FINAL PROJECTS IN ERGODIC THEORY

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1. SOME SKEW-PRODUCTS

Problem 1.1. Let (X,T) be a minimal dynamical system, K a compact Hausdorff group (not necessarily commutative), $f : X \to K$ a continuous map. Consider the dynamical system $(\hat{X},\hat{T})$ where

$$\hat{X} = X \times K, \qquad \hat{T}(x,k) = (Tx, f(x)k).$$

Prove that $(\hat{X},\hat{T})$ is minimal iff there does not exist closed proper subgroup $L < K$ and a continuous map $g : X \to K$ so that

$$g(Tx)^{-1}f(x)g(x) \in L \qquad (\forall x \in X).$$

(*Hint*: consider a $\hat{T}$-minimal subset $M \subset \hat{X}$ and $L = \{t \in K \mid (x,k) \in M \implies (x,kt) \in M\}$).

In the case of $K = S^1$ deduce that $(\hat{X},\hat{T})$ is minimal unless there exists a continuous $h : X \to S^1$ and $n \neq 0$ so that

$$f(x)^n = h(Tx)/h(x) \qquad (\forall x \in X). \tag{1.1}$$

Problem 1.2. Let (X,μ,T) be an ergodic system, $f : X \to S^1$ a measurable map. Consider the dynamical system $(\hat{X},\hat{\mu},\hat{T})$ where

$$\hat{X} = X \times S^1, \qquad \hat{\mu} = \mu \times \lambda \qquad \hat{T}(x,z) = (Tx, f(x)z).$$

Prove that $(\hat{X},\hat{\mu},\hat{T})$ is ergodic iff there does not exist a measurable $h : X \to S^1$ and $n \neq 0$ with

$$f(x)^n = h(Tx)/h(x) \qquad \mu\text{-a.e.}$$

(*Hint:* write a $\hat{T}$-invariant function as $F(x,z) = \sum_{n\in\mathbf{Z}} a_n(x)\cdot z^n$.)

Problem 1.3. Prove that there exists $\alpha \notin \mathbf{Q}$ and a C^∞-smooth $\phi : S^1 \to \mathbf{R}$ so the following holds for the rotation $T : x \mapsto e^{2\pi\alpha i}x$ on $X = S^1$:

- There exists $g \in L^2(S^1)$ with $\phi(x) = g(Tx) - g(x)$ a.e.
- There exists $p \in (0,\infty)$ so that $f : X \to S^1$ given by $f(x) = e^{p\cdot\phi(x)}$ does not satisfy (1.1) for any $n \neq 0$ and continuous $h : S^1 \to S^1$.

Deduce that the C^∞- skew-product diffeomorphism $\hat{T} : (x,z) \mapsto (e^{2\pi\alpha i}x, f(x)z)$ of the torus $S^1 \times S^1$ is minimal but not uniquely ergodic.

Hints:

(1) If $f(x) = g(Tx)/g(x)$ a.e., then g is determined a.e. uniquely up to a constant multiple.

(2) A function $f : S^1 \to \mathbf{R}$ is C^∞-smooth iff its Fourier coefficients $\hat{f}(n)$ decay faster than any polynomial.

(3) If $g : S^1 \to \mathbf{R}$ is continuous then the Cesaro averages of its Fourier series converge pointwise to g, in particular, $\frac{1}{n}\sum_{k=1}^{n}\hat{g}(k) \to g(0)$.

(4) Choose α carefully and $g \in L^2(S^1)$ through its Fourier coefficients $\hat{g} \in \ell^2(\mathbf{Z})$.

2. Minimality and unique ergodicity

Problem 2.1. Let X be a compact metric space and $T : X \to X$ a continuous map.

(1) Prove that (X,T) is *minimal*, i.e., X contains no closed T-invariant proper subsets, iff for any non-empty open set U there is $n \in \mathbf{N}$ so that $X = \bigcup_{i=0}^{n} T^{-i}U$.

(2) Let A be a finite set, T the shift on $A^{\mathbf{Z}}$, $(Tx)_i = x_{i+1}$. Prove that for a point $x \in A^{\mathbf{Z}}$ and $X = \overline{\{T^n x : n \in \mathbf{Z}\}}$ the system (X,T) is minimal iff for any word $w = (x_k,\dots,x_m)$ in x appears in x with bounded gaps: there exists $n = n(w)$ so that for any $i \in \mathbf{Z}$ there is $j \in \{i,\dots,i+n(w)\}$ with $(x_j,x_{j+1},\dots,x_{j+m-k}) = w$.

Problem 2.2. Let $T : X \to X$ be a continuous map (homeomorphism) of a compact metrizable space X, and $\mathrm{Prob}^T(X) = \{\mu \in \mathrm{Prob}(X) \mid T_*\mu = \mu\}$ - the space of T-invariant probability measures. Given a continuous function $f : X \to \mathbf{R}$ define

$$I_*(f) = \inf_{\mu \in \mathrm{Prob}^T(X)} \int_X f\,d\mu, \qquad I^*(f) = \sup_{\mu \in \mathrm{Prob}^T(X)} \int_X f\,d\mu.$$

(1) Explain why inf and sup can be replaced by min and max, and the set of T-invariant measures by the subset of T-ergodic ones.

(2) Prove that for any $f \in C(X,\mathbf{R})$ the following inequalities hold for *every* $x \in X$ and *uniformly* on X:

$$I_*(f) \le \liminf_{n\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x), \qquad \limsup_{n\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \le I^*(f).$$

(3) Assume that (X,T) is minimal. Check that any T-invariant μ has full support on X. Prove that for any $f \in C(X,\mathbf{R})$ one has

$$\left\{ x \in X \mid \int f\,d\mu \quad \text{is a limit point of} \quad \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) \right\}$$

is a dense G_δ-set in X.

(4) Deduce, that if (X,T) is minimal then for $f \in C(X,\mathbf{R})$ the set

$$\left\{ x \in X \mid I_*(f) = \liminf_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x), \quad \limsup_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) = I^*(f) \right\}$$

is a dense G_δ-set in X.

Denote $V = \{f \in C(X,\mathbf{R}) \mid I_*(f) = I^*(f)\}$. Then (2) implies that for every $f \in V$ the ergodic averages converge to $I_*(f) = I^*(f)$ for *every* $x \in X$ and *uniformly* on X; for minimal systems for $f \notin V$ fact (4) shows that the ergodic averages diverge on a dense G_δ-set.

3. Subshifts of Finite Type

Let A be a $d \times d$-matrix with entries 0 or 1 and such that $(A^n)_{i,j} > 0$ for all $n \geq n_0$. Denote $X_A = \left\{ x \in \{1, \ldots, d\}^{\mathbf{Z}} \;\middle|\; \forall i : \; A_{x_i, x_{i+1}} = 1 \right\}$ and let T be the shift $(Tx)_i = x_{i+1}$.

Problem 3.1. Recall that a Markov measure $\mu_{P,\mathbf{q}}$ on X_A is associated to a non-negative $d \times d$ stochastic[1] matrix P subordinate[2] to A and a P-stationary[3] vector $\mathbf{q}$ by the formula

$$\mu_{P,q}((\ldots, i_1, i_2, \ldots, i_n, \ldots)) = q_{i_1} P_{i_1, i_2} \cdots P_{i_{n-1}, i_n}.$$

Prove that such $\mu_{P,q}$ is T-invariant, and ergodic (in fact, mixing) if P satisfies $(P^n)_{ij} > 0$ for all $n \geq n_0$. Show that the entropy of such a system is given by

$$h(X_A, \mu_{P,\mathbf{q}}, T) = -\sum_{i,j} q_i P_{ij} \log P_{ij}.$$

Problem 3.2. Using Lagrange multipliers prove that among all Markov measures $\mu_{P,\mathbf{q}}$ subordinate to A the maximal entropy is attained by the Shannon-Parry measure corresponding to

$$P_{ij} = A_{ij} \frac{u_j}{\lambda u_i}, \qquad q_i = u_i \cdot v_i,$$

where $\lambda > 1$ the dominant eigenvalue of A, and $\mathbf{u}, \mathbf{v} \in \mathbf{R}^d$ are positive eigenvectors with

$$A\mathbf{u} = \lambda \mathbf{u}, \qquad A^{tr}\mathbf{v} = \lambda \mathbf{v}, \qquad \langle \mathbf{u}, \mathbf{v} \rangle = 1.$$

Here one has

$$h(X_A, \mu_{P,\mathbf{q}}, T) = h^{\text{top}}(X_A, T) = \log \lambda.$$

Problem 3.3. Let $\mu_{P,\mathbf{q}}$ be the Shannon-Parry measure. Prove that $\mu_{P,\mathbf{q}}$ is the weak-* limit of

$$\mu_n = \frac{1}{\#\{x : T^n x = x\}} \sum_{T^n x = x} \delta_x.$$

Hint: It suffices to prove that $\mu_n(E) \to \mu_{P,\mathbf{q}}(E)$ for cylindrical sets $E = \{(\ldots, i_1, i_2, \ldots, i_n, \ldots)\}$. Start with $n = 1$ and $n = 2$.

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[1] P is stochastic if $P\mathbf{1} = \mathbf{1}$, i.e. $\sum_j P_{ij} = 1$ for each i

[2] meaning that $A_{ij} = 0$ implies $P_{ij} = 0$, i.e. $P \leq A$ componentwise.

[3] probability vector satisfying $P^{tr}\mathbf{q} = \mathbf{q}$, i.e., $q_j = \sum_i q_i P_{ij}$.

4. The measure of maximal entropy for a mixing Subshift of Finite Type

Problem 4.1. Let (X_A, T) be a SFT as in Problems 3.1–3.3. Prove that if μ is a T-invariant probability measure on X_A satisfying

$$h(X_A, \mu, T) = \log\lambda = h^{\text{top}}(X_A, T)$$

then μ is the Shannon-Parry Markov measure $\mu_{P,\mathbf{q}}$.

Hint: Prove that $\mu(E) = \mu_{P,\mathbf{q}}(E)$ for cylindrical sets $E = \{(\dots, i_1, i_2, \dots, i_k, \dots)\}$. Start with $k = 1$ and $k = 2$. Use the Lemma below.

Fix a finite integer $d \geq 2$. For integers $k \ll n$ and any $x \in \{1, \dots, d\}^n$ let μ_x denote the probability distribution on $\{1, \dots, d\}^k$ given by

$$\mu_x(i_1, \dots, i_k) = \frac{\#\{j \in \{0, \dots, n-k\} \mid x_{j+1} = i_1, \dots, x_{j+k} = i_k\}}{n-k+1}.$$

So μ_x is the *empirical distribution* of k-words inside x. Denote by $H_k(\mu_x)$ the entropy of this distribution on k-letters

$$H_k(\mu_x) = -\sum_{y \in \{1, \dots, d\}^k} \mu_x(y) \cdot \log\mu_x(y).$$

Lemma 4.2. *Given $d \geq 2$, $k \in \mathbf{N}$, h and $\varepsilon > 0$ there exists N so that for $n \geq N$*

$$\frac{1}{n}\log\#\left\{x \in \{1, \dots, d\}^n \,\middle|\, \frac{1}{k}H_k(\mu_x) \leq h\right\} < h + \varepsilon.$$