for spectral radius

COR 1: For $A \geq 0$ irreducible,
$\mathbf{d} = (d_1, \ldots, d_n) > \mathbf{0}$, $D = D(\mathbf{d}) := \mathrm{diag}(d_1, \ldots, d_n)$

$$\rho(D(\mathbf{d})A) \geq \rho(A) \prod_{i=1}^{n} d_i^{x_i(A) y_i(A)}$$

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Original motivation: Population genetics
A - stochastic matrix describing Markov process of genes, $\mathbf{d}$ the strength of genes. When is $\rho(MD) > 1$?

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  is the supporting hyperplane of $f(\mathbf{x})$ at $\mathbf{u}$

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$$D(\mathbf{c})AD(\mathbf{d})\mathbf{u} = \mathbf{u}, \quad \mathbf{v}^\top D(\mathbf{c})AD(\mathbf{d}) = \mathbf{v}$$

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Example 1: $A = \begin{bmatrix} * & * \\ * & 0 \end{bmatrix}$ is not a pattern of doubly stochastic matrix

Example 2: $A = \begin{bmatrix} 0 & * \\ * & 0 \end{bmatrix}$ always rescalable to doubly stochastic with many more solutions than in THM 2.

Rescaling of irreducible matrices and applications

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$$\max_{\mathbf{z}>0} \sum_{i=1}^n w_i \log \frac{z_i}{(A\mathbf{z})_i} = \sum_{i=1}^n w_i \log(c_i d_i),$$

where $\mathbf{u} = (1, \dots, 1)^\top, \mathbf{v} = \mathbf{w}$ and $\mathbf{c}, \mathbf{d}$ are given in THM 3.

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Proof:
$\sum_{i=1}^n w_i \log \frac{d_i y_i}{(AD(\mathbf{d})\mathbf{y})_i} = \sum_{i=1}^n w_i \log \frac{y_i}{(D(\mathbf{c})AD(\mathbf{d})\mathbf{y})_i} + \sum_{i=1}^n w_i \log(c_i d_i)$

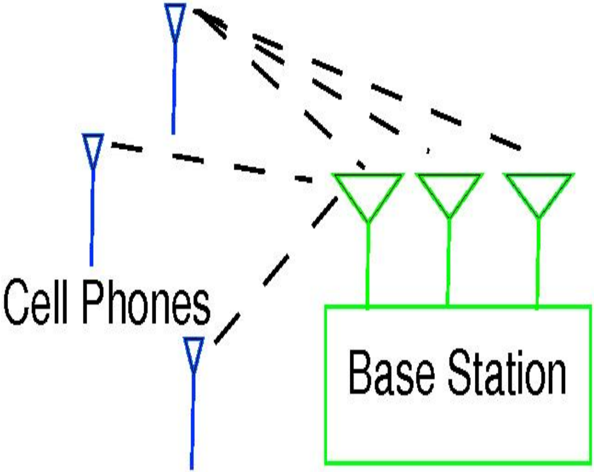

Figure: Cell phones communication

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Maximizing sum rates in Gaussian interference-limited channel

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If $\sum_{j \neq i} w_j > w_i > 0$ for $i = 1, \ldots, n$
relaxed maximal problem can be solved by THM 4.

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maximization of convex function on closed unbounded convex set

Approximation methods-I

Approximation 1:
For $K \gg 1$ $\mathcal{D}_K := \{\mathbf{x} \in \mathcal{D},\ \mathbf{x} \geq -K\mathbf{1} = -K(1,\ldots,1)^\top\}$
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$$\mathcal{D}(\xi_1,\dots,\xi_N,K) = \{\mathbf{x} \in \mathbb{R}^n, H_{j,k}(\mathbf{x}) \leq H_{j,k}(\xi_k), j \in \mathcal{A}_k, k \in \langle N\rangle, \xi \geq -K\mathbf{1}\}$$

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Use Simplex Method or Ellipsoid Algorithm

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E.g., divide $[\mathbf{0},\mathbf{p}]$ by a mesh, and choose all boundary points with positive coordinates

$\xi_k = \gamma(\mathbf{p}_k)$ and $\mathcal{A}_k$ all j s.t. $p_{j,k} = \bar{p}_j$

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Study $\max_{\mathbf{0}\le\mathbf{p}\le\bar{\mathbf{p}}} \Phi_{\mathbf{w}}(\mathbf{p}) = \Phi_{\mathbf{w}}(\mathbf{p}^\star)$

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Local minimum conditions at $\mathbf{0} \neq p^\star \in \partial[\mathbf{0}, \bar{\mathbf{p}}]$

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Assumption $\mathbf{0} < \mathbf{w} \in \Pi_n$

Local minimum conditions at $\mathbf{0} \neq p^\star \in \partial[\mathbf{0}, \bar{\mathbf{p}}]$

1. $\partial_i \Phi_{\mathbf{w}}(\mathbf{p}^\star) = 0$ if $0 < p_i^\star < \bar{p}_i$

Direct methods

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Apply gradient methods and their variations

References

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