Normal matrices and the completion problem

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Abstract

We show that there exists a normal matrix with an arbitrary upper triangular part.

1 Introduction

Let $\mathbf{M}(n, \mathbb{F})$ be the algebra of $n \times n$ matrices over a field $\mathbb{F}$, where $\mathbb{F}$ is either the field of real or complex numbers ($\mathbb{R}$ or $\mathbb{C}$) respectively. An inverse problem for matrices (IPFM) is the existence of a matrix A of certain class belonging to a given variety $\mathcal{A} \subset \mathbf{M}(n, \mathbb{F})$ of dimension m. Most of the IPFM are given by l polynomial conditions

$$p(A) = \alpha, \quad \alpha \in \mathbb{F}^l. \tag{1.1}$$

($p : \mathbf{M}(n, \mathbb{F}) \to \mathbb{F}^l$ is a polynomial map.) The most common IPFM are the inverse eigenvalue problems (IEP). As an example of IPFM (IEP) recall the result of Farahat and Ledermann [FL]: For a matrix $A \in \mathbf{M}(n, \mathbb{C})$ whose top left-hand corner is nonderogatory, every characteristic polynomial can be achieved through the choice of entries in the last row and column. Note that we have $2n - 1$ parameters to choose to satisfy n polynomial conditions. If $l = m$ and $p|\mathcal{A}$ is dominant, (the m

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polynomial conditions are algebraically independent over $\mathcal{A}$), then IPFM (IEP) is called *tight* IPFM (IEP) and is denoted by TIPFM (TIEP). The most known TIEP is determination of a real symmetric tridiagonal matrix from two spectra, which was first solved by M.G. Krein [GK, Appendix II] and rediscovered by many authors. This problem, as well as the TIEP discussed in [Fr3], can be solved in a constructive way by reducing the problem to a similar problem in $\mathbf{M}(n-1, \mathbb{R})$.

Assume first that $\mathbb{F} = \mathbb{C}$. Let $\mathcal{A}$ be a complex irreducible affine variety of complex dimension at least l. If $p|\mathcal{A}$ is a dominant map then (1.1) is solvable for all $\alpha \in \mathbb{C}^l \backslash W$, for some subvariety $W \subset \mathbb{C}^l$. (IPFM is generically solvable.) The dominancy of polynomial maps related to the IEP are studied in [HRW] and [RW]. To show that (1.1) is solvable for every $\alpha \in \mathbb{C}^l$ ($W = \emptyset$) requires stronger tools, e.g. the intersection theory from the algebraic geometry [Fr2] or the degree theory [Fr1].

Assume now that $\mathbb{F} = \mathbb{R}$. Then the theory of dominant maps does not apply. That is, even though $p|\mathcal{A}$ is dominant there may be an open (semialgebraic set) of $\alpha \in \mathbb{R}^l$ for which (1.1) is not solvable, e.g. [Fr2]. In the known nonconsructive solution to the TIEP for real symmetric Toeplitz matrices one uses the degree theory [Lan] (see also [Fr4]).

In this paper we solve a TIPFM over $\mathbb{R}$ which deals with normal complex valued matrices with prescribed upper triangular entries. Let $\mathcal{N}_n \subset \mathbf{M}(n, \mathbb{C})$ be the real variety of normal matrices. It is known that $\mathcal{N}_n$ is an irreducible variety of real dimension $n^2 + n$ [Ik2]. The purpose of this paper is twofold. First we describe in more details the structure of $\mathcal{N}_n$, the manifold of its regular (smooth) points $\mathcal{N}_n^r$ and its covering space induced by the spectral decomposition of a normal matrix. Second, using the degree theory, we show that any upper triangular $A \in \mathbf{M}(n, \mathbb{C})$ can be completed to a normal matrix $X \in \mathbf{M}(n, \mathbb{C})$. That is, for any $\binom{n+1}{2}$ complex numbers $a_{ij},\ i = 1, ..., n,\ j = i, ..., n$ there exists $X = (x_{ij})_1^n \in \mathcal{N}_n$, such that $x_{ij} = a_{ij}$ for $i = 1, ..., n,\ j = i, ..., n$. The cases $n = 2, 3$ were settled by Ikramov [Ik1]. The completion problem was raised by L. Elsner.

2 Remarks on the variety of normal matrices

Let $\mathbb{U}_n$ be the group of $n \times n$ unitary matrices. As $\mathbb{U}_n$ is a connected Lie group it follows that $\mathbb{U}_n$ is parallelizable and hence orientable. The symmetric group $\mathcal{S}_n$ acts on $\mathbb{U}_n$ by permuting the columns of $U \in \mathbb{U}_n$.

Lemma 2.1 *The action of $\mathcal{S}_n$ on $\mathbb{U}_n$ preserves the orientation of $\mathbb{U}_n$.*

Proof. We identify the tangent space of $\mathbb{U}_n$ with the Lie algebra of skew hermitian matrices:

$$\mathbb{A}_n := \{A = (a_{ij})_1^n \in \mathbf{M}(n, \mathbb{C}) : \quad A^* = -A\}.$$

Fix a frame (basis) in $\mathbb{A}_n$ which is given by the unit vectors in the positive direction of the imaginary parts of the diagonal elements of A and the real and the imaginary parts of the entries $a_{ij},\ 1 \le i < j \le n$. We view this frame as $\mathbb{R}^n \times \mathbb{C}^{\frac{n(n-1)}{2}}$. Orient this frame by giving the standard orientations on $\mathbb{R}^n$ and $\mathbb{C}^{\frac{n(n-1)}{2}}$. The n^2 vectors fields of $\mathbb{U}_n$ at U are given by transporting the frame of $\mathbb{A}_n$, at the identity I_n, by the left multiplication by U. Let $\Pi_n \subset \mathbb{U}_n$ be the subgroup of permutation matrices. We identify $\mathcal{S}_n$ with Π_n. Then Π_n acts on $\mathbb{U}_n$ by the right multiplication:

$$U \mapsto UP, \quad P \in \Pi_n,\ U \in \mathbb{U}_n. \tag{2.1}$$

Observe next that

$$e^A P = P e^{P^T A P}, \quad P \in \Pi_n,\ A \in \mathbb{A}_n.$$

Thus the transformation (2.1) preserves the orientation iff the tranformation

$$A \mapsto P^T A P, \quad P \in \Pi_n,\ A \in \mathbb{A}_n \tag{2.2}$$

preserves the orientation of $\mathbb{A}_n$. As any element of $\mathcal{S}_n$ is a product of transpostions, it is enough to show that that (2.2) preserves the orientation for any transposition P. Assume for simplicity that P corresponds to the transposition which transposes 1 and 2. Then the map (2.2) acts separately on $\mathbb{R}^n$ and $\mathbb{C}^{\frac{n(n-1)}{2}}$. On $\mathbb{R}^n$ it transposes a_{11} and a_{22}. Hence it reverses the orientation on $\mathbb{R}^n$. On $\mathbb{C}^{\frac{n(n-1)}{2}}$ the map (2.2) is a composition of the map $a_{12} \mapsto -\bar{a}_{12}$ and the permutations $a_{1j} \leftrightarrow a_{2j},\ j = 3, ..., n$ for $n > 2$. Clearly, the above permutations preserve the orientation of $\mathbb{C}^{\frac{n(n-1)}{2}}$. The map $a_{12} \mapsto -\bar{a}_{12}$ reverses the orientation of $\mathbb{C}^{\frac{n(n-1)}{2}}$. Hence the action of the above transposition preserves the orientation of $\mathbb{U}_n$. The same arguments apply to any transposition P. $\square$

Recall that $\mathbb{U}_n$ is a smooth compact real algebraic variety

$$\mathbb{U}_n = \{X + \sqrt{-1}Y : \quad XX^T + YY^T = I_n,\ -XY^T + YX^T = 0,\ X, Y \in \mathbf{M}(n, \mathbb{R})\}.$$

$\mathcal{S}_n$ acts freely on $\mathbb{U}_n$. Hence $\mathbb{U}_n/\Pi_n$ can be realized as a smooth compact real algebraic variety.

Corollary 2.2 *$\mathbb{U}_n/\Pi_n$ is orientable.*

Let $\mathcal{F}_n$ be the space of full flags in $\mathbb{C}^n$:

$$V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n. \tag{2.3}$$

Let $\mathbb{UD}_n < \mathbb{U}_n$ be the subgroup of diagonal unitary matrices. Then $\mathcal{F}_n$ is isomorphic to the homogeneous space $\mathbb{U}_n/\mathbb{UD}_n$. Indeed, to each coset $U\mathbb{UD}_n$ associate the following flag: V_k is the subspace spanned by the first k columns of U for $k = 1, ..., n$. Clearly, this flag is independent of the representative $U' \in U\mathbb{UD}_n$. Vice versa, assume that $\mathbb{C}^n$ is an inner product space with $< x, y >= y^* x$. Then a flag (2.3) defines an orthonormal basis of column vectors $v_1, ..., v_n \in \mathbb{C}^n$, such that $V_k = \text{span}\ (v_1, ..., v_k)$ for $k = 1, ..., n$. Let $U = (v_1, ..., v_n)$. Then U defines a unique coset $U\mathbb{UD}_n$. As $P\mathbb{UD}_n = \mathbb{UD}_n P$ for any $P \in \Pi_n$ it follows that $\mathcal{S}_n$ acts on $\mathbb{U}_n/\mathbb{UD}_n$:

$$U\mathbb{UD}_n \mapsto UP\mathbb{UD}_n, \quad P \in \Pi_n.$$

Equivalently $\mathcal{S}_n$ acts on $\mathcal{F}_n$ by permuting the orthonormal basis $v_1, ..., v_n$ corresponding to the flag (2.3):

$$\{v_1, ..., v_n\} \mapsto \{v_{\sigma(1)}, ..., v_{\sigma(n)}\}, \quad \sigma \in \mathcal{S}_n.$$

The following lemma is probably well known and we bring its proof for completeness.

Lemma 2.3 *$\mathcal{F}_n$ is a fibre bundle with the basis $\mathbb{P}^{n-1}$ and the fibre $\mathcal{F}_{n-1}$. Hence $\mathcal{F}_n$ is an orientable manifold of dimension $n(n-1)$. The action of $\mathcal{S}_n$ on $\mathcal{F}_n$ does not preserve the orientation of $\mathcal{F}_n$.*

Proof. The choice of v_1 in the orthonormal basis $\{v_1, ..., v_n\}$ is equivalent to the choice of a one dimensional subspace in $\mathbb{C}^n$ spanned by v_1. The space of one dimensional subspaces is $\mathbb{P}^{n-1}$. Fix one dimensional subspace $V_1 \subset \mathbb{C}^n$. Let $V_1^\perp$ be the orthogonal complement of V_1 in $\mathbb{C}^n$. Then the choices of all possible orthonormal bases in $V_1^\perp$ is $\mathcal{F}_{n-1}$. Hence $\mathcal{F}_n$ is a fibre bundle over $\mathbb{P}^{n-1}$ with the fibre $\mathcal{F}_{n-1}$. ($\mathcal{F}_1$ is a space consisting of one point.) We prove the rest of the lemma by induction. Let $n = 2$. Then $\mathcal{F}_2$ is identified with the Riemann sphere $\mathbb{P}$. Hence $\mathcal{F}_2$ is orientable of (real) dimension 2 . Let $V_1 \subset \mathbb{C}^2$ be a one dimensional subspace which is not spanned either by $e_1 = (1, 0)^T$ or by $e_2 = (0, 1)^T$. Then

$$V_1 = \text{span}\ ((1, z)^T), \quad V_1^\perp = \text{span}\ (1, -\frac{1}{\bar{z}})^T), \quad z \in \mathbb{C}\backslash\{0\}.$$

The action of the permutation $P \in \Pi_2$ is equivalent to the map $V_1 \mapsto V_1^T$. For V_1 of the above form this action is given by the map $z \mapsto -\frac{1}{\bar{z}}$. Clearly this map reverses the orientation of $\mathbb{P}$.

Assume by induction that $\mathcal{F}_n$ is orientable of dimension $n(n-1)$ and the action of $\mathcal{S}_n$ does not preserve the orientation of $\mathcal{F}_n$. Consider $\mathcal{F}_{n+1}$ which is a fibre bundle over $\mathbb{P}^n$ with the fibre $\mathcal{F}_n$. Clearly

$$\dim \mathcal{F}_{n+1} = \dim \mathbb{P}^n + \dim \mathcal{F}_n = 2n + n(n-1) = (n+1)n.$$

As $\mathbb{P}^n$ and $\mathcal{F}_n$ are orientable it follows straightforward that $\mathcal{F}_{n+1}$ is orientable. Fix $V_2 \subset V_3 \subset ... \subset V_{n+1} = \mathbb{C}^{n+1}$. Without loss of generality we identify V_2 with $\mathbb{C}^2$. Consider all possible choices of a one dimensional subspace $V_1 \subset V_2$. Let $P \in \Pi_{n+1}$ be the transposition which transpose 1 and 2. Then the above arguments show that the action of P reverses the orientation of $\mathcal{F}_{n+1}$. $\square$

Recall that

$$\mathcal{N}_n := \{A \in \mathbf{M}(n, \mathbb{C}) : \quad AA^* = A^*A\}.$$

By writing $A = X + \sqrt{-1}Y,\ X, Y \in \mathbf{M}(n, \mathbb{R})$ we obtain that

$$\begin{aligned}&\mathcal{N}_n = \{A = X + \sqrt{-1}Y : \\ &XX^T + YY^T = X^TX + Y^TY,\ -XY^T + YX^T = X^TY - Y^TX, \quad X, Y \in \mathbf{M}(n, \mathbb{R})\}.\end{aligned} \tag{2.4}$$

Let $\mathbb{D}(n, \mathbb{C})$ be the set of complex diagonal matrices. Let

$$\mathbb{D}^r(n, \mathbb{C}) = \{D = \text{diag}\,(d_1, ..., d_n) \in \mathbb{D}(n, \mathbb{C}), \quad d_i \neq d_j \quad \text{for } i \neq j\}.$$

Recall that any normal A has the form UDU^* where $D \in \mathbb{D}(n, \mathbb{C})$ and $U \in \mathbb{U}_n$. As $UDU^* = D$ for any $U \in \mathbb{UD}_n$ it follows that

$$\mathcal{N}_n = \{A : \quad A = UDU^*, \quad D \in \mathbb{D}(n, \mathbb{C}),\ U \in \mathbb{U}_n/\mathbb{UD}_n\}. \tag{2.5}$$

Some of the following results are stated explicitly in [Ik2]:

Lemma 2.4 *The set $\mathcal{N}_n$ is an irreducible real homogeneous algebraic variety of dimension $n(n+1)$. The set of smooth points $\mathcal{N}_n^r$ of $\mathcal{N}_n$ corresponds to all normal matrices with pairwise distinct eigenvalues. $\mathcal{N}_n^r$ is connected. The natural projection $p : \mathbb{D}(n, \mathbb{C}) \times \mathbb{U}_n/\mathbb{UD}_n \to \mathcal{N}_n$ is a branched covering. The symmetric group $\mathcal{S}_n$ acts on $\mathbb{D}(n, \mathbb{C}) \times \mathbb{U}_n/\mathbb{UD}_n$ by the permutation of the diagonal on the first factor and by column permutation on the second factor (simultaneously). This action commutes with p. Furthermore, for any $A \in \mathcal{N}_n^r$ $p^{-1}(A)$ is an $\mathcal{S}_n$-orbit consisting of exactly $n!$ elements.*

Proof. Clearly $p^{-1}(A)$ is a finite set iff A has n distinct eigenvalues. Hence $\mathcal{N}_n^r = p(\mathbb{D}^r(n,\mathbb{C}) \times \mathbb{U}_n/\mathbb{UD}_n)$. Furthermore

$$p : \mathbb{D}^r(n,\mathbb{C}) \times \mathbb{U}_n/\mathbb{UD}_n \to \mathcal{N}_n^r \tag{2.6}$$

is a covering map. Observe that $\mathbb{D}(n,\mathbb{C})\backslash\mathbb{D}^r(n,\mathbb{C})$ is a complex algebraic subvariety. Hence $\mathbb{D}^r(n,\mathbb{C})$ and $\mathbb{D}^r(n,\mathbb{C}) \times \mathbb{U}_n/\mathbb{UD}_n$ are connected sets. Hence $\mathcal{N}_n^r$ is a connected manifold. Therefore $\mathcal{N}_n$ is an irreducible variety. Let $A \in \mathcal{N}_n^r$. Then $p^{-1}(A)$ is an orbit of $\mathcal{S}_n$. The dimension of $\mathcal{N}_n$ is equal to the dimension of $\mathcal{N}_n^r$. As $\mathbb{D}^r(n,\mathbb{C}) \times \mathbb{U}_n/\mathbb{UD}_n$ is a finite cover of $\mathcal{N}_n^r$

$$\dim \mathcal{N}_n^r = \dim \mathbb{D}^r(n,\mathbb{C}) + \dim \mathbb{U}_n/\mathbb{UD}_n = 2n + \dim \mathcal{F}_n = 2n + n(n-1) = n(n+1).$$

□

Recall Ikramov's result [Ik1] that $\mathcal{N}_n^r \cap \mathbf{M}(n,\mathbb{R})$ is not connected for $n \geq 2$. One can deduce this result by showing that there is no path in $\mathcal{N}_n^r \cap \mathbf{M}(n,\mathbb{R})$ connecting any symmetric matrix with any skew symmetric matrix.

Lemma 2.5 *$\mathcal{N}_n^r$ is not orientable for $n \geq 2$.*

Proof. The action of $\mathcal{S}_n$ preserves the orientation of $\mathbb{D}^r(n,\mathbb{C})$. Lemma 2.3 claims that the action of $\mathcal{S}_n$ on $\mathbb{U}_n/\mathbb{UD}_n$ does not preserves the orientation of $\mathbb{U}_n/\mathbb{UD}_n$. Use Lemma 2.4 to deduce that $\mathcal{N}_n^r$ is not orientable. □

In what follows we need to consider compact models of $\mathcal{N}_n$. The first way is to consider the real projective variety $\mathcal{PN}_n \subset \mathbb{PR}^{2n^2-1}$, by identifing a line through the origin $L \subset \mathcal{N}_n$ with a point $\lambda \in \mathcal{PN}_n$. That is, $\mathcal{PN}_n$ is defined by the equations (2.4) where $X = (x_{ij})_1^n$, $Y = (y_{ij})_1^n$ are viewed as the homogeneous coordinates in the projective space $\mathbb{PR}^{2n^2-1}$. The second way is by considering the variety of normal matrices with the Frobenius norm equal to 1:

$$\mathcal{N}_{n,2} := \{A \in \mathcal{N}_n : \quad \text{trace } (AA^*) = 1\}.$$

Note that $\mathcal{N}_{n,2}$ is a double cover of $\mathcal{PN}_n$ obtained by identifying the points A and $-A$. Lemma 2.4 implies that the varieties $\mathcal{N}_{n,2}$, $\mathcal{PN}_n$ have singular points. It is of interest to find the CW decomposition of these varieties, and in particular the homology groups of $\mathcal{N}_{n,2}$ and $\mathcal{PN}_n$. A useful observation that can help to answer these questions is

$$p^{-1}(\mathcal{N}_{n,2}) = S^{2n-1} \times \mathbb{U}_n/\mathbb{UD}_n.$$

Here we identified the variety of all $n \times n$ complex diagonal matrices of Frobenius norm 1 with the $2n-1$ dimensional sphere S^{2n-1}. Hence $\mathcal{N}_{n,2}$ and $\mathcal{PN}_n$ are irreducible varieties, and the quasivariety of its smooth points are connected nonorientable manifolds.

3 Main result

Theorem 3.1 *Let $a_{ij},\ i = 1, ..., n,\ j = i, ..., n$ be any set of $\frac{n(n+1)}{2}$ complex numbers. Then there exists a normal matrix $X = (x_{ij})_1^n \in \mathcal{N}_n$ such that $x_{ij} = a_{ij}$ $i = 1, ..., n,\ j = i, ..., n$.*

Proof. Let $\tau : \mathcal{N}_n \to \mathbb{C}^{\frac{n(n+1)}{2}}$ be the projection

$$\tau((x_{ij})_1^n) = (x_{11}, ..., x_{1n}, x_{22}, ..., x_{2n}, ..., x_{nn}).$$

We claim that

$$\tau^{-1}(\tau(D)) = D \quad \text{for all} \quad D \in \mathbb{D}(n, \mathbb{C}). \tag{3.1}$$

Suppose that

$$\tau(X) = \tau(D), \quad X = (x_{ij})_1^n,\ D = \text{diag}\,(d_1, ..., d_n).$$

Then X is a lower triangular matrix with $x_{ii} = d_i,\ i = 1, ..., n$. Hence $d_1, ..., d_n$ are the eigenvalues of X. Since X is a lower triangular normal matrix it follows that

$$\sum_{i=1}^{n} |d_i|^2 = \text{trace}\,(XX^*) = \sum_{1 \le j \le i \le n} |x_{ij}|^2.$$

Hence $X = D$. In particular $\tau^{-1}(0) = 0$.

Let $\hat{\tau} : \mathcal{N}_{n,2} \to S^{n(n+1)-1}$ be given by

$$\hat{\tau}(X) = \frac{\tau(X)}{||\tau(X)||_2}, \quad X \in \mathcal{N}_{n,2}.$$

Here by $||x||_2$ we denote the l_2 norm of a vector $x \in \mathbb{C}^m$. Since $\mathcal{N}_n$ is real homogeneous variety our theorem is equivalent to the statement that $\hat{\tau}$ is onto map. To show that $\hat{\tau}$ is onto we apply the degree theory. The standard *mod* 2 degree theory considers a continuous map $\tilde{\tau} : \tilde{\mathcal{N}}_{n,2} \to S^{n(n+1)-1}$, where $\tilde{\mathcal{N}}_{n,2}$ is a smooth manifold. Recall that $\mathcal{N}_{n,2}$ is a real variety given by the equations (2.4) and the equation trace $(XX^T + YY^T) = 1$. Hironaka's resolution of singularities [GH] implies that it is possible to blow up the singularities of $\mathcal{N}_{n,2}$ to obtain a compact smooth algebraic variety $\tilde{\mathcal{N}}_{n,2}$. That is, there is a projection $\pi : \tilde{\mathcal{N}}_{n,2} \to \mathcal{N}_{n,2}$ with the following properties. Let $\mathcal{N}^r_{n,2} := \mathcal{N}_{n,2} \cap \mathcal{N}^r_n$ be the quasivariety of smooth points of $\mathcal{N}_{n,2}$. Then

$$\pi : \pi^{-1}(\mathcal{N}^r_{n,2}) \to \mathcal{N}^r_{n,2}$$

is a diffeomorphism. Furthermore, $\tilde{\mathcal{N}}_{n,2} \backslash \pi^{-1}(\mathcal{N}^r_{n,2})$ is a strict subvariety of $\tilde{\mathcal{N}}^r_{n,2}$. Set

$$\tilde{\tau} = \hat{\tau} \circ \pi : \tilde{\mathcal{N}}^r_{n,2} \to S^{n(n+1)-1}.$$

Let $D \in \mathbb{D}^r(n, \mathbb{C})$ and assume that D has the Frobenius norm 1. Then $\tau(D) \in S^{n(n+1)-1}$. (3.1) yields that $\hat{\tau}^{-1}(\tau(D)) = D$. Since $D \in \mathcal{N}^r_{n,2}$ it follows that $\pi^{-1}(D)$ consists of one point. We claim that $\hat{\tau}|\mathcal{N}^r_{n,2}$ is an immersion at any $D = \mathrm{diag}\ (d_1, ..., d_n) \in \mathbb{D}^r(n, \mathbb{C}) \cap \mathcal{N}^r_{n,2}$.

Indeed, let $A \in \mathcal{N}_n$ be a normal matrix close to D. Then A has eigenvalues $d_1 + z_1, ..., d_n + z_n$ such that $d_i + z_i \neq d_j + z_j$ for $i \neq j$. Hence $A \in \mathcal{N}^r_n$. Let $Z := \mathrm{diag}\ (z_1, ..., z_n)$. Then $A = U(D + Z)U^*$, $U \in \mathbb{U}_n$. Recall that $U = e^V$, $V^* = -V \in \mathbf{M}(n, \mathbb{C})$. Let $V = (v_{ij})^n_1$. Hence the tangent space of $\mathcal{N}^r_n$ at D is given by the matrices of the form

$$\begin{aligned} &A - D = Z + DV - VD = \mathrm{diag}\ (d_1 + z_1, ..., d_n + z_n) + (v_{ij}(d_i - d_j))^n_{i,j=1}, \\ &v_{11} = ... = v_{nn} = 0, \quad v_{ji} = -\overline{v}_{ij} \text{ for } j > i. \end{aligned} \tag{3.2}$$

Here the $n(n+1)$ real coordinates of the tangent space are the real and the imaginary parts of $z_1, ..., z_n$ and v_{ij} for $j > i$. Hence the tangent space of $\mathcal{N}^r_{n,2}$ at D is given by the matrices of the form (3.2) with an additional condition

$$\sum_{i=1}^n \mathrm{Re}\, z_i \overline{d}_i = 0.$$

Therefore $\hat{\tau}|\mathcal{N}^r_{n,2}$ is an immersion at D. This implies that the local degree of $\hat{\tau}|\mathcal{N}^r_{n,2}$ is ± 1. Hence the *mod* 2 degree of $\tilde{\tau}$ is 1 [Mil]. In particular $\tilde{\tau}$ is onto. Hence $\hat{\tau}$ is onto. □

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