Math 533, Final Exam

Solve 4 out of the following 9 problems. **Due Wed 2008-12-03 2pm.**

1. (25 points) Problem 42acd on page 109 (section 3.5).
2. (25 points) Problem 58 on page 177 (Section 5.5 Hilbert spaces).
3. (25 points) Problem 67 on page 178 (Section 5.5 Hilbert spaces).
4. (30 points) Problem 15 on page 187 (Section 6.1 L^p spaces).
5. (30 points) Problem 18 on page 191 (proof of Lebesgue and Radon-Nikodym theorem using Hilbert space techniques).
6. Let $f : [0,1] \to [0,1]$ be a measurable function with $\int_0^1 f\,dx > \frac{1}{2}$.
   (a) (15 points) Prove that $A = \{x \in [0,1] \mid f(x) > x\}$ has $m(A) > 0$.
   (b) (5 points) Assuming f is continuous, prove that there exists a Borel set $B \subset [0,1]$ with
   $$m(B) > 0, \qquad \sup(B) < \inf(f(B))$$
   (c) (10 points) Using Lusin's theorem prove the conclusion of (b) for measurable f as above.
7. (25 points) Let (X,μ) be a measure space with $\mu(X) = 1$ and $f : X \to [0,1]$ a measurable function with $\int f\,d\mu = \alpha$. Prove that: $\mu\{x \in X : f(x) \geq \frac{\alpha}{2}\} \geq \frac{\alpha}{2}$.
8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying $f(x+y) = f(x) + f(y)$ for every $x, y \in \mathbb{R}$[1]. Prove
   (a) (5 points) $f(0) = 0$ and $f(nx) = nf(x)$ for $n \in \mathbb{N}$. Deduce that $f(\frac{m}{n}x) = \frac{m}{n}f(x)$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$.
   (b) (5 points) If f is continuous on $\mathbb{R}$ then $f(x) = cx$ where $c = f(1)$.
   (c) (5 points) If f is continuous at $x = 0$ then f is continuous on all of $\mathbb{R}$.
   (d) (15 points) If f is measurable, then f is continuous[2].

   So linear maps $f(x) = cx$ are the only measurable homomorphisms $\mathbb{R} \to \mathbb{R}$.
9. Given two measurable functions f, g on $\mathbb{R}$ their *convolution* $f * g$ is the function defined by the formula
   $$(f * g)(x) = \int f(x-t)g(t)\,dt$$
   at all points where the integrand (as a function of t) is in L^1. Prove:
   (a) (15 points) If $f, g \in L^1$ then $f * g$ is defined a.e. and $\|f * g\|_1 \leq \|f\|_1 \cdot \|g\|_1$.
   (b) (15 points) If $f, g, h \in L^1$ then $(f * g) * h = f * (g * h)$ and $f * g = g * f$ a.e.

[1] Using axiom of choice one shows that there are uncountably many non-linear functions like this.
[2] Hint for (d): show $m\{x \in \mathbb{R} : |f(x)| < n\} > 0$ for some n and use (a) and Problem 31 on page 40.