Matchings, permanents and their random approximations

Shmuel Friedland
Univ. Illinois at Chicago

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Overview

- Matchings in graphs
- Number of k-matchings in bipartite graphs as permanents
- Lower and upper bounds on permanents
- Exact lower and upper bounds on k-matchings in 2-regular graphs
- Probabilistic methods
- Expected number of k-matchings in r-regular bipartite graphs
- p-matching and total matching entropies in infinite graphs
- Asymptotic lower and upper matching conjectures
- Plots and results

Uri N. Peled

Uri was born in Haifa, Israel, in 1944.

Education:

Hebrew University, Mathematics-Physics, B.Sc., 1965.
Weizmann Institute of Science, Physics, M.Sc., 1967
University of Waterloo, Mathematics, Ph.D., 1976
University of Toronto, Postdoc in Mathematics, 1976–78

Appointments:

1978–82, Assistant Professor, Columbia University
1982–91, Associate Professor, University of Illinois at Chicago
1991–2009, Professor, University of Illinois at Chicago

Areas of research: Graphs, combinatorial optimization, boolean functions.

Uri published about 57 paper

Uri died September 6, 2009 after a long battle with brain tumor.

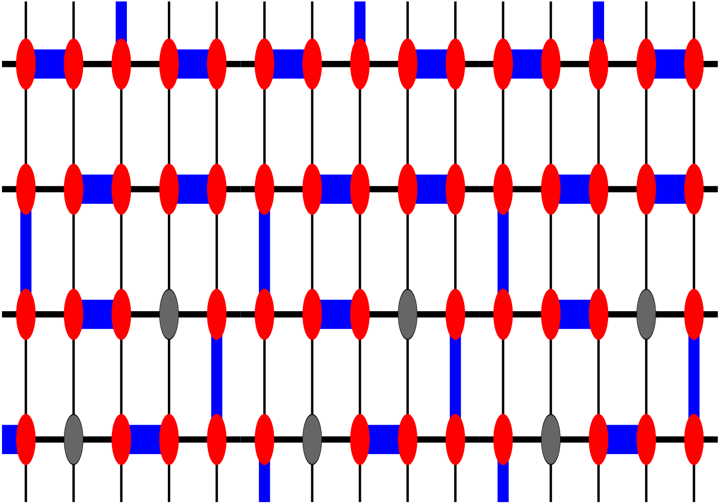

Figure: Matching on the two dimensional grid: Bipartite graph on 60 vertices, 101 edges, 24 dimers, 12 monomers

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- M is k-matching $\iff$ $\#M=k$.

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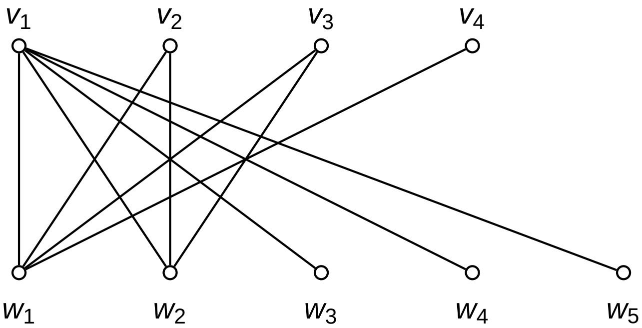
Birkhoff-Egerváry-König-Steinitz theorem (1946-1931-1916-1897)

Bipartite graphs

Figure: An example of a bipartite graph



Representation matrix
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

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Tverberg permanent conjecture 1963:

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- There are new simple proofs using nonnegative hyperbolic polynomials e.g. Friedland-Gurvits 2008

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$$\textit{Con FKM}\ 2006:\ \phi(k,G)\ge\binom{n}{k}^2\left(\frac{nr-k}{nr}\right)^{nr-k}\left(\frac{kr}{n}\right)^k,\ G\in\mathcal{G}(r,2n)$$

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$$\textit{Cor}:\ \phi(n,G) \ge \frac{r!}{r^r}\left(\frac{r}{r-1}\right)^{r(r-1)}\left(\frac{(r-1)^{r-1}}{r^{r-2}}\right)^n$$

$$\textit{Con FKM}\ 2006:\ \phi(k,G) \ge \binom{n}{k}^2\left(\frac{nr-k}{nr}\right)^{nr-k}\left(\frac{kr}{n}\right)^k,\ G \in \mathcal{G}(r,2n)$$

F-G 2008 showed weaker inequalities

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Upper perfect matching bounds for general graphs

$G = (V, E)$ Non-bipartite graph on $2n$ vertices

$$\phi(n, G) \leq \prod_{v \in V} ((\deg v)!)^{\frac{1}{2 \deg v}}$$

If $\deg v > 0, \forall v \in V$ equality holds iff G is a disjoint union of complete balanced bipartite graphs

Kahn-Lóvasz unpublished, Friedland 2008-arXiv, Alon-Friedland 2008-arXiv, Egorichev 2007

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Notation:

$$f(x) = \sum_{i=0}^{N} a_i x^i \preceq g(x) = \sum_{i=0}^{N} b_i x^i \iff$$

$$a_i \le b_i \text{ for } i = 1, \ldots, N.$$

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If n even G multi-bipartite 2-regular graph then $\Phi_G(x) \succeq \Phi_{C_n}(x)$.

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$A = [a_{ij}] \in \mathbb{R}_+^{n\times n}$, $X(A) := [\sqrt{a_{ij}}x_{ij}]$,
x_{ij} independent random variables $E(x_{ij}) = 0, E(x_{ij}^2) = 1$
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Concentration results
A. Barvinok 1999 -
1. x_{ij} real Gaussian $\Rightarrow \det X(A)^2$ with high probability $\in [c^n \,\mathrm{perm}\, A, \mathrm{perm}\, A]$ $c \approx 0.28$
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Friedland-Rider-Zeitouni 2004:
$0 < a \le a_{ij} \le b$, x_{ij} real Gaussian $\Rightarrow \det X(A)^2$ with high probability $\in [(1-\varepsilon_n)\,\mathrm{perm}\, A, \mathrm{perm}\, A]$ $\varepsilon_n \to 0$

Probabilistic Methods II

FRZ results use concentration for $\log_\varepsilon \det Z(A) = \operatorname{tr} f(Z(A))$,
$Z(A) = X(A)^\top X(A) \succeq 0, f = \log_\varepsilon x = \log\max(x,\varepsilon)$.
or $\log_\varepsilon \det Y(A)$, $Y(A) = \begin{bmatrix} 0 & X(A) \\ X(A)^\top & 0 \end{bmatrix}$

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$E(x_{ij}) = 0, E(x_{ij}^2) = 1$
$E(\det Y(B)) = \mathrm{haf}\ A$ -
total weight of weighted matchings in induced graph by A

$E(\det(\sqrt{t}I + Y(B)) = \Phi_{G_w}(t)$ - the weighted matching polynomial of $G(A)$.

Thm: Concentration of $\log\det(\sqrt{t}I + Y(A))$ around expected value $\log\tilde{\Phi}_{G_w}(t), t > 0$ which less $\log\Phi_{G_w}(t)$

$\frac{1}{n}\log\tilde{\Phi}(t, G_\omega) \le \frac{1}{n}\log\Phi(t, G_\omega) \le \frac{1}{n}\log\tilde{\Phi}(t, G_\omega) + \min(\frac{\max_{i,j}|a_{ij}|}{2t}, 1.271)$

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A dichotomy: some $\#P$ complete problem have fpras and some do not

Expected values of k-matchings

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- Permutation $\sigma : \langle nr \rangle \to \langle nr \rangle$ induces $G(\sigma) \in \mathcal{G}_{\mathrm{mult}}(r, 2n)$ and vice versa
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- $1 \le k_l \le n_l, l = 1, \dots,$ increasing sequences of integers s.t. $\lim_{l \to \infty} \frac{k_l}{n_l} = p \in [0, 1]$. Then

$$\lim_{l \to \infty} \frac{\log E(k_l, n_l, r)}{2n_k} = f(p, r)$$

$$f(p, r) := \frac{1}{2}(p \log r - p \log p - 2(1-p)\log(1-p) + (r-p)\log(1 - \frac{p}{r}))$$

p-matching entropy

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$p \in [0, 1]$-matching entropy, (p-dimer entropy) of G

$$h_G(p) = \sup_{\text{on all sequences}} \limsup_{l \to \infty} \frac{\log \phi(k_l, G_l)}{\# V_l}$$

$G_l = (E_l, V_l), l \in \mathbb{N}$ a sequence of finite graphs converging to G, and

$$\lim_{l \to \infty} \frac{2k_l}{\# V_l} = p$$

Asymptotic Lower and Upper Matching conjectures

FKLM 06:

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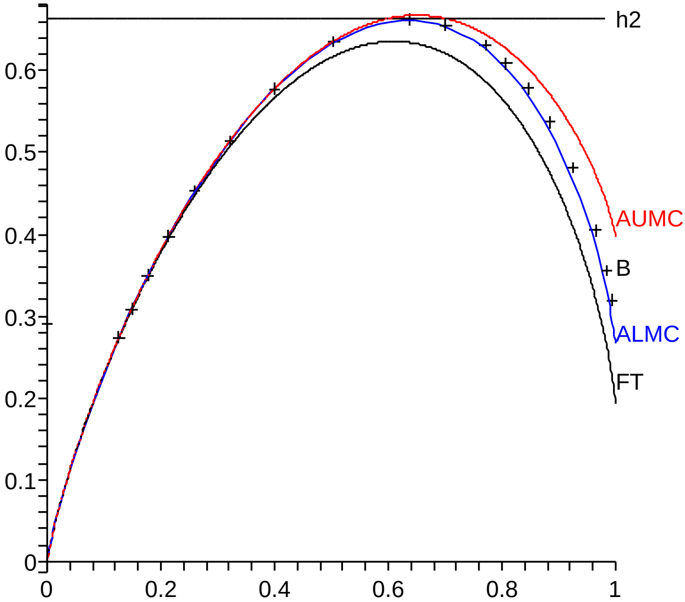
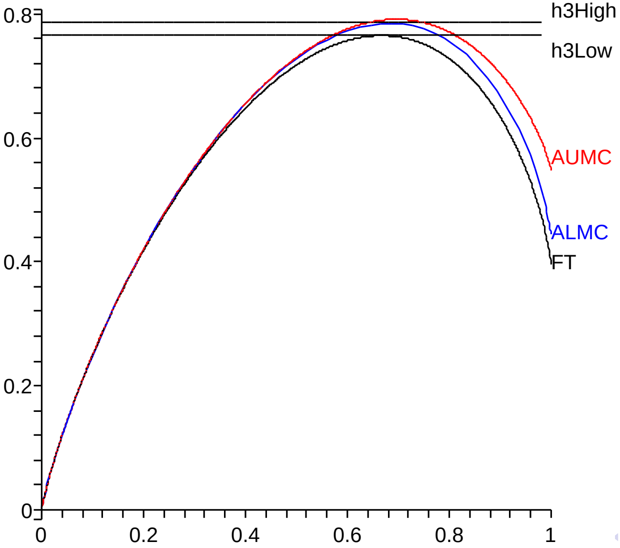
AUMC: $\mathrm{upp}_r(p) = h_{K(r)}(p)$, $K(r)$ countable union of $K_{r,r}$

$r = 4$



$r = 6$



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