Math 533, Midterm 2-II (take home)

due 2008-11-10

There are 4 questions for a total of 110 points. (10 bonus).
Due Monday, Nov 10, 1pm.

Name: ____________________

1. (25 points) Let f_n, g_n, f, g be functions in $L^1(X, \mathcal{M}, \mu)$.
   Assume that μ-a.e. $g_n \to g$, $f_n \to f$ and $|f_n| \le g_n$.
   Prove that if $\int g_n \to \int g$ then $\int f_n \to \int f$.

2. (25 points) Lusin's Theorem
   (a) State Lusin's theorem.
   (b) Prove Lusin's theorem, relying on Egoroff's theorem.

3. (25 points) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $f \in L^+(X, \mathcal{M})$. Let

$$A = \{(x, y) \in X \times [0, \infty] \ \mid \ y \le f(x)\}.$$

   (a) Prove that A is $\mathcal{M} \otimes \mathcal{B}_{\mathbf{R}}$-measurable.
   (b) Prove that $\mu \times m(A) = \int f \, d\mu$.

4. (35 points) Let $\{E_n\}_{n=1}^{\infty}$ a sequence of measurable subsets in a measure space $(X, \mathcal{M}, \mu)$, and $f_n(x) = \mathbf{1}_{E_n}(x)$ their characteristic functions.
   Prove that the following are equivalent:
   (a) The sequence $\{f_n\}$ is Cauchy in $L^1(X, \mathcal{M}, \mu)$.
   (b) The sequence $\{f_n\}$ is Cauchy in measure.
   (c) If, in addition $\mu(X) < \infty$, then (a) and (b) are also equivalent to

$$\mu\left((\limsup_{n\to\infty} E_n) \setminus (\liminf_{n\to\infty} E_n) \right) = 0.$$

   Remarks: For (c) Fatous' Lemma can be useful.