Results and problems for 3-tensors

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NIU LA'09 , 14 August, 2009

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6. CUR decompositions for tensors

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$\mathcal{T} = f_r(\mathbf{x}_1,\mathbf{y}_1,\mathbf{z}_1,\ldots,\mathbf{x}_r,\mathbf{y}_r,\mathbf{z}_r) := \sum_{i=1}^r \mathbf{x}_i\otimes\mathbf{y}_i\otimes\mathbf{z}_i$,
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Normalization: $2\leq m\leq n\leq l\leq mn$

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COR $\operatorname{grank}_{\mathbb{C}}(m,n,(m-1)(n-1)) = (m-1)(n-1)+1$.

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The minimal $r \ge \lceil \frac{mnl}{(m+n+l-2)} \rceil$ s.t. $\text{rank}\, J(f_r)(\mathbf{w}_r) = mnl$
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I checked the conjecture up to $m,n,l \le 14$

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For $2 \le m \le n \le l < mn - 1$, there exist $V_1, \ldots, V_{c(m,n,l)} \subset \mathbb{R}^{m \times n \times l}$ pairwise disjoint open connected semi-algebraic sets s.t.
Closure$(\cup_{i=1}^{c(m,n,l)} V_i) = \mathbb{R}^{m \times n \times l}$
$\mathrm{rank}\ \mathcal{T} = \mathrm{grank}_{\mathbb{C}}(m,n,l)$ for each $\mathcal{T} \in V_1$
$\mathrm{rank}\ \mathcal{T} = \rho_i$ for each $\mathcal{T} \in V_i$
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Examples [2]
$m = n \ge 2,\ l = (m-1)(n-1)+1$.
$m = n = 4,\ l = 11, 12$

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Problem: $\rho_i \le \mathrm{grank}_{\mathbb{C}}(m,n,l) + 1$?

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Upper bounds for generic and maximal rank

$L \cap \mathcal{R}(m,n,k,\mathbb{C}) \supsetneq \{0\}$ if codim $L < \dim \mathcal{R}(m,n,k,\mathbb{C}) = k(n+m-k)$.

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$\mathrm{mrank}(n,m,m) \le$
$\sum_{i=1}^{\lfloor \sqrt{n-1} \rfloor} (2i-1)(m-i+1) + (m - \lfloor \sqrt{n-1} \rfloor^2)(m - \lfloor \sqrt{n-1} \rfloor)$

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$\mathbb{R}^{m\times n\times l}$ IPS: $\langle \mathcal{A}, \mathcal{B}\rangle = \sum_{i=j=k}^{m,n,l} a_{i,j,k} b_{i,j,k}, \ \|\mathcal{T}\| = \sqrt{\langle \mathcal{T}, \mathcal{T}\rangle}$

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Rank one approximations

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Equivalent: $\max_{\|\mathbf{x}\|=\|\mathbf{y}\|=\|\mathbf{z}\|=1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

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λ singular value, $\mathbf{x}, \mathbf{y}, \mathbf{z}$ singular vectors
How many distinct singular values are for a generic tensor?

ℓ_p maximal problem and Perron-Frobenius

ℓ_p maximal problem and Perron-Frobenius

$\|(x_1, \ldots, x_n)^\top\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$

Problem: $\max_{\|\mathbf{x}\|_p = \|\mathbf{y}\|_p = \|\mathbf{z}\|_p = 1} \sum_{i=j=k}^{m,n,l} t_{i,j,k} x_i y_j z_k$

Lagrange multipliers: $\mathcal{T} \times \mathbf{y} \otimes \mathbf{z} := \sum_{j=k=1} t_{i,j,k} y_j z_k = \lambda \mathbf{x}^{p-1}$
$\mathcal{T} \times \mathbf{x} \otimes \mathbf{z} = \lambda \mathbf{y}^{p-1}$, $\mathcal{T} \times \mathbf{x} \otimes \mathbf{y} = \lambda \mathbf{z}^{p-1}$ ($p = \frac{2t}{2s-1}, t, s \in \mathbb{N}$)

ℓ_p maximal problem and Perron-Frobenius

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$p = 3$ is most natural in view of homogeneity

ℓ_p maximal problem and Perron-Frobenius

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$p = 3$ is most natural in view of homogeneity

Assume that $\mathcal{T} \geq 0$. Then $\mathbf{x}, \mathbf{y}, \mathbf{z} \geq 0$

For which values of p we have an analog of Perron-Frobenius theorem?

Yes, for $p \geq 3$, No, for $p < 3$,
Friedland-Gauber-Han [3]

Outline of the proof

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Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

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$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\ldots,n$$

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Assume $\sum_{j=k=1}^{n,l} f_{i,j,k} > 0, i = 1,\ldots,m$,
$\sum_{i=k=1}^{m,l} f_{i,j,k} > 0, j = 1,\ldots,n$, $\sum_{i=j=1}^{m,n} f_{i,j,k} > 0, k = 1,\ldots,l$

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Assume $\sum_{j=k=1}^{n,l} f_{i,j,k}>0, i=1,\ldots,m$,
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F 1-homogeneous monotone, maps open positive cone $\mathbb{R}_+^m\times\mathbb{R}_+^n\times\mathbb{R}_+^l$ to itself.

$\mathcal{F}=[f_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle,\langle n\rangle,\langle l\rangle$:
$i\in\langle m\rangle$ connected to $j\in\langle n\rangle$ and $k\in\langle l\rangle$ iff $f_{i,j,k}>0$, sim. for j,k

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If tri-partite graph is connected then F has unique positive eigenvector

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If tri-partite graph is connected then F has unique positive eigenvector

If F completely irreducible, i.e. F^N maps nonzero nonnegative vectors to positive, nonnegative eigenvector is unique and positive

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Define: $F : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$:

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$$F((\mathbf{x},\mathbf{y},\mathbf{z}))_{j,2} = \left(\|\mathbf{y}\|_p^{p-3} \sum_{i=k=1}^{m,l} f_{i,j,k} x_i z_k\right)^{\frac{1}{p-1}}, j = 1,\dots,n$$

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Assume $\sum_{j=k=1}^{n,l} f_{i,j,k} > 0, i = 1,\dots,m$,
$\sum_{i=k=1}^{m,l} f_{i,j,k} > 0, j = 1,\dots,n$, $\sum_{i=j=1}^{m,n} f_{i,j,k} > 0, k = 1,\dots,l$

F 1-homogeneous monotone, maps open positive cone $\mathbb{R}^m_+ \times \mathbb{R}^n_+ \times \mathbb{R}^l_+$ to itself.

$\mathcal{F} = [f_{i,j,k}]$ induces tri-partite graph on $\langle m\rangle, \langle n\rangle, \langle l\rangle$:
$i \in \langle m\rangle$ connected to $j \in \langle n\rangle$ and $k \in \langle l\rangle$ iff $f_{i,j,k} > 0$, sim. for j, k

If tri-partite graph is connected then F has unique positive eigenvector

If F completely irreducible, i.e. F^N maps nonzero nonnegative vectors to positive, nonnegative eigenvector is unique and positive

$p < 3$ numerical counterexamples $m = n = l = 2$

Scaling of nonnegative tensors to tensors with given row, column and depth sums

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$\mathbf{r} = (r_1, \ldots, r_m)^\top, \mathbf{c} = (c_1, \ldots, c_n)^\top, \mathbf{d} = (d_1, \ldots, d_l)^\top > \mathbf{0}$:

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$f_{\mathcal{T}}$ is strictly convex implies $\mathcal{T}$ is not decomposable: $\mathcal{T} \ne \mathcal{T}_1 \oplus \mathcal{T}_2$.

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Yes for Menon, unknown for Brualdi

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Do QR on each two columns successively to obtain:
$t_{2,1,1} = t_{1,2,1} = t_{1,1,2} = 0$.

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Friedland-Mehrmann-Miedlar-Nkengla 08: $U = A[I,J]^\dagger$-numerical

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CUR approximation $C \in \mathbb{R}^{m\times p}, R \in \mathbb{R}^{q\times n}$ some submatrices of A.

$C(C^\dagger A R^\dagger)R$ best rank $\le min(p,q)$ approximation matrix based on C, R submatrices of A.

Goreinov-Tyrtyshnikov-Zmarashkin 95: for $p = q$ choose $U = A[I,J]^{-1}$
(corresponds to best CUR approximation on the entries read)
good approximation when the corresponding $\det A[I,J]$ is maximal

Friedland-Mehrmann-Miedlar-Nkengla 08: $U = A[I,J]^\dagger$-numerical
choose several random choices of I, J set of rows and columns of A
such that $A[I,J]$ has maximal product of significant singular values

Extension to 3-tensors I:

$\langle n \rangle := \{1, \ldots, n\}$

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For given $\mathcal{A} \in \mathbb{R}^{m \times n \times l}, F \in \mathbb{R}^{m \times p}, E \in \mathbb{R}^{n \times q}, G \in \mathbb{R}^{l \times r}$,
where $\langle p \rangle \subset \langle n \rangle \times \langle l \rangle, \langle q \rangle \subset \langle m \rangle \times \langle l \rangle, \langle r \rangle \subset \langle m \rangle \times \langle l \rangle$

$\min_{\mathcal{U} \in \mathbb{R}^{p \times q \times r}} \|\mathcal{A} - \mathcal{U} \times F \times E \times G\|_F$ achieved for $\mathcal{U} = \mathcal{A} \times E^\dagger \times F^\dagger \times G^\dagger$

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CUR approximation of $\mathcal{A}$ obtained by choosing E, F, G submatrices of unfolded $\mathcal{A}$ in the mode $1, 2, 3$.

Extensions to 3-tensors: II

$\mathcal{A} = [a_{i,j,k}] \in \mathbb{R}^{m\times n\times \ell}$ - 3-tensor
given $I \subset \langle m\rangle, J \subset \langle n\rangle, K \subset \langle \ell\rangle$ define
$R := \mathcal{A}_{\langle m\rangle,J,K} = [a_{i,j,k}]_{\langle m\rangle,J,K} \in \mathbb{R}^{m\times(\#J\cdot\#K)}$,
$C := \mathcal{A}_{I,\langle n\rangle,K} \in \mathbb{R}^{\langle n\rangle\times(\#I\cdot\#K)}$,
$D := \mathcal{A}_{I,J,\langle \ell\rangle} \in \mathbb{R}^{l\times(\#I\cdot\#J)}$
Problem: Find 3-tensor $\mathcal{U} \in \mathbb{R}^{(\#J\cdot\#K)\times(\#I\cdot\#K)\times(\#I\cdot\#J)}$
such that $\mathcal{A}$ is approximated by the Tucker tensor
$\mathcal{V} = \mathcal{U} \times_1 C \times_2 R \times_3 D$
where $\mathcal{U}$ is the least squares solution

$$\mathcal{U}_{\text{opt}} \in \arg \min_{\mathcal{U}\in\mathbb{R}^{\text{three tensor}}} \sum_{(i,j,k)\in\mathcal{S}} \left(a_{i,j,k} - (\mathcal{U} \times_1 C \times_2 R \times_3 D)_{i,j,k}\right)^2$$

$\mathcal{S} = (\langle m\rangle \times J \times K) \cup (I \times \langle n\rangle \times K) \cup (I \times J \times \langle \ell\rangle)$

For $\#I = \#J = p,\ \#K = p^2,\ I \subset \langle m\rangle, J \subset \langle n\rangle, K \subset \langle \ell\rangle$
generically there is an exact solution to $\mathcal{U}_{\rm opt} \in \mathbb{R}^{p^3 \times p^3 \times p^2}$
obtained by unfolding in third direction
View $\mathcal{A}$ as $A \in \mathbb{R}^{(mn)\times \ell}$ by identifying
$\langle m\rangle \times \langle n\rangle \equiv \langle mn\rangle,\ I_1 = I \times J,\ J_1 = K$ and apply CUR again.

More generally, given $\#I = p,\ \#J = q,\ \#K = r$.
For $L = I \times J$ approximate $\mathcal{A}$ by $\mathcal{A}_{\langle m\rangle,\langle n\rangle,K} E^{\dagger}_{L,K} \mathcal{A}_{I,J,\langle \ell\rangle}$
Then for each $k \in K$ approximate each matrix $\mathcal{A}_{\langle m\rangle,\langle n\rangle,\{k\}}$ by
$\mathcal{A}_{\langle m\rangle,J,\{k\}} E^{\dagger}_{I,J,\{k\}} \mathcal{A}_{I,\langle n\rangle,\{k\}}$

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