Combinatorial Optimization
Solution of Selected problems

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1 Linear Programming - Appendix of [1]

Problem A.1. Prove that there exists a vector $\mathbf{x} \ge \mathbf{0}$, , such that $A\mathbf{x} \le \mathbf{b}$, if and only if for each $\mathbf{y} \ge \mathbf{0}$ satisfying $\mathbf{y}^\top A \ge \mathbf{0}$ one has $\mathbf{y}^\top \mathbf{b} \ge 0$. (We assume that $A = [a_{ij}] \in \mathbb{R}^{m\times n}$, and $\mathbf{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n, \mathbf{y} = (y_1, \dots, y_m)^\top, \mathbf{b} = (b_1, \dots, b_m)^\top \in \mathbb{R}^m$.)
Solution. The **if part.** We assume that there exists a vector $\mathbf{x} \ge \mathbf{0}$ such that $A\mathbf{x} \le \mathbf{b}$. Suppose that $\mathbf{y} \ge \mathbf{0}$ satisfying $\mathbf{y}^\top A \ge \mathbf{0}$. First, note that since $\mathbf{x} \ge \mathbf{0}$ and $\mathbf{y}^\top A \ge \mathbf{0}$ we get that $(\mathbf{y}^\top A)\mathbf{x} \ge 0$. Second, note that $A\mathbf{x} \le \mathbf{b}$ is equivalent to the coordinate inequalities: $(A\mathbf{x})_i \le b_i$ for each $i \in [m] = \{1, \dots, m\}$. $\mathbf{y} \ge \mathbf{0}$ means $y_i \ge 0$ for each $i \in [m]$. Hence $y_i(A\mathbf{x})_i \le y_i b_i$ for $i \in [m]$. Summing on all $i \in [m]$ we obtain

$$0 \le (\mathbf{y}^\top A)\mathbf{x} = \mathbf{y}^\top (A\mathbf{x}) = \sum_{i=1}^m y_i (A\mathbf{x})_i \le \sum_{i=1}^m y_i b_i = \mathbf{y}^\top \mathbf{b}.$$

The **only if part** Suppose there is no $\mathbf{x} \ge \mathbf{0}$ such that $A\mathbf{x} \le \mathbf{b}$. This is equivalet to the statement that the system

$$\hat{A}\mathbf{x} \le \hat{\mathbf{b}}, \quad \hat{A} = \begin{bmatrix} A \\ -I_n \end{bmatrix} \in \mathbb{R}^{(m+n)\times n}, \ \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0}_n \end{bmatrix} \in \mathbb{R}^{m+n}. \tag{1.1}$$

is not solvable. (Here $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix and $\mathbf{0}_n \in \mathbb{R}^n$ is the zero vector.) By Farkas Lemma (Theorem A.1) there exists $\hat{\mathbf{y}} \in \mathbb{R}^{m+n}$ such that $\hat{\mathbf{y}} \ge \mathbf{0}$, $\hat{\mathbf{y}}^\top \hat{A} = \mathbf{0}$ and $\hat{\mathbf{y}}^\top \hat{\mathbf{b}} < 0$. Write $\hat{\mathbf{y}}^\top = (\mathbf{y}_1^\top, \mathbf{y}_2^\top), \mathbf{y}_1 \in \mathbb{R}^m, \mathbf{y}_2 \in \mathbb{R}^n$. Then $\hat{\mathbf{y}} \ge \mathbf{0}$ is equivalent to $\mathbf{y}_1, \mathbf{y}_2 \ge \mathbf{0}$. As $\hat{\mathbf{y}}\hat{A} = \mathbf{y}_1^\top A - \mathbf{y}_2^\top I_n = \mathbf{y}_1^\top A - \mathbf{y}_2$ it follows that the condition $\hat{\mathbf{y}}^\top \hat{A} = \mathbf{0}$ is equivalent to $\mathbf{y}_1^\top A = \mathbf{y}_2^\top$. As $\mathbf{y}_2 \ge \mathbf{0}$ it follows that $\mathbf{y}_1^\top A \ge \mathbf{0}$. Next observe that $\hat{\mathbf{y}}^\top \hat{\mathbf{b}} = \mathbf{y}_1^\top \mathbf{b}$. In conclusion we showed that there exists $\mathbf{y}_1 \ge \mathbf{0}$ such that $\mathbf{y}_1^\top A \ge \mathbf{0}$ and $\mathbf{y}_1^\top \mathbf{b} < 0$. This concludes the proof of **only if part**.

Problem A.2. Prove that there exists a vector $\mathbf{x} > \mathbf{0}$ that that $A\mathbf{x} = \mathbf{0}$ if and only if for each $\mathbf{y}$ satisfying $\mathbf{y}^\top A \ge \mathbf{0}$ one has $\mathbf{y}^\top A = \mathbf{0}$.

Solution. The **if part.** We assume that there exists a vector $\mathbf{x} > \mathbf{0}$ that that $A\mathbf{x} = \mathbf{0}$. Suppose that there exists $\mathbf{y}$ satisfying $\mathbf{y}^\top A \geq \mathbf{0}$. Observe that

$$0 = \mathbf{y}^\top \mathbf{0} = \mathbf{y}^\top (A\mathbf{x}) = (\mathbf{y}^\top A)\mathbf{x} = \sum_{i=1}^n (\mathbf{y}^\top A)_i x_i.$$

As $x_i > 0$ and $(\mathbf{y}^\top A)_i \geq 0$ we deduce that $(\mathbf{y}^\top A)_i x_i \geq 0$ and equality holds if and only if $(\mathbf{y}^\top A)_i = 0$. Hence $(\mathbf{y}^\top A)\mathbf{x} = \sum_{i=1}^n (\mathbf{y}^\top A)_i x_i \geq 0$ and equality holds if and only if $\mathbf{y}^\top A = \mathbf{0}$. As we showed above$(\mathbf{y}^\top A)\mathbf{x} = 0$. Hence $\mathbf{y}^\top A = \mathbf{0}$.
The **only if part** Suppose that there is no vector $\mathbf{x} > \mathbf{0}$ that that $A\mathbf{x} = \mathbf{0}$. This is equivalent to the following statement that for each $\varepsilon > 0$ there is no solution to the system $A\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \geq \varepsilon \mathbf{1}_n$, where $\mathbf{1}_n = (1, \ldots, 1)^\top \in \mathbb{R}^n$. The nonsolvability of the above system is equivalent to the nonsolvability of the following system of inequalities:

$$\hat{A}\mathbf{x} \leq \hat{\mathbf{b}}, \quad \hat{A} = \begin{bmatrix} A \\ -A \\ -I_n \end{bmatrix} \in \mathbb{R}^{(2m+n)\times n},\ \hat{\mathbf{b}} = \begin{bmatrix} \mathbf{0}_m \\ \mathbf{0}_m \\ -\varepsilon \mathbf{1}_n \end{bmatrix} \in \mathbb{R}^{2m+n}$$

By Farkas Lemma (Theorem A.1) there exists $\hat{\mathbf{y}} \in \mathbb{R}^{2m+n}$ such that $\hat{\mathbf{y}} \geq \mathbf{0}$, $\hat{\mathbf{y}}^\top \hat{A} = \mathbf{0}$ and $\hat{\mathbf{y}}^\top \hat{\mathbf{b}} < 0$. Write $\hat{\mathbf{y}}^\top = (\mathbf{y}_1^\top, \mathbf{y}_2^\top, \mathbf{y}_3^\top), \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^m, \mathbf{y}_3 \in \mathbb{R}^n$. Then $\hat{\mathbf{y}} \geq \mathbf{0}$ is equivalent to $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \geq \mathbf{0}$. As $\hat{\mathbf{y}}\hat{A} = \mathbf{y}_1^\top A - \mathbf{y}_2^\top A - \mathbf{y}_3^\top I_n = (\mathbf{y}_1 - \mathbf{y}_2)^\top A - \mathbf{y}_3$ it follows that the condition $\hat{\mathbf{y}}^\top \hat{A} = \mathbf{0}$ is equivalent to $(\mathbf{y}_1 - \mathbf{y}_2)^\top A = \mathbf{y}_3^\top$. As $\mathbf{y}_3 \geq \mathbf{0}$ it follows that $(\mathbf{y}_1 - \mathbf{y}_2)^\top A \geq \mathbf{0}$. Next observe that $\hat{\mathbf{y}}^\top \hat{\mathbf{b}} = (\mathbf{y}_1 - \mathbf{y}_2)^\top \mathbf{0}_m - \varepsilon \mathbf{y}_3^\top \mathbf{1}_n = -\varepsilon \mathbf{y}_3^\top \mathbf{1}_n$. Hence the conditions $\hat{\mathbf{y}}^\top \mathbf{b} < 0$ is equivalent to $\mathbf{y}_3^\top \mathbf{1}_n > 0$. So $\mathbf{y}_3 \gneq \mathbf{0}$. In conclusion we showed that there exists $\mathbf{y}_4 = \mathbf{y}_1 - \mathbf{y}_2 \in \mathbb{R}^m$ such that $\mathbf{y}_4^\top A = \mathbf{y}_3^\top \gneq \mathbf{0}$. This concludes the proof of **only if part**.

Problem A.6. Prove that

$$\max\{\mathbf{c}^\top \mathbf{x} \ :\ \mathbf{x} \geq \mathbf{0},\ A\mathbf{x} \leq \mathbf{b}\} = \min\{\mathbf{y}^\top \mathbf{b} \ :\ \mathbf{y} \geq 0, \mathbf{y}^\top A \geq \mathbf{c}^\top\} \tag{1.2}$$

assuming that both sets are nonempty. We now show that the above equality follows from Theorem A.5 (Duality Theorem). We first use the solution of Problem A.1 to show that the system $\mathbf{x} \geq \mathbf{0}, A\mathbf{x} \leq \mathbf{b}$ is equivalent to (1.1). Hence the left-hand side of (1.2) is $\max\{\mathbf{c}^\top \mathbf{x}\ :,\ \hat{A}\mathbf{x} \leq \hat{\mathbf{b}}\}$, and the given set is nonempty. The duality theorem for this maximum problem is that the above maximum is equal to the following minimum: $\min\{\hat{\mathbf{y}}^\top \hat{\mathbf{b}}\ :\ \hat{\mathbf{y}} \geq 0, \hat{\mathbf{y}}^\top \hat{A} = \mathbf{c}^\top\}$, provided that the second set is also nonempty. Write $\hat{\mathbf{y}}^\top = (\mathbf{y}_1^\top, \mathbf{y}_2^\top)$, where $\mathbf{y}_1 \in \mathbb{R}^m, \mathbf{y}_2 \in \mathbb{R}^n$. So $\hat{\mathbf{y}} \geq \mathbf{0}$ is equivalent to $\mathbf{y}_1, \mathbf{y}_2 \geq \mathbf{0}$, Note that $\hat{\mathbf{y}}^\top \hat{A} = \hat{\mathbf{y}}_1 A - \mathbf{y}_2^\top$. So $\hat{\mathbf{y}}\hat{A} = \mathbf{c}^\top$ is $\mathbf{y}_1^\top A - \mathbf{y}_2^\top = \mathbf{c}^\top$. Equivalently, $\mathbf{y}_2^\top = \mathbf{y}_1^\top A - \mathbf{c}^\top$. Hence the assumption that $\mathbf{y}_2 \geq \mathbf{0}$ is equivalent to $\mathbf{y}_1^\top A \geq \mathbf{c}^\top$. Therefore the solvalbility of the system given in the right-hand side of (1.2) is equivalent to the solvability of $\hat{\mathbf{y}} \geq 0, \hat{\mathbf{y}}\hat{A} = \mathbf{c}^\top$. Thus we showed that (1.2) follows from the duality theorem for the maximum problem $\max\{\mathbf{c}^\top \mathbf{x}\ :,\ \hat{A}\mathbf{x} \leq \hat{\mathbf{b}}\}$.

Problem A.9.(i) Prove that for ay matrix $A \in \mathbb{R}^{m\times n}$ and vectors $\mathbf{b} \in \mathbb{R}^m, \mathbf{c} \in \mathbb{R}^n$ one has

$$\sup\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \leq \mathbf{b}\} = \inf\{\mathbf{y}^\top \mathbf{b} : \mathbf{y} \geq \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\}, \tag{1.3}$$

provided that at least one set is nonempty.

Solution. If both set are nonempty then the above equality follows from the Duality Theorem (Theorem A.5). So we assume now that one set is feasible and another one is not. Assume for example that the set $\{A\mathbf{x} \le \mathbf{b}\}$ is not empty, while the set $\{\mathbf{y} \ge \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\}$ is empty. We claim that this is equivalent to the equality $sup\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \le \mathbf{b}\} = \infty$. Example: $x \le 0$ and $c = -1$. (Here $m = n = 1$ and $A = [1], \mathbf{b} = 0$. Then $y \ge 0 andy = c$ is not feasible. Clearly, if $sup\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \le \mathbf{b}\} = \infty$ the set $\{\mathbf{y} \ge \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\}$ is empty by the weak duality theorem. So it is left to show that if $sup\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \le \mathbf{b}\} = \delta < \infty$ then the set $\{\mathbf{y} \ge \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\}$ is nonempty. Indeed, apply Corollary A.3. Hence, to conclude the proof of (1.3) we need to define that $\inf\{\mathbf{y}^\top \mathbf{b} : \mathbf{y} \ge \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\} = \infty$ if the set $\{\mathbf{y} \ge \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\}$ is empty.

Similarly we define $\sup\{\mathbf{c}^\top \mathbf{x} : A\mathbf{x} \le \mathbf{b}\} = -\infty$ if the set $\{\mathbf{x}, A\mathbf{x} \le \mathbf{b}\}$ is empty. Then in a similar way we prove that if the set $\{\mathbf{x}, A\mathbf{x} \le \mathbf{b}\}$ is empty and the set $\{\mathbf{y} \ge \mathbf{0}, \mathbf{y}^\top A = \mathbf{c}^\top\}$ is nonempty then (1.3) holds.
(ii) Assume that $m = n = 1$, $A = 0$ and $b = c = -1$.

Problem: Proof of Corollary A.6:

$$\max\{\mathbf{c}^\top \mathbf{x} : \mathbf{x} \ge 0, A\mathbf{x} = \mathbf{b}\} = \min\{\mathbf{y}^\top \mathbf{b} : \mathbf{y}^\top A \ge \mathbf{c}^\top\}. \tag{1.4}$$

Prove Corollary A.6 from Theorem A.5 by. Do the following substitutions in (1.4). Call $\mathbf{y} = -\mathbf{x}', \mathbf{x} = \mathbf{y}', A' = A^\top, \mathbf{b}' = -\mathbf{c}, \mathbf{c}' = \mathbf{b}$. Then $y^\top \mathbf{b} = -(\mathbf{x}')^\top \mathbf{c} = -(\mathbf{c}')^\top \mathbf{x}'$ and $\mathbf{y}^\top A \ge \mathbf{c}^\top$ is equivalent to $A'\mathbf{x}' \le \mathbf{b}'$. Hence the right-hand side of (1.4) is $-\max\{(\mathbf{c}')^\top \mathbf{x}', A'\mathbf{x}' \le \mathbf{b}'\}$. Next $\mathbf{c}^\top \mathbf{x} = -(\mathbf{b}')^\top \mathbf{y}'$ subject to $\mathbf{y}' \ge 0, (\mathbf{y}')^\top A' = (\mathbf{c}')^\top$. Hence (1.4) is equivalent to

$$-\min\{(\mathbf{b}')^\top \mathbf{y}', \mathbf{y}' \ge \mathbf{0}, (\mathbf{y}')^\top A' = (\mathbf{c}')^\top\} = -\max\{(\mathbf{c}')^\top \mathbf{x}', A'\mathbf{x}' \le \mathbf{b}'\},$$

which is equivalent to Theorem A.5.

2 Chapter 2

2.1 §2.1: Minimum spanning trees

Problem 2.1: *Kruskal algorithm*: The choice of edges and their weight: ((p,d),2); ((h,g),3); ((a,r),4); ((d,f),5); (a,q),7); ((r,h),8); ((g,f),9);((b,r),13). Total weight 51.
Prim algorithm from r: ((r,a),4); ((a,q),7); ((r,h),8);((h,g),3); ((g,f),9);((f,d),5);((d,p),2);((r,b),13). Total weight 51.
Problem 2.6: I. *Use Kruskal algorithm* until one is left with no vertices or isolated vertices.
II. *Use Prim algorithm* from a vertex r unitl it stops. Remove all the verttices on this tree, to obtain a subgraph G' of G. If some vertices are left, restart Prim's algo on the corresponding subgraph. Continue in this manner until one is left with no vertices or isolated verstices.
Problem 2.7: Assume that $G = (V, E)$. Let E' be the set of all edges with negative and zero costs. Let $G(E') = G' = (V', E')$ be the induced subgraph G by E'. If $V' = V$ and G' is connected we are done. Otherwise let $G_i = (V_i, E_i'), i = 1, \ldots, k$ be the connected components of G'. Let $\hat{G} = (\hat{V}, \hat{E})$ be the induced subgraph of G,

where we view each set of vertices V_i as one vertex $\{V_i\}, i = 1, \ldots, k$. If $v \in V$ is not a vertex in V' we have this vertex in $\hat{V}$. An edge $\hat{e} \in \hat{E}$, if it is was edge $e \in E$ from some vertex $u \in V_i$, or $u \in V \setminus V'$ to another vertex different $v \in V_i$, or $v \in V \setminus V'$. Note that now $\hat{G}$ may have multiple edges. Now find in $\hat{G}$, where the weight of each edge is positive, a MST. Add the edges in G, corresponding to MST in $\hat{G}$, to E' to obtain a solution to the connector problem.
Problem 2.8: Recall that a spanning tree of $G = (V, E)$ has exaclty $|V| - 1$ edges. Assume that $c : E \to \mathbb{R}$ are the costs of edges. Let $c' : E \to \mathbb{R}$ be a another cost of edges, where $c'(e) = c(e) + t$ for some fixed t and all $e \in E$. Then for each spanning tree $T = (V, E(T))$ of e have that $c'(T) = c(T) + (|V| - 1)t$. Henc a minimum spanning tree of G with respect to the cost c is also a minimum spanning tree with respect to c'. Now choose t big enough, for example $t = 1 - \min\{c(e), e \in E, c(e) < 0\}$ to reduce the MST problem with positive costs.
Problem 2.9: Consider the subgraph of MST $H' = (V, T \setminus e)$. It has teo connected components $H_1 = (V_1, T_1), H_2 = (V_2, T_2)$, where each H_i is a tree, and $V_1 \cup V_2 = V, T_1 \cup T_2 = T \setminus e$. Let $D = \delta(V_1, V_2) \subset E$ be the cut between V_1 and V_2. So $e \in D$. Let $f \in D$ be an edge with the minimal weight in D. So $c_f \leq c_e$. Clearly, $T' = T_1 \cup T_2 \cup \{f\}$ is a spanning tree of G. Also $c(T) - c(T') = c_e - c_f \geq 0$. But T is a MST. So $c(T) = c(T')$, hence $c_e = c_f$.
Problem 2.10. Add an edge $e = vw \in E \setminus T$ to T. We have now a cycle in $H' = (V, T \cup \{e\}$. we have a unique simple path from v to w in T and an edge vw. Suppose that on this path from v to w in T we had an edge f such that $c_f > c_e$. Remove this edge from $T \cup \{e\}$ to obtain a new tree $T' = (T \cup \{e\}) \setminus \{f\}$. Clearly $c(T') = c(T) + c_e - c_f < c(T)$, which contradicts that T is MST. Hence $c_f \leq c_e$ for each edge f in T in the path from v to w on T.

Assume now that we have a spanning tree T with the property that for each edge $e = vw \in E \setminus T$ and each edge $f \in T$ in the path from v to w $c_f \leq c_e$. We claim that T is a MST. We show that our tree is obtained by using Kruskal's algorithm. Assume that $f_1 \in T$ is has minimal cost from all edges of T. We claim that f_1 has the minimum cost in E. Suppose not. So there exists $e_1 = v_1w_1 \in E$ such that $c_{e_1} < c_{f_1}$. So $e_1 \notin T$. According to our assumption for each edge f in the path for v_1w_1 in T we have that $c_e \geq c_f \geq c_{f_1}$ which is contradiction. Assume now that in the stage $k-1$ of Kruskal's algorithm we can choose an edge from our tree T. It is left to show that in the stage k we can choose the next edge from our tree. Suppose not. So we chose the edge $e_k = v_kw_k$ of the smallest cost that is connecting between different connected components of the forest spanned by the subtrees of our tree T. There is a path in T from v_k to w_k. In this path there is at least one edge f_k that also connects the forest spanned by our tree in the $k-1$ stage of Kruskal's algorithm. By our assumption we $c_{f_k} \leq c_{e_k}$. This contradicts that we can not choose in the stage k of Kruskal's algorithm an edge from T. Hence, our T is given by Kruskal's algorithm, hence it is a MST.
Problem 2.16: Claim: MST $H = (V, T)$ is also a solution to the min max problem for finding a spanning tree whose maximal cost edge is minimal. For a spanning tree $H' = (V, T')$ let $\mu(T') = \max\{c_{e'}, e' \in T'\}$. Suppose not. Then there exists a spanning tree $H' = (V, T')$ such $\mu(T') = c_{e'} < \mu(T) = c_e$. Consider the disconnected graph $H_1 = (V, T \setminus e)$ which has two connected components. Since H' is connected, there is $f' \in T'$ which connects these two connected components. So $c_{f'} \leq c_{e'} < c_e$. Let $T_2 = (T \setminus e) \cup \{f'\}$. Then $H_2 = (V, T_2)$ is a spanning tree and $c(T_2) = c(T) -$

$c_e + c_{f'} < c(T)$, contradicting that T is a MST.

A solution to min max problem does not have to be a minimal spanning tree. Assume that $G = K_3 = (\{u, v, w\}, \{(u, v), (u, w), (v, w)\}$, where $c_{(u,v)} = c_{(u,w)} = 2, c_{(v,w)} = 1$. Then $T' = \{(u, v), (u, w)\}$ solves the min max problem but is not a MST.

2.2 §2.2: Shortest paths

Problem 2.18. Consider the digraph $G = (V, E)$ with $V = \{r, a, b, c\}$ and the diedges with the corresponding costs: $((r, a), 2), ((r, b), 4), ((a, b), 3), ((b, c), 4)$. A minium spanning ditree is $((r, a), 2), ((a, b), 3), ((b, c), 4)$. However, teh shortest path from r to b is the arc $(r, b), 4)$. The arcs on the spanning tree $((r, a), 2), ((r, b), 4), ((b, c), 4)$ from a spanning ditree, so each path on it of the minimal length. However, this is not a MST.

Problem 2.21. An acyclic sort: $h = 9, j = 8, g = 7, f = 6, d = 5, k = 4, b = 3, a = 2, r = 1$. Then the triples: $v, y_v, r(v)$ are given by

$$(r, 0, 0), (a, 2, r), (b, 5, r), (k, 7, r), (d, 9, k), (f, 6, a), (g, 9, f), (j, 12, g), (h, 13, g).$$

Problem 2.22. Add new vertices r, s to $G = (V, E)$ and the following arcs with the zero cost. Arcs (r, v) for each $v \in R$, and arcs (w, s) for each $w \in S$. Call the new graph $G' = (V', E')$. Find the least-cost dipath from r to s in G'. It is of the form rPs, where P is the least cost dipath in G from R to S.

Problem 2.23. There are fou possibilities for two arcs that are incident to w. Here u and v are two distinct vertices in the digraph $G = (V, E)$ which are different from w (1): $(u, w), (v, w)$; (2) $(u, w), (w, v)$; (3) $(w, u), (w, v)$; (4) $(u, w), (w, u)$.

So we delete vertex w from G, and add a new edge (u, v) with the cost $c'_{(u,v)} = c_{(u,w)} + c_{(w,v)}$ in the case (2). If we had already a diedge (u, v) in our digraph, then we choose the new cost of this edge as $\min(c_{(u,v)}, c'_{(u,v)})$. Now find minimal cost dipaths in the smaller graph from r to any vertex in the smaller graph. In the case (1) the smallest cost dipath form r to w is either to go to u and then to w or go to v and then to w, whatever is the smallest cost. In the case (2) , and (4) it is the cost to go to u and then from u to w. In the case (3) w is not reachable from r.

Problem 2.34. Label vertices $V = \{1, \ldots, n + 1\}$. Then consider the following diedges with the following costs: First $((i, i + 1), a_i)$ for $i = 1, \ldots, n$. Then diedges $((1, i), 0)$ for $i = 2, \ldots, n$, and $((i, n + 1), 0)$ for $i = 2, \ldots, n$. This is an acyclic graph. All dipaths from $r = 1$ to $s = n + 1$ are of the form $(1, j), (j, j + 1), \ldots, (k - 1, k), (k, n + 1)$. The cost of each dipath is $\sum_{k=i}^{j-1} a_k$. for all $1 \le i \le j \le n + 1$. (Note that if $i = j$ we agree that the above sum is 0.) Thus we need to find the minimum cost path. Note that teh number of edges is $n + (n - 1) + (n - 1) = 3n - 2$. Hence the complexity of finding the smallest cost path is $O(m) = O(n)$.

Problem 2.35. First we construct the following graph on $2k$ vertices $\{t_1, t'_1, \ldots, t_k, t'_k\}$. We first have diedge (t_i, t'_i) with cost p_i. We have a didedge from (t'_i, t_j) if job i needed to be performed before the job j. We assume that we have an acyclic graph. We also assume that for each j on has at most one job i that proceeds it. (Otherwise everything becomes much more complicated. See the last paragraph of the solution.) Recall that in each acyclic graph $G = (V, E)$ we must have a set nonempty set of sources $U \subseteq V$, which the set of all $v \in V$ such that there is no $u \in V$ such that $(u, v) \in E$. In our case the set of sources are of the form t_i for some $i \in I$, where

$I \subseteq \{1, \ldots, k\}$. Those are the jobs whose completion do not need other jobs to be completed. Introduce a new vertex r and the diedges (r, t_i) where $i \in I$. The cost of these edges is 0. The cost of the edge (t_i, t_i') is p_i. The cost of (t_i', t_j) IS ALSO 0. Then we have a family of dipaths which do no have any common vertices except of the beginning vertex r. On each dipath from r we are doing the processing sequentially. The time to finish all jobs is the cost of the longest path.

If however, a job j needs more than one preceding jobs to finish, then we have a more complicated situation. For each j let $P(j)$ be the set of all preceding jobs. Then we have to introduce additional vertices in our graph, denoted as $P(j)$, where $P(j)$ is not empty. If $P(j) = \{i\}$, then we identify $P(j)$ with t_i' as before. Otherwise $P(j)$ is an extra vertex. We have edges $(t_i', P(j))$ for each $i \in P(j)$. Now we need to give COSTS to diedges $(t_i', P(j))$. Suppose we already computed the the cost of the minimum dipath from r to t_i' for $i \in P(j)$, which we denote by y_i. The we take the maximum of y_i for all $i \in P(j)$ and call it μ_j. Then the cost of $(t_i', P(j))$ is $\mu_j - y_i$. Then the cost of the path from r to $P(j)$ is μ_j, which is the time we need to wait before we can do the job j. Finally we have an arc $(P(j), t_j)$ with cost 0.

3 Chapter 3

3.1 §3.1: Network Flow Problems

Problem 3.3: Prove directly that there is no maximum flow if and only if there is an (r, s) dipath , each of arcs has $u_e = \infty$. **Proof**. Assume first that there is a dipath $v_0 v_1 \cdots v_k$ such that $v_0 = r, v_k = s$ and $u_{v_{i-1}v_i} = \infty$. Let $x_{v_{i-1}v_i} = M$ for $i = 1, \ldots, k$ and $x_e = 0$ for all other diedges. Then x is a flow, and it satisfies the condition $0 \leq x_e \leq u_e$ on all diedges of G. Clearly, $f_x(s) = M$. SInce we can choose M as big as we wish, there is no maximum flow.

Assume now that there is no maximum flow. That is any (r, s)-cut has inifinite value, i.e., any (r, s)-cut contains an edge with with an infinite capacity. Let $G' = (V, E')$ be the subgraph of $G = (V, E)$ such that E' is the subset of E of all edges with infinite capacity. Let $R \subseteq V$ be the set of all vertices in V with are reachable by a dipath from r. So $r \in R$. We claim that R contains s. Suppose not. Then R is an (r, s) cut. Clearly, $\delta'(R)$ the set of all diedges from R to $\bar{R} = V \setminus R$ is empty. Otherwise R is not the set of all vertices in V which can be reach by a dipath from r in G'. Hence $\delta(R)$ in G contains only diedges with a finite arc. But then there is a maximum flow. This contradiction yields that $s \in R$. Hence there exists a dipath from r to s, whose all edges have infinite capacities.

Problem 3.4: Construct

(a) many integral maximum flows and many minimum cuts. For each i consider the following digraph: the dipath $r a_i b_i c_i d_i s$ plus an edge $b_i d_i$ where each edge has capacity 1. Then the maximum value of the flow is 1. There are two maximum integer flows using the dipaths $r a_i b_i c_i d_i s$ and $r a_i b_i d_i s$. There are 3 minimal cuts $\{r\}, \{r, a_i\}, \{r, a_i, b_i, c_i, d_i\}$. Now to have an exponential number of minimal number of cuts and of minimal maximum flows just consider $G = (V, E)$, $V = \{r, s\} \cup_{i=1} \{a_i, b_i, c_i, d_i\}$ with the above edges and capaciities for $i = 1, \ldots, n$. Then the maximum flow has value n, exactly 2^n maximum integer flows, and at least 2^n minimal cuts consisting of either the vertex $\{a_i\}$ or $\{a_i, b_i, c_i, d_i\}$ for $i = 1, \ldots, n$ in addition to the vertex r.

(b) many integral maximum flows and one minimum cuts. Start with a directed bipartite matching: (v_i, w_i) for $i = 1, \ldots, n$. Capacity of $v_i w_i$ is ∞. Take r, and joint it with r' by diedge rr' with capacity 1. Now join r' with each v_i: $r'v_i$ with capacity ∞. Now join w_i with s with capacity ∞. The value of the maximum flow is 1. All maximum integer valued flows are $rr'v_i w_i s$ where the value on this dipath of the flow is 1. One minimal (r, s)-cut: $\{r\}$.
(c) one maximum flows and many minimum cuts. Take a digraph which is a dipath $rv_1 \cdots v_n s$. The capacity of each diedge is one. There is only one maximum flow: $x_e = 1$. Any r cut of the form $rv_1 \cdots v_i$ has capacity 1.
Problem 3.6: A minimum cut is $R = \{r, q, a\}$ with the value of 4. So for any maximal flow we must have the following values: $x_{rp} = 1, x_{qb} = 1, x_{bq} = 0, x_{as} = 1, x_{ap} = 1, x_{p,0} = 0$. Since the value of the maximum flow is 4 we get $x_{bs} = 3$. Hence $x_{pb} = 2$ and $x_{qa} = 2$. Finally $x_{rq} = 3$. So the flow is unique.
Problem 3.7: Assume that $\delta(R_1)$ and $\delta(R_2)$ are minimum cuts. It is straightforward to show the following inclusions:

$$\delta(R_1 \cap R_2), \delta(R_1 \cup R_2) \subseteq \delta(R_1) \cup \delta(R_2), \delta(\overline{R_1 \cap R_2}), \delta(\overline{R_1 \cup R_2}) \subseteq \delta(\bar{R}_1) \cup \delta(\bar{R}_2). \tag{3.1}$$

Assume that x is a maximum flow. Then $x_e = u_e$ for $e \in \delta(R_1) \cup \delta(R_2)$ and $x_e = 0$ for $e \in \delta(\bar{R}_1) \cup \delta(\bar{R}_2)$. The inclusions (3.1) yield that $x_e = u_e$ for $e \in \delta(R_1 \cap R_2)$ and $x_e = 0$ for $e \in \delta(\overline{R_1 \cap R_2})$. As

$$f_s(x) = x(\delta(R_1 \cap R_2)) - x(\delta(\overline{R_1 \cap R_2}) = u(\delta(R_1 \cap R_2)) - 0$$

it follows that $u(\delta(R_1 \cap R_2))$ is minimum cut. Similarly, (3.1) yields that $\delta(R_1 \cup R_2)$ is also a minimum cut.
Problem 3.5: Start with a maximum flow x given as in solution of Problem 3.6. Then construct the graph $G(x)$. Then the set of all vertices that are reachable from r in $G(x)$ is a minimum cut $R = \{r, q, a\}$. By Problem 3.6 x is unique. Suppose that there was another minimal cut R'. By Problem 3.7: $\delta(R \cap R')$ and $\delta(R \cup R')$ are minimal cuts. Suppose that R' is a strict subset of R. Then the reachable set from r is contained in R', which is false. So R' strictly contains R. Suppose that also $b \in R'$. Then as and bs are in this cut, but the sum of these two capacities is already 5, which is not minimal. So the only left possibility is $R' = R \cup \{p\}$. But the value of the maximal flow $x_{pb} =< 3 = u_{pb}$ so this R' is a minimum cut either.
Problem 3.8: We assume that we are given a fixed digraph $G = (V, E)$ with given capacities and the maximum flows x^1 and x^2. Suppose that $v \neq r, s$ and there is an incrementing path from r to v. Let us consider a new digraph $\hat{G}$ by adding an additional edge sv with capacity $f_x(s)$. Now define a new flow from r to v corresponding to x^1, x^2 denoted by $\tilde{x}^1, \tilde{x}^2$ by letting $\tilde{x}^i_{sv} = f_{x^i}(s)$ for $i = 1, 2$, and on the other arcs having gthe same values as the flows x^1, x^2 respectively. The new flows $\tilde{x}^i$ have values $f_{x^i}(s)$, i.e. the maximum flow value in G. Observe nex that an x^1 incrementing path in $G(x^1)$ from r to v is an augmenting path in $\hat{G}(\hat{x}^1)$. So $\hat{x}^1$ is not a maximal flow in $\hat{G}$. Therefore $\hat{x}^2$ is not a maximum flow in. Hence, there exists an augmented $r - v$ dipath from r v. This augmented $r - v$ path is an incrementing $r - v$ path in $G(x^2)$.
Problem 3.9 For each vertex $v \in V$ let

$$g_u(v) = (\sum_{vw \in E} u_{vw}) - (\sum_{w'v \in E} u_{w'v}).$$

Note that for subset $T \subseteq V$ we have that $\sum_{v \in T} g_u(v) = u(\delta(T)) - u(\delta(\bar{T}))$. Note that $R = \{r\} \cup T$, where $T \subseteq V \setminus \{r, s\}$. Let $\tilde{T}$ be the set of all $v \in V \setminus \{r, s\}$ such that $g_u(v) < 0$. Then an optimal R is $\{r\} \cup \tilde{T}$.

3.2 §3.3: Application of maximum flow and minimum cut

Problem 3.17: We assume that each $c_v > 0$. As on page 48, we add a vertex r which is connected to each $v \in P$, and the capacity of rv is s_v. We add vertex s so that for each $q \in Q$ we have a diedge qs with capacity y_q. We orient all edges of the bipartite graph from p to q and the capacity of pq in ∞. As $u(\delta(\{r\}) = \sum_{p \in P} y_p < \infty$ we have a maximum flow with is equal to the minimum capacity. Let us take a minimum (r, s) cut. It will be of the form $\delta(\{r\} \cup A)$ where $A \subseteq V$. Its capacity is $\sum_{p \in P \setminus A} y_p + \sum_{q \in Q \cap A} y_v$, where we assume that there are no edges from $A \cap P$ to $Q \setminus A$. So all edges in the bipartite graph are from $A \cap P$ to $Q \cap A$ and $P \setminus A$ to Q. Thus $(P \setminus A) \cup (A \cap Q)$ is a cover with capacity $\sum_{p \in P \setminus A} y_v + \sum_{q \in Q \cap A} y_v$. So minimum cover weight is found by maximum flow, and the corresponding minimum cut.
Problem 3.18: As in problem 3.17 we add a vertex r which is connected to each $p \in P$, and the capacity of rp is h_p. We connect each $q \in Q$ to s and the capacity of the diedge qs is d_q. Each diedge pq has capacity 1. The such a subdigraph exists if and only there is a maximum flow of capacity $\sum_{p \in P} d_p = \sum_{q \in Q} d_q$. So any (r, s) cut is $\delta(\{r\} \cup A)$ where $A \subseteq V$. Then the capacity of this cut is

$$\sum_{p \in P \setminus A} d_p + \sum_{q \in Q \cap A} d_q + \sum_{p \in P \cap A} (\text{ number of edges from p to } Q \setminus A).$$

This should be alway greater or equal to $\sum_{p \in P} d_p$. This is equivalent to the inequality

$$\sum_{q \in Q \cap A} d_q + \sum_{p \in P \cap A} (\text{ number of edges in G from p to } Q \setminus A) \geq \sum_{p \in P \cap A} d_p,$$

for any subset $A \subset V$.
Problem 3.21: This is a special case of flow feasibility problem discussed on pages 53-54, with reversing the sets P and Q. We set $a_p = 1$ for each $p \in P$ and $b_q = 1$ and each $q \in Q$. To satisfy the demand we need to match each q in Q with some $p \in P$. This is possible if and only if $a(N(C)) \geq b(C)$ for each subset $C \subseteq Q$. (Top of page 54.) This condition is $a(C) = |N(C)| \geq |C| = b(C)$.
Problem 3.23: Assume that we have a bipartite simple undirected graph $G = (V, E)$, where $V = P \cup Q$. Assume that G is $k - regular$, i.e., the degree of each vertex is $k \geq 1$. First we claim that $|P| = |Q|$. Since each vertex $p \in P$ has degree k $|E| = k|P|$. Similarly, $|E| = k|Q|$. Hence $|P| = |Q|$. Next we claim that G has a perfect match. We use Problem 3.21. We need to show that for each subset A of P we have $|N(A)| \geq |A|$. Let consider the subgraph $G' = (A \cup N(A), E')$ of G on the vertices $A \cup N(A)$. As before the number of edges in G' is the number of edges coming out A, which is $k|A = |E'|$. Now let us count the edges in G' coming out of $N(A)$. Since some edges from $N(A)$ can go to $P \setminus A$, the number of edges $|E'| \leq k|N(A)|$. Hence $|A| \leq |N(A)|$. By Problem 3.21 there is match M in G that matched all vertices in P. As $|P| = |Q|$ this match also matches all vertices in Q.

So $M \subset E$ is a perfect match, i.e. $|M| = |P| = |Q|$. Consider $\hat{G} = (V, E \setminus M)$. This is a bipartite graph where the degree of each bertex is $k-1$. Repeat the above arguments to deduce that G has k disjoint perfect matchings.

Problem 3.24: If $G = (V, E)$ is a simple undirected bipartite graph such that each vertex has degree at least k does not insure that G has a perfect matching. Indeed, assume that $P = \{v_1, \ldots, v_m\}, Q = \{w_1, \ldots, w_n\}$ where $m, n > 2k$. Connect each v_i to $w_1, \ldots, w_k$ and each w_j to $v_1, \ldots, v_k$ for each $j > k$. Then the degree of each vertex of G is at least k, but obviously, a maximum match can be at most of size $2k$.

Problem 3.26: For a family of $(S_1, \ldots, S_k)$ of subsets of Q a *system of different representatives* (SDR) is a set $\{q_1, \ldots, q_k\}$ distinct elements of Q such that $q_i \in S_i$ for $i = 1, \ldots, k$. Assume that $(T_1, \ldots, T_k)$ is another family of subsets of Q. Then $\{q_1, \ldots, q_k\}$ is a common SDR if it is is an SDR for the two families.

Assume $[k] = \{1, \ldots, k\}$. To find if a common SDR exists we construct a following digraph $G = (V, E)$ where the vertices of V are $r, s, s_1, \ldots, s_k, t_1, \ldots, t_k$ and the sets of vertices Q, Q' where Q' is another copy of Q. Then we have the following diedges with the following capacities. The diedges rs_i, t_is have capacity 1. We have a diedge s_iq for $q \in Q$ if and on if $q \in S_i$ for each $i \in [1, k]$ and $q \in Q$. The capacity of the s_iq is infinity. Each $q \in Q$ is connected to one $q' \in Q'$, the copy of q in Q' the capacity of such diedge qq' is 1. One has edge $q't_j$ if $q \in Q$, the copy of q' is in T_j. The capacity of $q't_j$ is infinity. We now claim that there exists a common SDR if and only if there is a maximum integer flow of value k.

Indeed, assume that we have a common SDR. After renaming the sets $T_1, \ldots, T_k$ we have that $q_i \in S_i \cap T_i$ for $i = 1, \ldots, k$. Then $x_{rs_i} = x_{s_iq_i} = x_{q_iq_i'} = x_{q_it_i} = x_{t_is} = 1$ for $i = 1, \ldots, k$. This gives a flow of value k.

Vice versa, assume that x is an integer flow of value k. Then $x_{rs_i} = x_{s_iq_i} = x_{q_iq_i'} = x_{q_i't_i} = x_{t_is} = 1$ for all $i = 1, \ldots, k$. This is an (r, s)-flow of value k. Vice versa, suppose we have an integer flow of value k. So $x_{rs_i} = x_{t_is} = 1$ for $i = 1, \ldots, k$. At vertex s_i we have only one $x_{s_iq_{j(i)}} = 1$. Since the capacity of $q_{j(i)}q_{j(i)}'$ is 1, the value of all other $x_{s_pq_{j(i)}} = 0$ if $q_{j(i)} \in S_p$. Also $x_{q_{j(i)}q_{j(i)}'} = 1$. So $x_{q_{j(i)}'t_\ell} = 1$ for some $\ell \in [k]$. This means that $x_{q_a't_\ell} = 0$ for $a \neq j(i)$. Hence $q_{j(i)}$ represents both S_i and T_ℓ. It follows now that $\{q_{j(1)}, \ldots, q_{j(k)}\}$ represents the two sets.

It is left to show that all (r, s)-cuts have capacity at least k. Let us take an (r, s) cut which has a finite capacity. It will have contain s_i for $i \in I$ and t_j for $j \in J_1$, where I, J_1 are subsets of $[k]$. So first r is connected to all s_i, where $i \notin I$. The capacity of all these diedges from r is $k - |I|$. Next let us consider all element in $\cup_{i \in I} S_i$. This is the set of all points in Q that are connected to vertices $s_i, i \in I$. Hence these vertices must be also in this cut, as the capacity of the edge s_iq where $q \in S_i$ is infinite. Similarly, all vertices in Q' which are equivalent to $\cup_{j \in J_1} T_j$ must be also in the cut. Now each t_j for $j \in J_1$ is connected to s. So the capacities of these diedges is $|J_1|$. So now we need to find the minimum capacities of all diedges from the points $q \in \cup_{i \in I} S_i$ that are not connected to vertices in Q' which are equivalent to $\cup_{j \in J_1} T_j$. By letting $J = \bar{J_1}$ we will get that we have at least the number of edges as in $(\cup_{i \in I} S_i) \cap (\cup_{j \in J} T_j)$ which yields the inequality

$$|(\cup_{i \in I} S_i) \cap (\cup_{j \in J} T_j)| \geq |I| + |J| - k.$$

Problem 3.31: How can we decide if it is possible for Buzzards to finish first or second? First, we see if Buzzards can win the series. This is discussed in subsection "Elimination of Sports Teams" pages 50 - 53. If yes, we are done. Suppose not.

Let T' be the set of all other players than Buzzards. Again, we assume that Buzzards won all other teams that it is left to play with. The number of wins for Buzzards is M. Let $t \in T'$, and assume that t is the winner of all other games that it is left to play in T'. Then t won M' games. If $M' \leq M$ then we do not consider this t. So assume that $M' > M$. Now set $T = T' \setminus \{t\}$ and use the construction of the flow on page 52. If such flow exists, Buzzards is the winner in T, which is a second in all the teams. We need to check all possible suitable $t \in T'$ to see if at least in one case Buzzards can come second.

Problem 3.35: In a round robbin tournament each player plays one again other opponents once. There are no ties. Assume that there are n players, labeled as $1, \ldots, n$. Let w_i is the number of wins of player i. Suppose that a vector $\mathbf{w} = (w_1, \ldots, w_n)$ is given. How can we determine if $\mathbf{w}$ arises from a such a tournament. Give a good algorithm and an good characterization.

Clearly, w_i is an integer satisfying $0 \leq w_i \leq n-1$.. Since the number of playes is $\frac{n(n-1)}{2}$ we have the condition that $\sum_{i=1}^n w_i = \frac{n(n-1)}{2}$. We now constract a digraph with vertices r, s $v_1, \ldots, v_n$ and $\frac{n(n-1)}{2}$ pairs $\{v_i, v_j\}$ for $1 \leq i < j \leq n$. r is connected to each v_i with diedge rv_i with capacity w_i. Each v_p is connected to $\{v_i, v_j\}$ by diedge $v_p\{v_i, v_j\}$ if $p = i$ or $p = j$. The capacity of such diedge is ∞. Each $\{v_i, v_j\}$ is connected to s by diedge $\{v_i, \mathbf{v}_j\}s$ with capacity 1. Then $\mathbf{w}$ arises from a tournament if and only if there is a maximum integral flow with value $\frac{n(n-1)}{2}$. Note that $x_{v_p\{v_i,v_j\}} = 1$ if $p \in \{i, j\}$ wins the play between v_i and v_j. Otherwise $x_{v_p\{v_i,v_j\}} = 0$.

A necessary and sufficient condition is given by mincut. Let us take a cut with r, $A_1 \subset V$ and A_2 subset of pairs. In order that this cut has finite value we must assume that A_2 has all pairs such that at least on of the players is in A_1. The value of the cut is

$$\sum_{i \in V \setminus A_1} w_i + (\text{number of pairs that either i or j} \in A_1) \geq \frac{n(n-1)}{2}.$$

Note that the number of pairs that either$i$ or j in A_1 is: the number of pairs that both i and j are in A_1 is $\frac{|A_1|(|A_1|-1)}{2}$ and teh number of pairs that $i \in A_1$ and $j \notin A_1$ is $|A_1|(n - |A_1|)$. Hence the above inequality is equivalent to

$$\frac{|A_1|(|A_1|-1)}{2} + |A_1|(n - |A_1|) \geq \sum_{i \in A_1} w_i,$$

for each subset A_1 of V. (Note that if $|A_1| = 1$ then the above inequality is $n-1 \geq w_i$. In general, the above inequality means that the number of wins of a group of A_1 players of cardinality $k = |A_1|$ is at most $\frac{k(k-1)}{2} + k(n-k)$ for the following reason. The number of wins when the players in A_1 play against themselves is $\frac{k(k-1)}{2}$. When the players in A_1 play against the players in $V \setminus A_1$ they can always win. Each player can in A_1 can win $(n-k)$ games. Hence the total maximal number of wins of players in A_1 is $\frac{k(k-1)}{2} + k(n-k)$!

3.3 §3.5: Minimum Cuts in Undirected Graphs

Problem 3.52: Minimum cut is the set of vertices $S = \{h, g, f\}$. (That is the correct answer!) The corresponding edge-cut is $\{ch, dh, dg, ef\}$. Hence the value (capacity) of the mincut is 7.

Problem 3.53: Legal ordering has the same complexity as Prim's algorithm. Choosing root in Prim's algo is equivalent to choose any vertex and call it v_1. In step $i-1$ in Prim's algorithm we have a subtree on V_{i-1} vertices. In legal ordering we have also V_{i-1}. In Prim's algo we choose v_i to be the vertex v_i as follows: First for each v_j, $j \in \{1, \ldots, i-1\}$. we choose $v(v_j)$ outside of V_{i-1} connected to v_j with minimum cost edge. Then we choose $v(v_j)$ with the minimum cost edge $(v_j, v(v_j))$. In legal ordering for each v not in V_{i-1} we have the positive cost of the capacities of all edges from v to V_{i-1} and v, which have to be computed. Then we choose the maximum one as v_i. So it is as similar to the case of the complete graph.

Problem 3.54: Note that we have three points $\{p, q, r\}$ and we choose all possible three pairs $\{p, q\}, \{p, r\}, \{q, r\}$ out of them. Hence by renaming p, q, r we can assume that $\lambda(G; p, q) \geq \lambda(G; p, r) \geq \lambda(G; q, r)$. Now $\lambda(G; q, r) \geq \min(\lambda(G; p, r), \lambda(G(p, q))$. Our assumption yields that $\min(\lambda(G; p, r), \lambda(G(p, q)) = \lambda(G; p, r)$. Furthermore, we also assumed that $\lambda(G; p, r) \geq \lambda(G; q, r)$. Hence $\lambda(G; p, r) = \lambda(G; q, r)$.

Problem 3.56: In this book a minimum cut corresponds to a subset of V. However a minimal cut is ment to be a minimal cut-edge. (Indeed, it can not by a set, since if S is a cut set, it corresponds to the partition of V, the sed of vertices to two complementary subsets: $S, \bar{S}$. So is S_1 is a strict subset of S then $\bar{S}$ is a strict subset of $\bar{S}_1$. Hence, one can not talk about minimal cut as a subset of V.) So minimal cut here is a minimal subset of edges!

One needs to analyze a minimal cut. Let $E' \subset E(G)$. Then E' is an cut-edge set if the graph $G \setminus E'$ is disconnected. Note that the weight of these edges if $u(E')$. E' is minimal if for each $e \in E'$ the set $E' \setminus \{e\}$ is not a cut- edge set. So if E' is not a minimal edge-cut set then there exists $e \in E'$ the set $E' \setminus \{e\}$ is a cut-edge set. As $u_e > 0$ it follows $u(E') > u(E' \setminus \{e\})$. Thus a minimal cost cut-edge set is a minimal cut. Assume that E' is a minimal cost cut-edge. We claim that $G \setminus E'$ contains exactly two connected components S, a nonempty strict subset of $V(G)$ and $V(G) \setminus S$. Otherwise, there would be exactly k-connected components on mutually disjoint nonempty strict subsets $S_1, \ldots, S_k$ for $k > 2$. So E' must contain some edges form each S_i to some other S_j. Suppose first that S_1 is not connected to $S_2, \ldots, S_k$ by edges in E'. Let E'' be the set of all edges in E' that connect S_1 to all $S_2, \ldots, S_k$. So E'' is a strict subset of E'. Clearly E'' is a cut-edge set. So E' is not a minimal cut edge set, contrary to our assumption. Assume that S_1 is connected to all $S_2, \ldots, S_k$ by edges in E'. Now let $\hat{E} \subset E'$ to be the set of all edges in E' such that S_k is connected to $S_1, \ldots, S_{k-1}$. Then $\hat{E}$ is a cut-edge and E' is not a minimal cut-edge, contrary to our assumption. Thus minimal cost cut-edge is a minimum cut-edge set of the form $\delta(S)$, where S and $V(G) \setminus S$ are connected. Hence the minimum contraction algorithm would return only a minimal cut (when the algorithm succeeds).

Problem 3.60: Assume that there are exactly $E_1, \ldots, E_k \subset E(G)$ distinct minimum cost cut-edges. (See Problem 3.56) Each has probability at least $\frac{2}{n(n-1)}$. So the sample space consists of k-elements and $1 = \sum_{i=1}^{k} Pr(X = E_i)$. Hence $1 \geq k\frac{2}{n(n-1)}$

which yields that $k \leq \frac{n(n-1)}{2}$.
Problem 3.66: Discussed the following fact in class. Let $A, B \subseteq V$. Consider all the edges in $\delta(A)$ and $\delta(B)$. Note the edges in $\delta(A) \cap \delta(B)$ appear twice in $\delta(A)$ and $\delta(B)$. Now consider the edges in $\delta(A \cup B)$ and $\delta(A \cap B)$. I showed that the edges that appear twice in $\delta(A \cup B)$ and $\delta(A \cap B)$ appear twice in $\delta(A \cup B)$ and $\delta(A \cap B)$. The the edges that appear once in $\delta(A \cup B)$ and $\delta(A \cap B)$ appear at least once in $\delta(A \cup B)$ and $\delta(A \cap B)$. But some edges in $\delta(A)$ and $\delta(B)$ may not appear at all in $\delta(A \cup B)$ and $\delta(A \cap B)$. This yields the inequality

$$u(\delta(A)) + u(\delta(B)) \geq u(\delta(A \cup B)) + u(\delta(A \cap B)).$$

Problem 3.68: Suppose that S and T cross. Observe that the four sets $A_1 = S \cap T, A_2 = S \cap \bar{T}, A_3 = \bar{S} \cap T, A_4 = \bar{S} \cap \bar{T}$ are nonempty disjoint sets whose union is the whole space. Note that

$$S = A_1 \cup A_2, \quad \bar{S} = A_3 \cup A_4, \quad T = A_1 \cup A_3, \quad \bar{T} = A_2 \cup A_4.$$

By interchanging S and T we can assume: $u(\delta(S)) \geq u(\delta(T))$. Also we can assume that $r \in S$ and $v \in T$. Observe that $u(S) = u(\bar{S}), u(T) = u(\bar{T})$ Recall Problem 3.36. Hence we have four inequalities

$$\begin{aligned}
u(\delta(S)) + u(\delta(T)) &\geq u(\delta(S \cup T)) + u(\delta(S \cap T)) = u(A_1 \cup A_2 \cup A_3) + u(\delta(A_1)), \\
u(\delta(S)) + u(\delta(\bar{T})) &\geq u(\delta(S \cup \bar{T})) + u(\delta(S \cap \bar{T})) = u(\delta(A_1 \cup A_2 \cup A_4)) + u(\delta(A_2)), \\
u(\delta(\bar{S})) + u(\delta(T)) &\geq u(\delta(\bar{S} \cup T)) + u(\delta(\bar{S} \cap T)) = u(\delta(A_1 \cup A_3 \cup A_4)) + u(\delta(A_3)), \\
u(\delta(\bar{S})) + u(\delta(\bar{T})) &\geq u(\delta(\bar{S} \cup \bar{T})) + u(\delta(\bar{S} \cap \bar{T})) = u(\delta(A_2 \cup A_3 \cup A_4)) + u(\delta(A_4)).
\end{aligned}$$

Now let us introduce a symmetric matrix a_{ij} that is the capacities of the edges from A_i to A_j for $i \neq j$ ($a_{ii} = 0$). We can translate the above inequalities, say the first one to:

$$a_{13} + a_{14} + a_{23} + a_{24} + a_{12} + a_{14} + a_{32} + a_{34} \geq a_{14} + a_{24} + a_{34} + a_{12} + a_{13} + a_{14}.$$

which are obvious, since $a_{ij} \geq 0$.

So we have 8 possibilities by choosing S or $\bar{S}$ and T or $\bar{T}$. For a choice S, T we have two possibilities: (a) $u(\delta(S \cup T)) \geq u(\delta(S \cap T))$.
(b) $u(\delta(S \cup T)) < u(\delta(S \cap T))$.

Possibility (a) yields that $u(\delta(S)) \geq u(\delta(S \cap T)) = \delta(B)$.

First possibility that each of the four sets contains exactly one of the vertices in $\{r, s, v, w\}$. Say $r \in A, w \in B, v \in C, s \in D$. It seems too tedious for me to analyze the cases, without some trick!

4 Chapter 4

4.1 §4.1: Minimum-Cost Flow Problems

Problem 4.3: We have $V = P \cup Q$. we direct the edged in the biparttite graph $G = (V, E)$ from P to Q. So $b_v = -1$ for each $v \in P$ and $b_v = 1$ for each $v \in Q$. Thus we are looking at the equations $f_x(v) = b_v$ where $0 \leq x_e \leq 1$ and integer flow. We are looking of the minimum of $\mathbf{c}^\top \mathbf{x}$.

Problem 4.4: It is better to prove slightly general theorem. Assume that for each digraph $G = (V, E)$ with satisfying $|\delta(v)| = |\delta(\bar{v})|$ we have the following. The set V decomposes to a finite number of pairwise disjoint nonemty sets: $V = \cup_{i=1}^{k} V_i$, such that each induced subraph $G_i = (V_i, E(V_i))$ is strongly connected, has directed Euler tour, and $E = \cup_{i=1}^{k} E(V_i)$. That is, there is no diedges in E between V_i and V_j for $i \neq j$. The proof is by induction on number of edges in G. (If $|\delta(v)| = |\delta(\bar{v})| = 0$, we assume that the Euler tour exists and is empty tour. Assume true where no diedges. True when the number of diedges $\leq m$, and supose that teh number of diedges is $m + 1$. Take a vertex with v_1 with $\delta(v_) \geq 1$. So follows from v_1 to v_2 by diedge $v_1 v_2$. As $|\delta(v_2)| = |\delta(\bar{v}_2)|$ it follws there is a diedge from v_2 to v_3 and cetera. Hence we have a dicircuit C: $v_i v_{i+1} \cdots v_j v_i$. Delete these diedges. The new digraph has less diedges so we can apply the induction to $G' = (V, E \setminus E(C))$. Start the Euler tour from v_i using the above dicycle. Now conitnue if possible with Euler tour in G'. This will prove the claim.

Problem 4.5: (My solution.) First, there is an Euler tour: Since the digraph is weakly connected, i.e. connected as undirected graph, each degree is even we have an undirected Euler tour, and by reversing the directions if needed we have a directed Euler tour. Consider the system $f_x(v) = 0$ for all $v \in V$. (So if $x_e = 1$ then we leave the edge as is, and if $x_e = -1$ we reversing the edge. Then $f_x(v) = 0$ is the condition $|\delta(v)| = |\delta(\bar{v})|$. So we give the bounds: $-1 \leq x_e \leq 1$ for each $e \in E$. Find $\min\{\sum_{e \in E} -c_e x_e\}$. First note that the minimum problems is equivalent to $\min\{\sum_{e \in E} -c_e(x_e - 1)\}$. So if $x_e = 1$ we do not have any contribution to the cost. But if $x_e = -1$, which means we change the direction then we pay $2c_e$. It is left to show that the minimum solution satisfies $x_e \in \{-1, 1\}$. (But why?) What happens if $x_e = 0$?

(Solution suggested by David Wang). Let $b_v = \frac{|\delta(\bar{(}v)| - |\delta(v)}{2}$. Note that since $|\delta(\bar{(}v)| + |\delta(v)|$ was even it follows that b_v is an integer. Also $\sum_{v \in V} b_v = 0$ as each diedge is counted once as exiting and one as entering. Now we look for the solution of the flow problem $f_y(v) = b_v$ for $0 \leq y_e \leq 1$. We calim that there exists a soltuion $y_e \in \{0, 1\}$ if and only if we can reverse some diedges to get Eulerian condition: the number of edges in is the same as the number of edges out. So $y_e = 0$ no reversal and $y_e = 1$ reversal. How it is related to my solution. We let $y_e = \frac{1 - x_e}{2}$. Note that it follows that $f_y(v) = b_v$. SAssume that $z_e = 1$ on all diidges. Then $f_z(v) = 2b_v!$. So since we have a solution $y_e \in \{0.1\}$ corresponding to reversal of edges we see that the system is solvable. However, as in my suggested solution how do we know that the minimum solution is also integer valued?

Theorem 3.19 gives the answer for the a flow which satisfies $x_e \geq l_e$, see page 56 (3.9). On page 98, it gives the reduction procedure which makes any problem $l_e \leq x_e \leq u_e$ to a problem $y_e \geq l'_e$ on a bigger graph.

In summary, both solutions are correct!

Problem 4.6: We consider again the system $f_x(v) = 0$ for all $v \in V$. Now the condition is $x_e \geq 1$. Assuming that x_e integer then $x_e - 1$ is the number of duplicated edges. So the cost of dulicating is $\sum_{e \in E} c_e(x_e - 1) = (\sum_{e \in E} c_e x_e) - \sum_{e \in E} c_e$. The second term is a fixed constant. We we minimize $\sum_{e \in E} c_e x_e$.

Problem 4.10: If $\mathbf{y} \in \mathbb{R}^V$ and it is feasible, it means that whenever $u_{vw} = \infty$ then we must have (4.3) which is $-y_v + y_w \leq c_{vw}$. Otherwise if $-y_v + y_w \leq c_{vw}$. We set $z_{vw} = 0$. If $-y_v + y_w > c_{vw}$ we set $z_{vw} = -y_v + y_w - c_{vw}$. Now continue as in the book: Set $\bar{c}_{vw} = c_{vw} + y_v - y_w$. Now we have Theorem 4.2. (Perhaps I am missing here something?)

Problem 4.11: Suppose we have a minimal solution and the corresponding maximal dual solution. We continue as [1, page 95]. Given dual variables y_v and z_{vw} for $u_{vw} < \infty$ we can express z_{vw} by the formula $\max(0, -\bar{c}_{vw})$, where $\bar{c}_{vw} = c_{vw} + y_v - y_w$. Then we let $x_e = 0$ if $\bar{c}_e > 0$ and $x_e = u_e$ if $\bar{c}_e < 0$. Let $E' \subseteq E$ be all set of diedges where $\bar{c}_e = 0$. Let us denote by $G' = (V, E')$ and by $G'' = (V, E \backslash E')$. Then $x''_e = 0$ if $\bar{c}_e > 0$ and $x''_e = u_e$ if $\bar{c}_e < 0$. Denote by $b''_v = f_{x''}(v)$. So we are left with a new system with variables x'_e where $e \in E'$. Clearly $f_x(v) = f_{x'}(v) + f_{x''}(v) = f_{x'}(v) + b''_v = b_v$. The system of equations are $f_{x'}(v) = b'_v$ where $b'_v = b_v - b''_v$. We now need to find condition where the system $f_{x'}(v) = b'_v, v \in V$ is solvable for $0 \le x'_e \le K$. The solubility of such a system is given by [1, Theorem 3.15]: $K|\delta(\bar{A})| \ge b'(A)$. The minimum of the left hand side is K, i.e., $|\delta(\bar{A})| = 1$. How big can be the right hand side? Note that $b'(A) = b(A) - b''(A)$. $b(A) \le \sum_{v \in V, b_v > 0} b_v$. How big $-b''(A)$ can be? Remember that for $e \in E \setminus E'$ $x_e \in \{0, u_e\}$. So the worst case is that the contribution of x_e to $-b''(A)$ is u_e. (Only once!) Hence $-b''(A) \le \sum_{e, u_e < \infty} u_e$. Therefore it is enough to choose

$$K = \sum_{v \in V, b_v > 0} + \sum_{e, u_e < \infty} u_e.$$

5 Homework 5

5.1 Aliabadi's problems for the week 10-31 – 11-2

Problem 1 Assume that there is a matching that saturates X. Take $S \subseteq X$. Each vertex $s \in S$ is matched with $y(s) \in Y$ and $y(s_1) \neq y(s_2)$ for $s_1 \neq s_2$. Hence $N(S) \supseteq \cup_{s \in S}\{y(s)\}$. Therefore

$$|N(S)| \ge |S| \text{ for each } S \subseteq X. \tag{5.1}$$

Vice versa, suppose that (5.1) holds. We show by induction on $|X|$ that there is a match that saturates X. For $|X| = 1$, $|N(X)| \ge |X| = 1$, so there is an edge xy from the only vertex $x \in X$ that connects to Y. This is our match. Suppose that the claim is true for each bipartite graph satisfying the condition (5.1) if $|X| \le M$. Assume that $|X| = M + 1$. Let $a \in X$. As $|N(\{a\})| \ge |\{a\}| = 1$, there is an edge $ab \in V(G)$. Consider the subgraph $G' = G \setminus \{a, b\}$. It is a bipartite graph with bipartition $X' = X \backslash \{a\}, Y' = Y \backslash \{b\}$. If G' satisfies the condition (5.1) we are done: Let M' be a matching that saturates X' in G'. Then $M = M' \cup \{ab\}$. So suppose that (5.1) is not satisfied in G'. So there exists $A \subseteq X'$ so that $|N_{G'}(A)| < |A|$. But $|N_G(A) \ge |A|$. So $N_G(A) = N_{G'}(A) \cup \{b\}$ and $|N_G(A)| = |A|$. Hence the subgraph $G(A \cup N(A))$ is a bipartite subgraph satisfying the condition (5.1). As $|A| \le M$ it has a perfect match M_1 by the induction hypothesis. Now, consider the subgraph $G_2 = G((X \cup Y) \setminus (A \cup N(A))$. It is straightforward to show that the condition (5.1) on G yields that G_2 satisfies (5.1). By induction, there is a match in M_2 which saturates $X \setminus A$. So $M_1 \cup M_2$ is a match that saturates X.

Problem 2: Every tree has at most one perfect matching. Clearly, it is enough to assume that the tree has an even number of vertices $T = (V, E)$. By induction on

$n' = \frac{|V|}{2}$. For $n' = 1$ exactly one perfect match. Suppose this is true for $n' \le N$. Suppose that $n' = N + 1$. Recall that every tree has at least two leaves. Assume that $v \in V$ is a leaf. So every perfect match has to have the unique edge vw, where $w \in V$. Consider $T \setminus \{v, w\}$. It decomposes to a finite number of trees. By induction each such tree has at most one perfect matching. Hence T can have at most one perfect matching.

Problem 3: Let μ_n be the size of the minimum maximal matching in the cycle $C_n = (V, E), V = \{v_1, \ldots, v_n\}$, where E has edges $v_i v_{i+1}$ for $i = 1, \ldots, n$. Here $v_{n+1} = v_0 = v_1$. Clearly: $\mu_3 = 1, \mu_4 = 2, \mu_5 = 2$. Now $\mu_6 = 2$. To obtain μ_n we should "pad" each edge of matching $v_i v_{i+1}$ with two exposed vertices v_{i-1} and v_{i+1}, as much as possible. So in C_6 we have a maximal matching $v_1 v_2, v_4 v_5$. Hence $\mu_{6k} = 2k$: $v_1 v_2, v_4 v_5, v_7 v_8, \ldots, v_{6k-2} v_{6k-1}$. Therefore $\mu_{6k+1} = 2k+1$, $\mu_{6k+2} = 2k+1$. Thus the answer is $\mu_n = \lceil \frac{n}{3} \rceil$.

Problem 4: Recall Birkhoff's theorem that each doubly stochastic $n \times n$ matrix A is of the form $\sum_{\sigma \in S_n} \theta(\sigma) P_\sigma$. Here $\sigma : [n] \to [n]$ is a permutation (bijection) on $[n] = \{1, \ldots, n\}$. S_n is the symmetric group of all permutations. (The cardinality of S_n is $n!$.) P_σ is the permutation matrix corresponding to σ. (So the (i, j) entry of P_σ is $\delta_{\sigma(i)j}$.) $\theta(\sigma) \ge 0$ for each $\sigma \in S_n$, and $\sum_{\sigma \in S_n} \theta(\sigma) = 1$. Clearly, $A^\top = \sum_{\sigma \in S_n} \theta(\sigma) P_\sigma^\top$ If A is symmetric then $A = A^\top = \frac{1}{2}(A + A^\top)$. Hence a symmetric doubly stochastic matrix is

$$A = \sum_{\sigma \in S_n} \theta(\sigma) \frac{1}{2} (P_\sigma + P_\sigma^\top).$$

It is not hard to show that each $\frac{1}{2}(P_\sigma + P_\sigma^\top)$ is an extreme points in the convex set of all $n \times n$ symmetric doubly stochastic matrices.

Problem 5: Suppose first $G = (X \cup Y, E$ is a bipartite graph. with a partition X, Y of vertices that has a perfect matching. So each $x \in X$ is matched wit $y \in Y$. Hence $|X| = |Y|$. Let $A \subseteq X, B \subseteq Y$. So $N(A) \subseteq Y, N(B) \subseteq X$.Then $N(A \cup B) = N(A) \cup N(B)$. By Hall's theorem $|N(A)| \ge |A|, |N(B)| \ge |B|$. Then $|N(A \cup B)| \ge |N(A)| + |N(B)| \ge |A| + |B|$. As $S = (S \cap X) \cup (S \cap Y)$ we deduce the result that $|N(S)| \ge |S|$.

Assume now that $|N(S)| \ge |S|$. So $|Y| \ge |N(X)| \ge |X|$ and $|X| \ge |N(Y)| \ge |Y|$. Hence $|X| = |Y|$. Clearly, for each $S \subseteq X$ we have that $|N(S)| \ge |S|$. Hence by Hall's theorem there is a matching that saturates X. This matching is perfect matching.

Take a complete graph $K_n = ([n], E)$, where n is odd. For each $A \subseteq [n]$ $|N(\{v\}| = n - 1$, and $|N(A)| = n$ if $|A| \ge 2$. So for $n \ge 3$ we have the condition that $|N(A) \ge |A|$. As n is odd there is no perfect matching.

5.2 §5.1

Problem 5.1: We assume that $G = (P \cup Q, E)$ a bipartite graph. We orient all edges $P \to Q$. We can assume that the capacities of edges $pq \in E, p \in P, q \in Q$ are ∞. We add a source r and connect it with with diedges rp with capacity 1 for each

$p \in P$. We connect each $q \in Q$ with s with diedge qs with capacity 1. This gives rise to the digraph G'. Suppose we have a match $M \subset E$ that we augment by an M-augmenting path. We can assume without loss of generality that an augmenting path startet at P and ended in Q: $p_0q_1p_1 \cdots q_kp_kq_{k+1}$. So M_1 composed of edges $q_1p_1, \ldots, q_kp_k$ was in the given match M while p_0 and q_{k+1} where M exposed. So the augmentation is to replace M_1 by $k+1$ edges $p_0q_1, p_1q_2, \ldots, p_kq_{k+1}$.

Now let us see the flow x corresponding to the match M: For $pq \in M$ we have $x_{pq} = 1$. Also $x_{rp} = 1, x_{qs} = 1$. If p is M exposed then $x_{rp} = x_{pq} = 0$ for all $pq \in E$. If q in M exposed then $x_{qs} = x_{pq} = 0$ for all $pq \in E$.

So the dipath $rp_0q_1p_1 \cdots q_kp_kq_{k+1}s$ is an x-augmenting in $G'(x)$. Indeed rp_0, p_0q_1 are in G' and $G'(x)$ since p_0 is M-exposed. Next, q_1p_1 is in $G'(x)$ but not in G', since the orientation in G' is p_1q_1 but $x_{p_1q_1} = 1 > 0$. (As $u_{p_1q_1} = \infty$ if follows that diedge p_1q_1 is also in $G'(x)$.) Next p_1q_2 is in G' and $G'(x)$. Next q_2p_2 is in $G'(x)$ but not in G'. Continuing in this manner we deduce that $rp_0q_1p_1 \cdots q_kp_kq_{k+1}s$ is an x-incrementing path. Hence we can augment the flow by 1.

Vice versa, suppose that $rp_0q_1p_1 \cdots q_kp_kq_{k+1}s$ is an x-incrementing path. First, p_0 is was M-exposed, since the integer flow in direction rp_0 was zero in x. (Otherwise, as $u_{rp_0} = 1$ there would be only a diedge p_0r in $G'(x)$. Since there is a diedge p_0q_1 it means that there was an edge p_0q_1 in E. Since there is an edge q_1p_1 in $G'(x)$ it means that $x_{p_1q_1} = 1$ so p_1q_1 where in the original M and cetera. So $p_0q_1p_1 \cdots q_kp_kq_{k+1}$ is an M-augmented path.

Now each M-alternating path again, can be viewed as $p_0q_1p_1 \cdots q_kp_k$ where p_0 is M-exposed and M_1 composed of edges $q_1p_1, \ldots, q_kp_k$. Hence $rp_0q_1p_1 \cdots q_kp_k$ is an x-incrementing path.

Problem 5.2: Let M be a match and M' is a maximum matching of cardinality p. We consider the symmetric difference $M\Delta M'$. It consists of paths and even cycles. The number of edges is $|M| + |M'| - 2|M \cap M'|$. In each even cycle the number of edges in M and in M' is the same. In each M-augmenting path the number of edges in M' is greater by 1 then the number of edges in M. Since M' is maximum matching each path can not be M'-augmenting. So each path is either M-augmenting, or it has an even number of edges:a half in M and a half in M'. Hence we must have exactly $p - |M|$ M-augmenting paths. They are all node disjoint, because each vertex of degree 2 is in M and M' and teh vertices on the end of M-augmenting paths are only on M' but not on M.

Problem 5.3: Let M' be a matching of cardinality $5,000$. Suppose that all augmenting paths are longer than 9. Since every augmenting path has an odd length each path is of length 11. From the arguments of the proof of Problem 5.2 we deduce we have at least $5,000 - 4,000 = 1,000$ M-augmenting paths. (Note that the maximum matching may have more than $5,000$ edges.) Hence we would have at least $1,000$ M-augmenting paths in $M\Delta M'$. All these paths are node-disjoint, and hence edge disjoint. So the total number of edges in these paths is at least $11,000$. But the $|M| + |M'| = 9,000$ which contradicts that $|M\Delta M'| \geq 11,000$. Hence at least one of the M-augmenting paths has at most length 9.

Problem 5.4: Suppose $|M| \leq p - \sqrt{p}$, where $p = |N|$, and N is a maximum matching. Exercise 5.2 yields that $M\Delta N$ has at least $p - |M|$ M-augmenting paths.

Suppose that each at least $\sqrt{p}+\varepsilon$ edges from M, for some $\varepsilon > 0$. Hence M wil have at least

$$(p - |M)(\sqrt{p}+\varepsilon) \geq \sqrt{p}(\sqrt{p}+\varepsilon) = p + \varepsilon\sqrt{p} > p - \sqrt{p},$$

contrary to our assumptions. Hence $M\Delta N$ has at least one augmenting path of lenght at most $\sqrt{p}$.

Problem 5.7: First we show that if $|C|$ is odd then $|\delta(C)| \geq 3$. If $C = \{v\}$ then $|\delta(v)| = 3$, the degree of each vertex. Consider $G' = G(C)$. The degree of each vertex in G' is at most 3. If there is an isolated vertex in v $G(C)$ then $\delta(v) \subseteq \delta(C)$ so $|\delta(C)| \geq |\delta(v)| = 3$, and we are done. So the degree of each vertex in $G(C)$ is at least one. It is impossible to have all vertices in $G(C)$ to be of degree 3, since the number of vertices of odd degree must be even. (The sum of degrees is twice the number of edges, hence even.) So suppose that $|\delta(C)| = 1$. This means that the only edge in $\delta(C)$ is e and $G \setminus \{e\}$ is disconnected to C and other components. (C may connect several connected components.) This contradicts teh assumptions of Petersen's theorem. Hence $|\delta(C)| \geq 2$. Suppose that $|\delta(C)| = 2$. So either exactly one vertex in $G(C)$ has degree 1 and all other degree 3 or exactly two vertices have degree 2 in $G(C)$ and all other vertices have degree 3. This will be a contradiction, since the number of vertices of odd degree is odd in both cases. Hence $\|\delta(C)| \geq 3$.

Using Tutte's theorem, it is now left to show that we have the inequality $|A| \geq oc(G \setminus A)$. Each odd component in C in $G \setminus A$ is connected to A. The number of edges coming out of each C is at least 3. So the number of edges coming to A from all the odd components is at least $3oc(G \setminus A)$. But the degree of each vertex in A in G is 3. So $3|A| \geq 3oc(G \setminus A)$, i.e. $|A| \geq oc(G \setminus A)$.

Problem 5.8: Let M be a maximum matching in bipartite graph $G = (P \cup Q, E)$. Assume that $vw \in E$. According to Lemma 5.5 if both vw are not essential then G has an odd cycle. But G does not have an odd cycle! So either u is essential or v is essential. By changing the names we can assume that u is essential. So every maximum matching covers u. Then we put u in C. Suppose that u and v are essential. So we get that either uv is in the maximum matching, or we have two edges uv' and $u'v$ in maximum matchings. We can not have these two possibilities be together otherwise, $u'vuv'$ would be an augmenting path for a maximum match that has the match uv. In the case that uv appears in each maximum matching we choose either u or v in C. In the other case we choose u and v in C. It now follows that C is a covering set.

Problem 5.9: Assume that $G = (V, E)$. I had difficulty in following the hint in the book. If v, w are inessential and $vw \in E$, then there exists a tight odd cycle C such that vw is in this cycle. We can shrink this cycle to obtain new graph $G' = G \times C$. As G was connected so is G'. Note that the vertex C an inessential node in G'. If we can show that every other node in G' is inessential, we are done by induction on the number of vertices. So we need to show that every node in G' which is not C is inessential. This node is w and w is not the node on the cycle C. Let us take a maximum match M in G that omits w. Some of the edges of in M connect to the vertices on the cycle C. If we shrink C we lose all these matchings except one.

As $v \in V$ is inessential, there is a maximum matching where v is exposed. Hence

$\nu(G) \le \frac{|V|-1}{2}$. Suppose that $\nu(G) < \frac{|V|-1}{2}$. So for each maximum match M there exist at least two M-exposed vertices v and w. So fix a maximum match M and v, w two uncovered by M. So assume that As V connected there exists a path connecting these two vertices, call it $P = u_0 u_1 \cdots u_k$, where $u_0 = v, u_k = w$. Note that this path has length 2 at least, i.e. $k \ge 2$. (If v, w are M-exposed and $vw \in E$ then M is not maximum.) Take a maximum match on P. Call it $N = \{u_0u_1, u_2u_3, \ldots, \}$. So v is covered by N. If N is maximum then $\nu(G) \ge \frac{|V|-1}{2}$ contrary to our assumption. So $|N| < |M|$. Consider $M \Delta N$. According to the proof of Exercise 5.2 it must have $|M| - |N|$ augmenting paths. So taking away the common edges in M and N from P we get l subpaths of P, $P_1, \ldots, P_l$. Then the matches in N sub-paths are extended to sub-paths in $M \Delta N$. The path starting from $u_0 = v$ must be of even length, otherwise M is not maximum. It now follows that by augmenting each M-augmenting path in $M \Delta N$, and leaving all other edges in N as is, we obtain a match M' that covers all get to covers at least by one more vertices than M. Contradiction?

5.3 §5.2

Problem 5.13: Let $\nu(G)$ be the cardinality of a maximum match $M \subseteq E$. So M covers V_1 vertices of cardinality $2|M|$. If $V = V_1$ then $M$is an edge cover of $G = (V, E)$ of cardinality $\frac{|V|}{2} = |V| - \nu(G)$. Suppose that V_1 is a strict subset of V. Let $V_2 = V \setminus V_1$. As G has isolarted vertez, for each $v \in V_2$ chose and edge $e(v)$ that covers v. Let $E_1 = M \cup) \cup_{v \in V_2} e(v)$. Then E_1 is an edge cover and

$$|E_1| = \nu(G) + |V_2| = \nu(G) + (|V| - 2\nu(G)) = |V| - \nu(G).$$

It is left to show that each edge cover is of cardinality at least $|V| - \nu(G)$. Assume that D is a minimal edge cover. So if $e \in D$, then $D \setminus e$ is not an edge cover. So $u(e) \in V$ is not in $D \setminus e$. We have two possilities: First $e = uv$ and $D \setminus e$ does not cover u and v. The set of such edges in D corresponds to a match M. $M \subset D$. So $|M| \le \nu(G)$. For each $e \in D \setminus M$ there is exaclty one vertex that is only covered by e. Hence $|D| = |M| + (|V| - 2|M|) = |V| - |M| \ge |V| - \nu(G)$.

Problem 5.18: Suppose M is a perfect match in G. First player choose $e_1 = v_1 v_2 \in M$. If the degree of v_1 and v_2 is 1 the second player lost. Otherwise, my renaming v_1, v_3 the second player chose an edge $v_2 v_3$. Then the first player choses $e_2 \in M$ such that $e_2 = v_3 v_4$. Continuing in this matter the second palyer must lose, since there is no augmenting path for M as it is maximum!

6 Homework 6

Problem 1: Problem 5.9 yields that $|V|$ is odd and $\nu(G) = \frac{|V|-1}{2}$. That is, for each node $v \in V$ the is a match $M \subset E$ such that M covers all vertices in V except v. So suppose the first player chose an edge $v_0 v_1$. As v_0 is not essential, then there exist a perfect match in $G_1 = G \setminus \{v_0\}$. The second player chooses an edge in M that covers v_1. It must be of the form $v_1 v_2$, where $v_2 \ne v_0$. Now we continue as in Problem 5.18, where the Second player has the role of the First player on the graph G_1.

Problem 5.22: Let $A(G) = [a_{ij}]_{i,j=1}^n$. Here $A(G)$ has entries 0 or 1,as $a_{ij} = 1$ if $v_i vJ \in E$ and otherwise $a_{ij} = 0$. Also $A(G)$ is symmetric and the diagonal entries $a_{ii} = 0$. The system (5.8) is equivalent to the statement that there exists a symmetric matrix $X = [x_{ij}]$ such that $x_{ij} = 0$ if $v_i v_j \notin E$ and $x_{ij} = x_{ji} = x(e) \geq 0$, where $e = v_i v_j$ such that $x(\delta(v)) = 1$ for each $v \in V$. This condition is equivalent to: X is a doubly stochastic matrix.

Suppose first that $G = (V, E)$ contains a subgraph $H = (V, E')$ where the degree of each vertex is 1 or 2. This is equivalent to the statement H consists of a match M and a union of cycles $C_1, \ldots, C_k$, which are vertex disjoint. Let us construct $X(H)$, a symmetric doubly stochastic matrix with zero diagonal corresponding to H. If $e \in M$ then let $x(e) = 1$. If $e \in E(C_i)$ then $x(e) = \frac{1}{2}$. All other entries of X are zero. We claim that X is doubly stochastic. If v is covered by M then the corresponding row and column of X to v contains only one nonzero entry equal to 1. Hence $x(\delta(v)) = 1$. If $v \in V(C_i)$ then the corresponding row and column of X to v has two nonzero entries, corresponding to two neighbors of v in $V(C_i)$. The value of each entry is $\frac{1}{2}$. Again $x(\delta(v)) = 1$. Therefore (5.8) is solvable.

Assume now that (5.8) is solvable. Hence there exists a symmetric doubly stochastic matrix $X = [x_{ij}]$ such that $x_{ij} = 0$ if $v_i v_j \notin E$. Recall the solution of Problem 4 in HW 5. X is a convex combination of matrices $\frac{1}{2}(P + P^\top$, where P is a permutation matrix. So there exists a permutation matrix $P = [p_{ij}]$ such that the nonzero entry of $P + P^\top$ are located on the edges of G. Consider the multigraph $G' = (V, E')$ corresponding to $P + P^\top$. Since G has no self loops, G' does not have self loops. As each row and column of $P + P^\top$ sums to 2 each vertex has degree 2. So G' consists of a match M where each edge appears twice and a union of cycles $C_1, \ldots, C_k$. Hence G contains a subgraph H as above.

Problem 5.40: Take $E(P)$. Suppose that $E(P) \neq E$. Then there is a vertex $v \in V(E(P))$ such that its degree in G is greater than the degree of v the subgraph induced by $E(P)$. If not $E(P)$ is a connected component of G. Since G is connected this is a contradiction to $E(P) \neq E$. Note that $G' = G \setminus E(P)$ has all vertices of even degree. Take this v and consider an edge disjoint path in G' starting form v. This path must end in v for the reason that this path is in a connected component of G'. Call this path $E(P_1)$. Now join $E(P_1)$ to $E(P)$ to get a bigger closed edge disjoint path in G. If $G = E(P)$ we are done. Otherwise repeat.

Problem 5.43: Suppose first that all $c_e \leq 0$. Then this problem is equivalent to find the minimum cut with nonnegative cost edges $-c_e$. Make a directed graph G' from G be making each undirected edge to two directed edges in two different directions. Fix $r \in V$. Choose $s \in V \setminus \{r\}$. Now find the minimum edge cut finding the maximum flow from r to s using $-c_e$ as the capacity of the directed edges. This minimum edge cut in G' correspond to an edge cut in G with the corresponding cost. Now vary $s \in V \setminus \{r\}$ to find the minimum edge cut.

Assume now that exactly one $c_f > 0$. Let $f = vw$, $G_1 = G \setminus \{f\}$ and G' be the oriented G_1. Assume the cost of edges is $-c_e$ for $e \neq f$. Suppose first that the maximum cut contains f. Then we need to find the minimum cut in G' where $r = v$ and $s = w$.

Suppose that f is not in the maximum cut. Then we find the minimum cut in G' with $r = v$ and s varies in $V \setminus \{v\}$. From all these we consider all minimum cuts of the above form. If f connects the two connected components in some choice then we have the maximum cut with c_f and we are done. Suppose that f connects two vertices in the same connected component for each minimal cut if G'. Now choose the maximum between the maximum cut conatinaing c_f and the maximum cut where c_f in one of the connected components.

Problem 5.45: Assume that $J \subset V$ is set of odd vertices in $G = (V, E)$. Clearly, if $J = \emptyset$ then $c(J) = 0$ and $\frac{1}{3}C(E) \geq c(J)$. Next observe that the assumption that the graph with edge connectivity two means that first the degree of each vertex is two at least. As the degree of each vertex $v \in J$ is odd the degree of each vertex in J is at least 3. We now claim that if we give a weight $\frac{1}{3}$ to each $e \in E$ we are going to satisfy the condition in (5.33) that $x(D) \geq 1$ for each $D \subset E$, $D = \delta(S)$, and $S \cap J$ is odd. So first $|\delta(S)| \geq 2$ since G is two connected. It is left to show that $|\delta(S)| > 2$. We claim that the condition $|\delta(S)| = 2$ would violate that the subgraph $G(S)$ has an even number of vertices of odd degrees. (Consider a number of possibilities to which vertices these two edges are connected!)

References

[1] W.J. Cook, W.H. Cunningham, W.R. Pulleyblank, A.Schrijver, *Combinatorial Optimization*, Wiley, 1998.