# Math 533 - Real Analysis. HW on cardinality

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Recall:

- For any two non-empty sets $X$ and $Y$ the set $Y^X$ stands for the collection of all functions $f : X \to Y$ (note the order).
- If $X$ is a non-empty set, then $P(X)$ denotes the collection of all its subsets. $P(X)$ is naturally identified with $\{0,1\}^X$.
- The cardinality of $\mathbf{N}$ is denoted by the Hebrew $\aleph_0$, the cardinality of the reals $\mathbf{R}$ by $\aleph$. The latter cardinality (of the reals) is also called the *continuum*.
- $\aleph = 2^{\aleph_0}$, or in other words $\mathrm{card}(\{0,1\}^{\mathbf{N}}) = \mathrm{card}(\mathbf{R})$.
- Cantor's "diagonal argument" proves that $\aleph_0 < \aleph$, continuum is uncountable. More generally, $\mathrm{card}(X) < \mathrm{card}(P(X))$.

1. For any three non-empty sets $X, Y, Z$ identify (i.e. build a "natural" bijection between) the sets
$$(Z^Y)^X \qquad \text{and} \qquad Z^{Y \times X}.$$

2. Prove that $\mathrm{card}(\mathbf{R}^{\mathbf{N}}) = \mathrm{card}(\mathbf{R}) = \aleph$ (Hint: use problem 1).

3. Prove that the space $C(\mathbf{R})$ of all continuous functions $\mathbf{R} \to \mathbf{R}$ has the cardinality of the continuum (Hint: a continuos function is determined by its values at, say rational points).

4. Prove that the collection of all open subsets of $\mathbf{R}$ has the cardinality of the continuum (Hint: first show that any open set $U \subset \mathbf{R}$ is a union of open intervals with rational endpoints).