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MEASURABLE RIGIDITY OF ACTIONS ON INFINITE MEASURE HOMOGENEOUS SPACES, II

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1. Introduction and statement of the main results

The starting point of our discussion is the following beautiful result of Yehuda Shalom and Tim Steger:

Theorem 1.1 (Shalom and Steger, [21]). *Measurable isomorphisms between linear actions on $\mathbf{R}^2$ of abstractly isomorphic lattices in $\mathbf{SL}_2(\mathbf{R})$ are algebraic.*

More precisely, if $\tau : \Gamma_1 \xrightarrow{\cong} \Gamma_2$ is an isomorphism between two lattices in $\mathbf{SL}_2(\mathbf{R})$ and $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a measure class preserving map with $T(\gamma x) = \gamma^\tau T(x)$ for a.e. $x \in \mathbf{R}^2$ and all $\gamma \in \Gamma_1$, then there exists $A \in \mathbf{GL}_2(\mathbf{R})$ so that $\gamma^\tau = A\gamma A^{-1}$ for all $\gamma \in \Gamma_1$ and $T(x) = Ax$ a.e. on $\mathbf{R}^2$.

The linear $\mathbf{SL}_2(\mathbf{R})$-action on $\mathbf{R}^2 - \{0\}$ is $G = \mathbf{SL}_2(\mathbf{R})$-action on the homogeneous space G/H where H is the *horocyclic* subgroup

$$H = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \,:\, s \in \mathbf{R} \right\}.$$

The action of Γ on G/H is closely related to its "dual" dynamical system—the action of H on G/Γ, which is an algebraic description of the horocycle flow on the unit tangent bundle SM to the Riemann surface $M = \mathbf{H}^2/\Gamma = K\backslash G/\Gamma$ (here we assume that Γ is torsion-free). In the 1980s, Marina Ratner discovered remarkable measurable rigidity properties of the horocycle flow, proving that all measurable isomorphisms [17], measurable quotients [18], and finally all joinings [19] of such flows are algebraic, i.e., G-equivariant. The Shalom-Steger result above can be viewed as a "dual companion" of Ratner's isomorphism theorem [17]. It is important to emphasize, however, that despite the similarities, Theorem 1.1 is not directly related to (neither implies nor follows from) any of the above results of Ratner; it also cannot be deduced from the celebrated Ratner classification of invariant measures theorem [20], which contains [17], [18], [19] as particular cases.

Shalom and Steger prove their Theorem 1.1 (and other rigidity results, such as Corollary 1.9 (1), (2) below) ingeniously using unitary representation techniques. The present paper grew out of an attempt to give an alternative, purely dynamical proof for this theorem and other related results from [21][1]. The technique that has been developed for this purpose—the alignment property—is quite general and

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[1]Hence the numeral II in the title of this paper.

turns out to be very powerful. We develop it as an abstract tool and show how to apply it to homogeneous spaces and spaces of horospheres. In the present work the rigidity phenomena of Theorem 1.1 are generalized in several directions: (1) we consider homogeneous spaces of all semi-simple groups, (2) we also consider spaces of horospheres in variable pinched negative curvature, (3) in the context of negative curvature we treat actions of discrete groups Γ which are not necessarily lattices, (4) in all the above examples we prove rigidity results not only for isomorphisms but also for quotients and more generally for joinings. In the forthcoming paper [2] further generalizations of these results are obtained using more direct arguments on homogeneous spaces.

Before stating the results, we need to set a few conventions concerning II_∞ *actions*—these are measure preserving, ergodic group actions on non-atomic *infinite measure* Lebesgue spaces (see section 2 for more details).

Definitions 1.2. Let (X_i, m_i, Γ) $(i = 1, 2)$ be two ergodic measure preserving actions of a fixed group Γ on infinite measure Lebesgue spaces (X_i, m_i) and $\tau \in \operatorname{Aut}\Gamma$ be a group automorphism. A *morphism* or (τ-twisted) *quotient map* between such systems is a measurable map $\pi : X_1 \to X_2$ such that

$$\pi_* m_1 = const \cdot m_2 \qquad \text{and} \qquad \pi(\gamma x) = \gamma^\tau \pi(x)$$

for all $\gamma \in \Gamma$ and m_1-a.e. $x \in X_1$. In particular, the first condition implies that the preimage $\pi^{-1}(E)$ of a set $E \subset X_2$ of finite m_2-measure has finite m_1-measure. A (τ-twisted) *isomorphism* is a measurable bijection with the same properties. A (τ-twisted) *joining* of such systems is a measure $\bar{m}$ on $X = X_1 \times X_2$ which is invariant under the (twisted) diagonal Γ-action

$$\gamma : (x_1, x_2) \mapsto (\gamma x_1, \gamma^\tau x_2)$$

and such that the projections $\pi_i : X \to X_i$ $(i = 1, 2)$ are morphisms, i.e.,

$$(\pi_i)_* \bar{m} = const_i \cdot m_i \qquad (i = 1, 2).$$

Two systems are *disjoint* if they admit no joinings[2]. Given an infinite measure preserving ergodic system (X, m, Γ), its measurable *centralizer* is defined to be the group of all measurable (possibly twisted) automorphisms of the Γ-action on (X, m). Similarly *self-joinings* are joinings of (X, m, Γ) with itself.

This framework of II_∞-actions in many respects parallels that of II_1-actions—the classical theory of ergodic *probability measure* preserving actions. For example, self-joinings control centralizers and quotients of a given system, and joinings between two systems control isomorphisms and common quotients (see section 2 for details).

We shall be mostly interested in actions of discrete subgroups $\Gamma < G$ on homogeneous spaces $X = G/H$, where G is a locally compact (always second countable) group and H is a closed subgroup so that G/H carries an infinite G-invariant measure $m_{G/H}$. The main results of the paper assert that

- measurable centralizers, quotients and self-joinings of $(G/H, \Gamma)$ and
- measurable isomorphisms and joinings between two such systems $(G_1/H_1, \Gamma)$ and $(G_2/H_2, \Gamma)$

are *algebraic*, i.e., essentially coincide with centralizers, quotients, isomorphisms, joinings, etc., for the transitive G-action on $X = G/H$. Let us describe algebraic centralizers and quotients more explicitly:

[2]Note that for infinite measure systems the product measure $m_1 \times m_2$ is a not a joining.

Example 1.3 (Algebraic centralizers and quotients)**.** Let $X = G/H$ be a homogeneous space with an infinite G-invariant measure m. For the transitive G-action on (X, m) we have the following:

Centralizers: The centralizer of the G-action on X in both the measurable and set-theoretic sense is the group $\Lambda = N_G(H)/H$, where $\lambda = n_\lambda H \in \Lambda$ acts on $X = G/H$ by $\lambda : x = gH \mapsto \lambda x = gn_\lambda H$.

Quotients: Any G-equivariant measurable quotient of G/H is G/H' via $\pi : gH \mapsto gH'$, where $H < H'$ and H'/H is compact.

We shall consider semi-simple Lie groups G and a class of closed unimodular subgroups $H < G$ which we call "super-spherical" (see Definition 1.7). For rank one real Lie group G, a super-spherical subgroup is any closed subgroup $H < G$ with $N < H < MN$, where N is the horospherical subgroup and $M < K$ is the centralizer of the Cartan A in K. The assumptions on $\Gamma < G$ will vary: requiring $\Gamma < G$ to be a lattice would be sufficient to establish rigidity for all the examples; for homogeneous spaces G/H of rank one real Lie groups G, a wider class of discrete subgroups $\Gamma < G$ can be shown to be rigid. We start with these latter cases.

Homogeneous spaces of rank one real Lie groups.

Theorem A (**Real rank one: Centralizers, self-joinings and quotients**)**.** *Let G be a real, connected, Lie group of rank one with trivial center, $N < G$ its horospherical subgroup, and $H < G$ a proper closed unimodular subgroup $N < H < MN$. Suppose that $\Gamma < G$ is a discrete subgroup acting ergodically on $(G/N, m_{G/N})$ and hence also on the homogeneous space $(X, m) = (G/H, m_{G/H})$.*

Then the Γ-action on (X, m) has only algebraic centralizers and quotients as described in Example 1.3, and any ergodic self-joining descends to an algebraic centralizer of the algebraic quotient G/MN.

Let us describe the scope of this theorem. The possibilities for $H < G$ as in the theorem are quite restricted: the spaces G/H are compact extensions of G/MN—the space of horospheres $\mathrm{Hor}(S\mathbf{H})$ in the unit tangent bundle $S\mathbf{H}$ to the symmetric space $\mathbf{H} \cong G/K$ of G. However, the condition on a discrete subgroup $\Gamma < G$ is quite mild. Examples of such subgroups include the following.

- Any lattice Γ in G (both uniform and non-uniform ones). Ergodicity of the Γ-action on $(G/H, m_{G/H})$ follows from Moore's ergodicity theorem (H is not precompact and hence acts ergodically on G/Γ).
- Let $\Lambda < G$ be a lattice and $\Gamma \triangleleft \Lambda$ so that Λ/Γ is nilpotent. Then Γ acts ergodically on $G/MN = \mathrm{Hor}(S\mathbf{H})$. This was proved by Babillot and Ledrappier [1] for the case where Λ/Γ is Abelian and $\Lambda < G$ is a uniform torsion-free lattice. In [11] Kaimanovich showed that in this context ergodicity of the Γ-action on the space of horospheres is equivalent to the ergodicity of the Γ-action on the sphere at infinity $\partial\mathbf{H} = G/MAN$ which, in turn, is equivalent to the lack of non-constant bounded harmonic functions on the regular cover $\bar{M} = \Gamma\backslash\mathbf{H}$ of the finite volume manifold $M = \Lambda\backslash\mathbf{H}$. For nilpotent covering group Λ/Γ the latter is well known (e.g., Kaimanovich [10]).

The next rigidity result requires a stronger assumption on a discrete group Γ in a rank one Lie group G. Let $\mathbf{H} = G/K$ denote the symmetric space of G and $\partial\mathbf{H} = G/MAN$ its boundary.

Definition 1.4. We shall say that a torsion-free Γ satisfies condition (E2) if the following equivalent conditions hold (the equivalence is due to Sullivan [22]):

(E2a) Γ acts ergodically on $\partial\mathbf{H} \times \partial\mathbf{H}$ with respect to the standard measure class (that of the $K \times K$-invariant measure).
(E2b) The geodesic flow is ergodic on $S\mathbf{H}/\Gamma$.
(E2c) The Poincaré series $\sum_{\gamma\in\Gamma} e^{-s\cdot d(\gamma p,p)}$ diverges at $s = \delta(\mathbf{H})$ where

$$\delta(\mathbf{H}) = \lim_{R\to\infty} \frac{1}{R} \log \mathrm{Vol}(B(p,R))$$

denotes the volume growth rate of the symmetric space $\mathbf{H}$.

These conditions are satisfied by any lattice $\Gamma < G$. In [8] Guivarc'h considered geodesic flows on regular covers of compact hyperbolic manifolds. His results (extending previous work of M. Rees) in particular imply that if $\Lambda < G$ is a uniform lattice and $\Gamma \triangleleft \Lambda$, then Γ satisfies (E2b) iff a simple random walk on Λ/Γ is recurrent, which occurs iff Λ/Γ is a finite extension of $\mathbf{Z}^d$ with $d \le 2$.

Theorem B (**Real rank one: Rigidity of actions).** *Let G_1, G_2 be real, connected, non-compact, rank one Lie groups with trivial centers, $N_i < G_i$ the horospheric subgroups, $H_i < G_i$ closed unimodular subgroups with $\check{H}_i = N_i < H_i < \hat{H}_i = M_i N_i$, and $(X_i, m_i) = (G_i/H_i, m_{G_i/H_i})$. Let $\Gamma_i < G_i$ be discrete subgroups satisfying condition* (E2) *and acting ergodically on $(G_i/N_i, m_{G_i/N_i})$. Assume that Γ_1 and Γ_2 are isomorphic as abstract groups $\tau : \Gamma_1 \xrightarrow{\cong} \Gamma_2$ and that (X_i, m_i, Γ_i) admit a τ-twisted (ergodic) joining $\bar{m}$. Then*

1. *$\tau : \Gamma_1 \to \Gamma_2$ extends to an isomorphism of the ambient groups $\tau : G_1 \xrightarrow{\cong} G_2$ which maps N_1 onto N_2,*
2. *the joining $\bar{m}$ descends to an algebraic isomorphism $T' : G_1/\hat{H}_1 \to G_2/\hat{H}_2$ between algebraic quotients of the original systems, with $\hat{H}_i = M_i N_i > H_i$,*
3. *if the actions admit not only an ergodic joining but actually a measurable τ-twisted isomorphism $T : X_1 \to X_2$, then the isomorphism $\tau : G_1 \to G_2$ as in* (1) *in addition maps H_1 onto H_2 and for some $\lambda \in N_{G_2}(H_2)$ we have almost everywhere $T(gH_1) = \lambda g^\tau H_2$.*

The above theorem in particular applies to $\Gamma_i < G_i$ being lattices. However, due to Mostow rigidity the only examples of abstractly isomorphic but not conjugate lattices occur in $G_1 = G_2 = \mathbf{PSL}_2(\mathbf{R})$. In these cases, $X_1 = X_2 = (\mathbf{R}^2 - \{0\})/x \sim \pm x$, but an easy modification gives a similar rigidity result for $\Gamma_i < \mathbf{SL}_2(\mathbf{R})$ acting linearly on $\mathbf{R}^2$. In this case statement (3) gives Shalom and Steger's result, Theorem 1.1.

In addition to lattices in $\mathbf{SL}_2(\mathbf{R})$, there are many examples of infinite covolume discrete subgroups in rank one G satisfying condition (E2); in particular, these examples include certain normal subgroups in uniform lattices, namely fundamental groups of $\mathbf{Z}$ or $\mathbf{Z}^2$ regular covers of compact locally symmetric spaces. This opens the possibility for the same group Γ to be embedded as a discrete subgroup satisfying (E2) in different rank one groups G_1 and G_2. It is known that most (conjecturally all) arithmetic lattices in rank one groups $G \simeq \mathbf{SO}_{n,1}(\mathbf{R})$ and $\mathbf{SU}_{n,1}(\mathbf{R})$ have a finite index subgroup Λ with infinite abelianization (cf. [12], [16]). In particular it would fit in an exact sequence $\Gamma \to \Lambda \to \mathbf{Z}$, and often the kernel Γ is expected to be a free group on infinitely many generators F_∞.

Another class of examples is obtained by embedding the fundamental group $\Gamma = \pi_1(S)$ of a closed orientable surface S of genus $g \geq 2$ in $\mathbf{PSL}_2(\mathbf{C})$. Let ϕ be a pseudo-Anosov diffeomorphism of S, and let $\Gamma_\phi = \mathbf{Z} \ltimes_{[\phi]} \Gamma$ denote the semi-direct product defined by $[\phi] \in \mathrm{Out}\Gamma$. Then Γ_ϕ is a fundamental group for the 3-manifold $M_\phi = S \times [0,1]/(x,0) \sim (\phi(x),1)$. By the famous hyperbolization theorem of Thurston ([15]) such an M_ϕ admits a hyperbolic structure, i.e., Γ_ϕ is a cocompact lattice in $G = \mathrm{Isom}_+(\mathbf{H}^3) = \mathbf{PSL}_2(\mathbf{C})$. It contains the surface group Γ as a normal subgroup with $\Gamma_\phi/\Gamma \cong \mathbf{Z}$. Thus Γ satisfies (E2). Therefore, surface groups Γ can appear as a discrete subgroup with condition (E2) in a variety of ways in $\mathbf{PSL}_2(\mathbf{R}) \cong \mathrm{Isom}_+(\mathbf{H}^2)$ and in $\mathbf{PSL}_2(\mathbf{C}) \cong \mathrm{Isom}_+(\mathbf{H}^3)$. In the former, there is a continuum of such embeddings—parametrized by the Teichmuller space; in the latter, there are (at least) countably many such embeddings defined by varying a pseudo-Anosov element ϕ in the mapping class group of S.

Remark 1.5. The rank one results can be extended to the geometric context of the spaces of horospheres in manifolds of variable negative curvature. We formulate this result (Theorem 5.2) in Section 5.

Homogeneous spaces of general semi-simple groups. It turns out that rigidity phenomena for actions of *lattices* are quite wide spread among homogeneous spaces G/H of semi-simple groups G and sufficiently large unimodular $H < G$. Before formulating general results, consider the following:

Example 1.6. Let k be a local field, i.e., $\mathbf{R}$, $\mathbf{C}$, or a finite extension of $\mathbf{Q}_p$ for a prime p, and let $G = \mathbf{SL}_n(k)$. Then $X = k^n \setminus \{0\}$ is the homogeneous space G/H for the unimodular closed subgroup

$$H = \{g \in \mathbf{SL}_n(k) \,:\, g_{11} = 1,\ g_{21} = \cdots = g_{n1} = 0\}.$$

More generally, given a partition $n = n_1 + \cdots + n_m$ (with $m > 1$ and $n_i \in \mathbf{N}$), consider the subgroup $Q < G = \mathbf{SL}_n(k)$ consisting of the upper triangular block matrices of the form

$$\begin{pmatrix} A_{11} & B_{12} & \cdots & B_{1m} \\ 0 & A_{22} & \cdots & B_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{mm} \end{pmatrix} \tag{1.i}$$

where $A_{ii} \in \mathbf{GL}_{n_i}(k)$ and $B_{ij} \in M_{n_i \times n_j}(k)$. Let $\check{H} \lhd Q$ denote the closed subgroup consisting of matrices with $\det A_{ii} = 1$ $(i = 1, \ldots, m)$; and let $\hat{H} \lhd Q$ denote a slightly larger subgroup consisting of block matrices as above with $|\det A_{ii}| = 1$, where $|\cdot| : k \to [0,\infty)$ denotes the absolute value on k. Then $\check{H} < \hat{H}$ and any intermediate closed subgroup $\check{H} < H < \hat{H}$ are unimodular and $X = G/H$ carries an infinite G-invariant measure $m = m_{G/H}$.

We shall now describe a more general setting for our rigidity results: G will be a semi-simple group (in a very general sense), while $H < G$ will be restricted to some class of subgroups, which we shall call "*super-spherical*"; this class includes the cases appearing in Theorems A and B and in Example 1.6. The formal definition/construction is the following:

Definition 1.7 (Super-spherical subgroups H in a semi-simple G)**.** Let A be a finite set. For $\alpha \in A$ let k_α be a local field and $\mathbf{G}_\alpha$ be some connected, semi-simple linear algebraic k_α-group. The product

$$G = \prod_{\alpha \in A} \mathbf{G}_\alpha(k_\alpha) \tag{1.ii}$$

of k_α-points of the corresponding k_α-groups taken with the Hausdorff topology is a localy compact second countable group. We shall refer to such groups just as *semi-simple*. For each $\alpha \in A$, choose a k_α-parabolic subgroup $\mathbf{Q}_\alpha < \mathbf{G}_\alpha$ and, taking the product of k_α-points of these groups, form a closed subgroup $Q = \prod_{\alpha\in A} \mathbf{G}_\alpha(k_\alpha)$ in G. To any such subgroup $Q < G$, which we call *parabolic*, we associate two closed unimodular subgroups $\check{H} \triangleleft \hat{H} < G$ with $\check{H} < [Q,Q] < \hat{H}$ and $\hat{H}/\check{H}$ compact; any intermediate closed subgroup $\check{H} < H < \hat{H}$ will be called *super-spherical*. We let $\hat{H}$ be the preimage under the abelianization epimorphism $Q \to Q/[Q,Q]$ of the maximal compact subgroup of the locally compact Abelian group $Q/[Q,Q]$, and let $\check{H} = \prod_{\alpha\in A} \check{H}_\alpha$ where $\check{H}_\alpha = [\mathbf{Q},\mathbf{Q}](k_\alpha)^+$. (For a k-algebraic group $\mathbf{L}$, $\mathbf{L}(k)^+$ denotes the normal subgroup generated by unipotent radicals of k-parabolic subgroups of $\mathbf{L}$; it is a normal cocompact subgroup of $\mathbf{L}(k)$ [4, 6.14].)

Remark 1.8. In a given semi-simple group G, the collection of all super-spherical subgroups $H < G$ splits into families of groups related to a given parabolic $Q < G$; in each such family the groups H share a common cocompact normal subgroup $\check{H}$ and a common compact extension $\hat{H}$ with $[Q,Q]$.

Theorem C (General case: Centralizers, self-joinings and quotients). *Let $G = \prod \mathbf{G}_\alpha(k_\alpha)$ be a semi-simple group and $H < G$ be a super-spherical subgroup as in Definition* 1.7 *associated to a parabolic $Q < G$. Let $\Gamma < G$ be a lattice acting by left translations on $(X,m) = (G/H, m_{G/H})$.*

Then the only measurable centralizers and quotients of the Γ-action on (X,m) are algebraic (as in Example 1.3*), and any ergodic self-joining descends to an algebraic automorphism of an algebraic quotient $(\hat{X},\hat{m}) = (G/\hat{H}, m_{G/\hat{H}})$ and is itself a quotient of an algebraic automorphism of an algebraic extension $(\check{X},\check{m}) = (G/\check{H}, m_{G/\check{H}})$.*

Corollary 1.9. *Let k be a local field, let $\Gamma < \mathbf{SL}_n(k)$ be a lattice and let (X,m) denote the vector space k^n with the Lebesgue measure, with the linear Γ-action. Then*

1. *the measurable centralizer of the system (X,m,Γ) consists only of homotheties: $x \mapsto \lambda x$, where $\lambda \in k^*$,*
2. *the only measurably proper quotients of (X,m,Γ) are of the form k^n/C where C is a closed subgroup of the compact Abelian group of k-units $U_k = \{u \in k : |u| = 1\}$,*
3. *the only ergodic self-joinings are on graphs of homotheties*

$$\{(x,\lambda x) \mid x \in k^n\} \qquad (\lambda \in k^*).$$

Remark 1.10. Items (1) and (2) in the above corollary were first proved by Shalom and Steger [21].

For higher rank groups our techniques are restricted to lattices. Due to Mostow-Margulis rigidity, we are not able to vary the embedding of a given lattice in a higher rank group G. Hence, we shall consider a fixed lattice $\Gamma < G$, but we will still be able to vary the homogeneous space G/H.

Theorem D (**General case: Rigidity for actions**). *Let $G = \prod \mathbf{G}_i(k_i)$ be a semi-simple group, $\Gamma < G$ be a lattice, and $H_1, H_2 < G$ be two super-spherical subgroups as in Definition* 1.7*. Assume that the* Γ*-actions on* $(X_i, m_i) = (G/H_i, m_{G/H_i})$ *admit an ergodic joining* $\bar{m}$.

Then X_1*,* X_2 *share a common algebraic quotient* $(\hat{X}, \hat{m}) = (G/\hat{H}, m_{G/\hat{H}})$ *and a common algebraic extension* $(\check{X}, \check{m}) = (G/\check{H}, m_{G/\check{H}})$*; the joining* $\bar{m}$ *descends to an algebraic automorphism of the* Γ*-action on* $(\hat{X}, \hat{m})$ *and is a quotient of an algebraic automorphism of* $(\check{X}, \check{m})$.

Furthermore, if the original Γ*-actions on* (X_i, m_i) *are isomorphic, say via* $T : X_1 \to X_2$*, then for some* $q \in Q$*:* $H_1 = qH_2q^{-1}$ *and* $T(gH_1) = gqH_2$ *for* m_1*-a.e.* $gH_1 \in X_1$.

Organization of the paper. Section 2 contains the basic properties of our setup, such as joinings of infinite measure preserving group actions. In section 3, the notion of *alignment* is introduced, and general rigidity results are proved for principal bundles with alignment. In section 4 we establish the alignment property for our main examples: homogeneous spaces and spaces of horospheres. In section 5 joinings between spaces of horospheres are studied. In section 6 the final results on homogeneous spaces are proved using the alignment property.

In the forthcoming paper [2], Theorems C and D are extended to more general homogeneous spaces by using more direct arguments.

2. Preliminaries

2.a. **Strictly measure class preserving maps.** We first discuss some technical points regarding II_∞-actions. Let (X_i, m_i, Γ) $(i = 1, 2)$ be two such actions of a fixed group Γ. A measurable map $T : (X_1, m_1) \to (X_2, m_2)$ will be called *strictly measure class preserving* if $T_* m_1 \sim m_2$ and the Radon-Nikodym derivative T' is almost everywhere positive and finite. Note that such a map as the projection $\mathbf{R}^2 \to \mathbf{R}$, $(x, y) \mapsto x$, is not *strictly* measure class preserving, although it is usually considered measure class preserving.

Any strictly measure class preserving Γ-equivariant map $X_1 \to X_2$ between ergodic measure preserving Γ-actions has a Γ-invariant, and hence a.e. constant positive and finite, Radon-Nikodym derivative. Therefore

Lemma 2.1. *If* (X_i, m_i, Γ)*,* $i = 1, 2$*, are two ergodic infinite measure preserving systems and some fixed* $\tau \in \operatorname{Aut}\Gamma$*, then the following hold.*

(1) *Any strictly measure class preserving map* $T : X_1 \to X_2$*, satisfying*

$$T(\gamma x) = \gamma^\tau T(x) \qquad (\gamma \in \Gamma)$$

for a.e. $x \in X_1$*, is a (*τ*-twisted) quotient map. If furthermore* T *is invertible, then* T *is a (*τ*-twisted) isomorphism.*

(2) *A measure* $\bar{m}$ *on* $X = X_1 \times X_2$ *invariant under the (*τ*-twisted) diagonal* Γ*-action*

$$\gamma : (x_1, x_2) \mapsto (\gamma x_1, \gamma^\tau x_2),$$

for which the projections $\pi_i : (X, \bar{m}) \to (X_i, m_i)$ *are strictly measure class preserving, is a (*τ*-twisted) joining of* (X_i, m_i, Γ)*,* $i = 1, 2$.

Lemma 2.2. *Let (X, m, Γ) be an ergodic, infinite measure preserving system, and suppose that L is a locally compact group with a faithful measurable action by measure class preserving transformations on (X, m), which commute with Γ. Then the following hold.*

(1) *There exists a continuous multiplicative character $\Delta : L \to \mathbf{R}_+^*$ so that $g_*^{-1}m = \Delta(g)m$ for all $g \in L$.*
(2) *Any compact subgroup $K < L$ acts by measure preserving transformations on (X, m), and the K-orbits define a quotient system*

$$P : (X, m, \Gamma) \to (X', m', \Gamma) \qquad \text{where} \qquad X' = X/K,\ m' = m/K$$

with $P : x \in X \mapsto Kx \in X'$.

Proof. (1) The derivative cocycle $\Delta(g, x) = \frac{dg_*^{-1}m}{dm}(x)$ is a measurable map $L \times X \to \mathbf{R}_+^*$ which is invariant under the Γ-action on X. By ergodicity, $\Delta(g, x)$ is a.e. constant $\Delta(g)$ on X. Hence Δ is a measurable character $L \to \mathbf{R}_+^*$. It is well known that any measurable homomorphisms between locally compact second countable groups are continuous.

(2) The multiplicative positive reals $\mathbf{R}_+^*$ do not have any non-trivial compact subgroups. Thus Δ is trivial on K, i.e., K acts by measure preserving transformations on (X, m). The space of K-orbits X' inherits (1) a measurable structure from X (because K is compact), (2) the action of Γ (because it commutes with K), and (3) the measure m', as required. ☐

2.b. **Joinings of II_∞ systems.** Let us point out some facts about joinings of ergodic infinite measure preserving systems:

(1) Any joining between two ergodic infinite measure preserving systems disintegrates into an integral over a probability measure space of a family of ergodic joinings.
(2) If $(X \times Y, \bar{m})$ is a τ-twisted joining of ergodic infinite measure preserving systems (X, m, Γ) and (Y, n, Γ) (where $\bar{m}$ projects as $c_1 \cdot m$ on X and as $c_2 \cdot n$ on Y), then there exist unique up to null sets measurable maps $X \to \mathrm{Prob}(Y)$, $x \mapsto \mu_x$, and $Y \to \mathrm{Prob}(X)$, $y \mapsto \nu_y$, so that

$$\bar{m} = c_1 \cdot \int_X \delta_x \otimes \mu_x \, dm(x) = c_2 \cdot \int_Y \nu_y \otimes \delta_y \, dn(y).$$

Furthermore, $\mu_{\gamma x} = \gamma * \mu_x$ and $\nu_{\gamma^\tau y} = \gamma_* \nu_y$ for all $\gamma \in \Gamma$ and m-a.e. $x \in X$ and n-a.e. $y \in Y$.
(3) In contrast to actions on probability spaces, group actions on infinite measure systems do not always admit a joining[3]. Existence of a joining is an *equivalence relation* between II_∞-actions of a fixed group Γ. Indeed, reflexivity is obvious; if $\bar{m}$ on $X_1 \times X_2$ is a (τ-twisted) joining of (X_1, m_1, Γ) with (X_2, m_2, Γ), then the image $\check{\bar{m}}$ of $\bar{m}$ under the flip $X_1 \times X_2 \to X_2 \times X_1$, $(x_1, x_2) \mapsto (x_2, x_1)$, is a ($\tau^{-1}$-twisted) joining of (X_2, m_2, Γ) with (X_1, m_1, Γ).

For transitivity, one can use the following "amalgamation" construction: given three systems (X_i, m_i, Γ) $(i = 1, 2, 3)$ and joinings $\bar{m}_{12}$ and $\bar{m}_{23}$ of

[3]Note that the product measure $m_1 \times m_2$ is not a joining.

the corresponding pairs, one can use the decompositions

$$\bar{m}_{12} = c_{12} \cdot \int_{X_2} \mu_y^{(1)} \otimes \delta_y \, dm_2(y), \quad \bar{m}_{12} = c_{23} \cdot \int_{X_2} \delta_y \otimes \mu_y^{(3)} \, dm_2(y)$$

with $\mu^{(i)} : X_2 \to \mathrm{Prob}(X_i)$ $(i = 1, 3)$ measurable functions in order to construct the "amalgamated" joining $\bar{m} = \bar{m}_{12} \times_{X_2} \bar{m}_{23}$ of X_1 with X_3 by setting

$$\bar{m} = \int_{X_2} \mu_y^{(1)} \otimes \mu_y^{(3)} \, dm_2(y).$$

Note that this amalgamated joining need not be ergodic, even if $\bar{m}_{12}$ and $\bar{m}_{23}$ are. If $\bar{m}_{12}$ is τ-twisted and $\bar{m}_{23}$ is σ-twisted, then the amalgamated joining $\bar{m}$ is $\sigma \circ \tau$-twisted.

(4) If (Y, n, Γ) is a common quotient of two systems (X_i, m_i, Γ) $(i = 1, 2)$, then one can form a *relatively independent* joining of X_1 with X_2 over Y by taking the measure $\bar{m}$ on $X_1 \times X_2$ to be

$$\bar{m} = \int_Y \mu_y^{(1)} \otimes \mu_y^{(2)} \, dn(y)$$

where $m_i = c_i \cdot \int_Y \mu_y^{(i)} \, dn(y)$ $(i = 1, 2)$ are the disintegration with respect to the projections.

(5) An isomorphism (or a τ-twisted isomorphism) $T : (X_1, m_1) \to (X_2, m_2)$ gives rise to the (τ-twisted) joining:

$$\bar{m} = \int_{X_1} \delta_x \otimes \delta_{T(x)} \, dm_1(x)$$

which is supported on the graph of T. In particular, any (non-trivial) element T of the *centralizer* of (X, m, Γ) defines a (non-trivial) self-joining of (X, m, Γ).

(6) Any (τ-twisted) measurable quotient $p : (X, m, \Gamma) \to (Y, n, \Gamma)$ gives rise to the *relatively independent* self-joining $(X \times X, \bar{m})$ of (X, m, Γ) given by

$$\bar{m} = \int_Y \mu_y \otimes \mu_y \, dn(y)$$

where $m = c \cdot \int_Y \mu_y \, dn(y)$ is the disintegration of m with respect to n into a measurable family $Y \to \mathrm{Prob}(X)$, $y \mapsto \mu_y$, of probability measures ($\mu_y(p^{-1}\{y\}) = 1$ and $\mu_{\gamma y} = \gamma_* \mu_y$ for n-a.e. $y \in Y$). This joining need not be ergodic.

Thus, understanding joinings between different systems and self-joinings of a given system provides useful information on isomorphisms between systems, quotients and centralizers.

2.c. **An auxiliary lemma.** We shall need the following technical lemma.

Lemma 2.3 (Pushforward of singular measures)**.** *Let (X, μ), (Y, ν) be measure spaces, Z be a standard Borel space, $\rho : Y \to Z$ and $x \in X \mapsto \alpha_x \in \mathrm{Prob}(Y)$ be measurable maps so that*

$$\int_X \alpha_x(B) \, d\mu(x) = 0 \qquad \text{whenever} \qquad \nu(B) = 0.$$

Define a measurable map $X \to \mathrm{Prob}(Z)$ by $x \in X \mapsto \beta_x = \rho_* \alpha_x \in \mathrm{Prob}(Z)$. Then the following hold.

(1) *The map* $\beta : X \to \mathrm{Prob}(Z)$ *is well defined in terms of* $\alpha : X \to Y$ *and* $\rho : Y \to \mathrm{Prob}(Z)$, *all up to null sets. More precisely, if* $x \in X \mapsto \alpha'_x \in \mathrm{Prob}(Y)$ *agrees* μ*-a.e. with* α_x *and* $\rho' : Y \to Z$ *agrees* ν*-a.e. with* ρ, *then the map* $x \in X \mapsto \beta'_x = \rho'_* \alpha'_x \in \mathrm{Prob}(Z)$ *agrees with* β_x *for* μ*-a.e.* $x \in X$.

(2) *If a countable group* Γ *acts measurably on* X, Y, Z, *preserving the measure class of* μ *on* X *and of* ν *on* Y *and such that*
$$\alpha_{\gamma x} = \gamma_* \alpha_x, \qquad \rho(\gamma y) = \gamma \rho(y)$$
for μ*-a.e.* $x \in X$, ν*-a.e.* $y \in Y$ *and all* $\gamma \in \Gamma$, *then* $\beta_{\gamma x} = \gamma_* \beta_x$ *for* μ*-a.e.* $x \in X$ *and all* $\gamma \in \Gamma$.

Proof. (1) For μ-a.e. equality $\beta_x = \beta'_x$ it suffices to show that for each $E \in \mathcal{B}(Z)$
$$\mu\left\{x \in X \,:\, \beta_x(E) \neq \beta'_x(E)\right\} = 0$$
because $\mathcal{B}(Z)$ is countably generated (Z is a standard Borel space). Let $F = \rho^{-1}(E)$, $F' = {\rho'}^{-1}(E) \in \mathcal{B}(Y)$. We have $\nu(F \vartriangle F') = 0$ and therefore
$$\int_X \alpha_x(F \vartriangle F')\, d\mu(x) = 0.$$
By Fubini, $\alpha_x(F) = \alpha_x(F')$ for μ-a.e. $x \in X$. At the same time μ-a.e. $\alpha_x(F') = \alpha'_x(F')$ and so
$$\beta_x(E) = \alpha_x(F) = \alpha'_x(F') = \beta'_x(E)$$
for μ-a.e. $x \in X$.

(2) For each γ we have a.e. $\alpha_{\gamma x} = \gamma_* \alpha_x$ and $\rho \circ \gamma = \gamma \circ \rho$ which give rise to
$$\beta_{\gamma x} = \rho_*(\gamma_* \alpha_x) = (\rho \circ \gamma)_* \alpha_x = (\gamma \circ \rho)_* \alpha_x = \gamma_* \beta_x,$$
justified by part (1). $\square$

3. Principal bundles with alignment properties

3.a. **Basic definitions.** The following notion, which we call the *alignment property*, will play a key role in the proofs of our results. Section 4 contains examples of the alignment property. Here we shall give the definition, basic properties and the main applications of this notion.

Definition 3.1. Let Γ be a group, (X, m) be a measure space with a measure class preserving Γ-action, B be a topological space with a continuous Γ-action, and $\pi : X \to B$ be a measurable Γ-equivariant map. We shall say that π has the *alignment property* with respect to the Γ-action if $x \mapsto \delta_{\pi(x)}$ is the only Γ-equivariant measurable map from (X, m) to the space $\mathrm{Prob}(B)$ of all regular Borel probability measures on B (as usual two maps which coincide m-a.e. are identified).

The alignment property depends only on the measure class $[m]$ on a Borel space X. However in all the examples of the alignment phenomena in this paper, X is a topological space with a continuous Γ-action, the map $\pi : X \to B$ is continuous and m is an infinite Γ-invariant measure on X. The notion seems to be related to the notions of strong proximality and boundaries (cf. Furstenberg [6], [7]).

Given a measurable map $\pi : (X, m) \to B$ on a Lebesgue space, there is a well defined measure class $[\nu]$ on B—the "projection" $[\nu] = \pi_*[m]$ of the measure class $[m]$ on X. It can be defined by taking the usual pushforward $\nu = \pi_* \mu$ of some *finite* measure μ equivalent to m (being Lebesgue, m is σ-finite). The measure class $[\nu]$ depends only on $[m]$, and $\nu(E) = 0$ iff $m(\pi^{-1}E) = 0$.

We start with a list of simple but useful observations about the alignment property.

Lemma 3.2 (Uniqueness). *Let $(\pi : (X, m) \to B; \Gamma)$ be an alignment system and ν a probability measure on B with $[\nu] = \pi_*[m]$. Then*

(1) *π is the unique, up to m-null sets, measurable Γ-equivariant map $X \to B$,*
(2) *$b \mapsto \delta_{\pi(b)} \in \mathrm{Prob}(B)$ is the unique, up to ν-null sets, measurable Γ-equivariant map $B \to \mathrm{Prob}(B)$,*
(3) *the identity map is the unique, up to ν-null sets, measurable Γ-equivariant map $B \to B$.*

Proof. Evident from the definitions. □

Lemma 3.3 (Conservative). *If $\pi : (X, m) \to B$ has the alignment property with respect to a Γ-action, with a non-trivial B, then Γ-action on (X, m) is conservative.*

Proof. Indeed, otherwise there exists a Borel subset $E \subset X$ with $m(E) > 0$ so that $m(\gamma_1 E \cap \gamma_2 E) = 0$ whenever $\gamma_1 \neq \gamma_2 \in \Gamma$. Choose an arbitrary measurable map $p : E \to B$ with $p(x) \neq \pi(x)$, extend it in a Γ-equivariant way to $\Gamma E = \bigcup \gamma E$ and let $p(x) = \pi(x)$ for $x \in X \setminus \Gamma E$. Then p is a measurable Γ-equivariant map which does not agree with π on a positive measure set ΓE, contradicting the alignment property. □

Lemma 3.4 (Intermediate quotients). *Let (X, m, Γ) and (X_0, m_0, Γ) be some measure class preserving measurable Γ-actions, $p : (X, m) \to (X_0, m_0)$ a Γ-equivariant measurable map, B a topological space with a continuous Γ-action, and $\pi_0 : (X_0, m_0) \to B$ a measurable Γ-equivariant map. If the composition map*

$$\pi : (X, m) \xrightarrow{p} (X_0, m_0) \xrightarrow{\pi_0} B$$

has the alignment property, then so does $\pi_0 : (X_0, m_0) \to B$.

Proof. Follows from the definitions. □

Lemma 3.5 (Compact extensions). *Let $p : (X_1, m_1) \to (X, m)$ be a compact group extension of a Γ-action on (X, m), i.e., a compact group K acts on (X_1, m_1) by measure preserving transformations commuting with the Γ-action so that $(X, m) = (X_1, m_1)/K$ with p being the projection. If $\pi : (X, m) \to B$ has the alignment system with respect to Γ, then the map*

$$\pi_1 : (X_1, m_1) \xrightarrow{p} (X, m) \xrightarrow{\pi} B$$

has the alignment property too.

Proof. If $y \in X_1 \mapsto \nu_y \in \mathrm{Prob}(B)$ is a measurable Γ-equivariant map, then

$$x \in X \mapsto \mu_x, \qquad \mu_{p(y)} = \int_K \nu_{ky}\, dk$$

is a Γ-equivariant map, and by the alignment property $\mu_x = \delta_{\pi(x)}$ for m-a.e. $x \in X$. Since Dirac measures are extremal points of $\mathrm{Prob}(B)$, it follows that $\nu_y = \delta_{\pi(p(y))}$ for m_1-a.e. $y \in X_1$. □

Lemma 3.6 (Finite index tolerance). *Let (X, m, Γ) be a measure class preserving action of a countable group Γ, B a topological space with a continuous Γ-action, and $\pi : X \to B$ a measurable Γ-equivariant map. Let $\Gamma' < \Gamma$ be a finite index subgroup.*

Then $\pi : (X, m) \to B$ has the alignment property with respect to Γ if and only if it has the alignment property with respect to Γ'.

Proof. Let γ_i $(i = 1, \dots, k)$ be the representatives of Γ' cosets. Suppose $x \in X \mapsto \nu_x \in \mathrm{Prob}(B)$ is a Γ-equivariant measurable map. Then $x \mapsto \mu_x = k^{-1} \sum_1^k \nu_{\gamma_i x}$ is a Γ'-equivariant map $X \to \mathrm{Prob}(B)$, and therefore it is $\mu_x = \delta_{\pi(x)}$. The fact that $\delta_{\pi(x)}$ are extremal points of $\mathrm{Prob}(B)$ implies that $\nu_x = \delta_{\pi(x)}$ for m-a.e. $x \in X$. $\square$

Lemma 3.7 (Products)**.** *A direct product of alignment systems is an alignment system.*

Proof. For $i = 1, 2$ let $\pi_i : (X_i, m_i) \to B_i$ be alignment systems with respect to actions of some groups Γ_i. Define

$$(X, m) = (X_1 \times X_2, m_1 \otimes m_2), \qquad B = B_1 \times B_2, \qquad \Gamma = \Gamma_1 \times \Gamma_2.$$

We demonstrate that $\pi(x, y) = (\pi_1(x), \pi_2(y))$ has the alignment property.

Suppose that $(x, y) \in X \mapsto \mu_{x,y} \in \mathrm{Prob}(B_1 \times B_2)$ is a measurable Γ-equivariant map. Choose probability measures m_i' in the measure classes of m_i. Define measurable maps $\nu^{(i)} : (X_i, m_i) \to \mathrm{Prob}(B_i)$ by

$$\nu_x^{(1)}(E) = \int \mu_{x,y}(E \times B_2)\, dm_2'(y), \qquad \nu_y^{(2)}(F) = \int \mu_{x,y}(B_1 \times F)\, dm_1'(x).$$

Then $\nu^{(i)} : (X_i, m_i) \to \mathrm{Prob}(B_i)$ are measurable and Γ_i-equivariant. Thus $\nu_x^{(i)} = \delta_{\pi_i(x)}$ and since these are extremal points, we conclude that a.e.

$$\mu_{x,y}(E \times B_2) = \delta_{\pi_1(x)}(E), \qquad \mu_{x,y}(B_1 \times F) = \delta_{\pi_2(y)}(F).$$

This readily gives $\mu_{x,y} = \delta_{\pi_1(x)} \otimes \delta_{\pi_2(y)}$. $\square$

We shall be interested in examples where $\pi : X \to B$ is a *principal bundle*, in the sense that $\pi : X \to B$ is a surjective continuous map between topological spaces, and L is a locally compact group acting continuously and freely on X so that the L-orbits are precisely the fibers of $\pi : X \to B$. An automorphism of a principal L-bundle is a homeomorphism of X which commutes with the L-action and therefore descends to a homeomorphism of $B = X/L$.

Definition 3.8. If $\pi : X \to B$ is a principal L-bundle, Γ a group of bundle automorphisms, m a measure on X so that both Γ and L act on (X, m) by measure class preserving transformations, and $\pi : (X, m) \to B$ has the alignment property with respect to the Γ-action, we shall say that $(\pi : (X, m) \to B; \Gamma)$ is a *principal L-bundle with alignment.*

3.b. **Rigidity properties of principal bundles with alignment property.** We start by showing how to compute the measurable centralizer, quotients and self-joinings of any II_∞ system (X, m, Γ) admitting a structure of a principal bundle with an alignment property relative to its base.

Theorem 3.9 (Centralizers, self-joinings, quotients)**.** *Let (X, m, Γ) be an ergodic infinite measure preserving system and $\pi : X \to B$ be a principal L-bundle with alignment. Then the following hold.*

(1) *Let $T : X \to X$ be some Borel map, satisfying m-a.e. on X*

$$T(\gamma x) = \gamma\, T(x) \qquad (\gamma \in \Gamma).$$

Then there exists a unique $\lambda_0 \in L$ *so that* $T(x) = \lambda_0 x$ *for* m*-a.e.* $x \in X$. *In particular, the measurable centralizer of* (X, m, Γ) *is* L.

(2) *Any ergodic self-joining of the* Γ*-action on* (X, m) *is given by the measure* $c \cdot m_{\lambda_0}$ *on* $X \times X$ *for some* $0 < c < \infty$ *and* $\lambda_0 \in L$, *where*

$$m_{\lambda_0} = \int_X \delta_x \otimes \delta_{\lambda_0 x} \, dm(x)$$

(3) *The only measurably proper* Γ*-equivariant quotients of* (X, m) *are of the form* $(X, m)/K$ *where* $K < L$ *is a compact subgroup.*

Note that T in (1) is not assumed a prioi to preserve the measure class of m.

Corollary 3.10. *If an ergodic infinite measure preserving system* (X, m, Γ) *can be viewed as a principal* L*-bundle with alignment, then the base action* (B, ν, Γ), *the quotient map* $\pi : X \to B$, *and the structure group* L *are uniquely determined by* (X, m, Γ).

Proof. Theorem 3.9 (1) allows us to identify L acting on (X, m) as the measurable centralizer of the Γ-action and $(B, [\nu])$ as the space of the ergodic components $(X, m)/\!\!/L$ of the centralizer of Γ. □

Proof of Theorem 3.9. (1) The map $X \xrightarrow{T} X \xrightarrow{\pi} B$ is Borel and Γ-equivariant. By the alignment property we have $\pi(T(x)) = \pi(x)$ for m-a.e. $x \in X$. This allows us to define a measurable function $\lambda : X \to L$ by $T(x) = \lambda_x x$. For m-a.e. $x \in X$ and all $\gamma \in \Gamma$,

$$\gamma \lambda_{\gamma x} x = \lambda_{\gamma x} \gamma x = T(\gamma x) = \gamma T(x) = \gamma \lambda_x x.$$

In view of the freeness of the L-action, we get m-a.e. $\lambda_{\gamma x} = \lambda_x$. Ergodicity of the Γ-action on (X, m) implies that λ_x is m-a.e. a constant $\lambda_0 \in L$ and $T(x) = \lambda_0 x$ as claimed.

(2) Given an ergodic self-joining, $(X \times X, \bar{m})$, disintegrate the measure $\bar{m}$ with respect to its projections as

$$\bar{m} = c_1 \cdot \int_X \delta_x \otimes \mu_x \, dm(x) = c_2 \cdot \int_X \nu_x \otimes \delta_x \, dm(x)$$

with $0 < c_1, c_2 < \infty$ and $\{\mu_x\}$ and $\{\nu_x\}$ being measurable families of probability measures on X, indexed by $x \in X$. These measures satisfy m-a.e.

$$\mu_{\gamma x} = \gamma_* \mu_x, \qquad \nu_{\gamma x} = \gamma_* \nu_x \qquad (\gamma \in \Gamma).$$

By Lemma 2.3 we can define a measurable Γ-equivariant map

$$X \xrightarrow{\mu_\cdot} \mathrm{Prob}(X) \xrightarrow{\pi_*} \mathrm{Prob}(B), \qquad \text{by} \qquad x \mapsto \nu_x \mapsto \pi_* \nu_x.$$

By the alignment property for m-a.e. x the measure μ_x is supported on $\pi^{-1}(\pi(x)) = Lx$. Thus $\lambda x \in Lx \mapsto \lambda \in L$ maps μ_x to a probability measure σ_x on L. Note that $\{\sigma_x\}$, $x \in X$, is a measurable family of probability measures on L, which satisfies for m-a.e. $x \in X$, every $\gamma \in \Gamma$, and every Borel set E on L:

$$\begin{aligned} \sigma_{\gamma x}(E) &= \mu_{\gamma x}(\{\lambda \gamma x \,:\, \lambda \in E\}) = \gamma_* \mu_x(\{\gamma \lambda x \,:\, \lambda \in E\}) \\ &= \mu_x(\{\lambda x \,:\, \lambda \in E\}) = \sigma_x(E). \end{aligned}$$

Ergodicity of the Γ-action on (X, m) implies that σ_x is a.e. equal to a fixed probability measure σ on L.

The fact that a.e. $\pi_*\mu_x = \delta_{\pi(x)}$ means that the measure $\bar{m}$ is supported on the set

$$\mathcal{F} = \{(x,y) \in X \times Y \,:\, \pi(x) = \pi(y)\}\,. \tag{3.i}$$

Given a Borel set $E \subseteq L$, define $F_E = \{(x, \lambda x) \,:\, x \in X,\ \lambda \in E\} \subseteq \mathcal{F}$ and observe that

- F_E is invariant under the diagonal Γ-action,
- $\bar{m}(F_E) = 0$ if and only if $\sigma(E) = 0$,
- $\mathcal{F} \setminus F_E = F_{L\setminus E}$.

If $\bar{m}$ is ergodic with respect to the diagonal Γ-action, for each measurable $E \subset L$ either $\bar{m}(F_E) = \sigma(E) = 0$ or $\bar{m}(F_{L\setminus E}) = \sigma(L \setminus E) = 0$. This implies that σ is a Dirac measure δ_{λ_0} at some $\lambda_0 \in L$, and consequently $\bar{m}$ is supported on $\{(x, \lambda_0 x) \,:\, x \in X\}$ as in the statement of the theorem.

(3) Let $p : (X,m) \to (Y,n)$ be a Γ-equivariant measurably proper quotient. Then m can be disintegrated with respect to the quotient map

$$m = \int_Y \mu_y \, dn(y)$$

where $\mu_y \in \mathrm{Prob}(X)$ and $\mu_y(p^{-1}(\{y\})) = 1$ for n-a.e. $y \in Y$. Consider the *independent joining relative* to $p : X \to Y$, given by

$$\bar{m} = \int_Y \mu_y \otimes \mu_y \, dn(y). \tag{3.ii}$$

The disintegration of $\bar{m}$ into Γ-ergodic components consists of ergodic self-joinings of (X, m, Γ):

$$\bar{m} = \int_L m_\lambda \, d\sigma(\lambda), \qquad \text{where} \qquad m_\lambda = \int_X \delta_x \otimes \delta_{\lambda x} \, dm(x)$$

and σ is a probability measure on L. In particular $\bar{m}$ is supported on the set

$$\mathcal{F} = \{(x,x') \,:\, \pi(x) = \pi(x')\} = \{(x, \lambda x) \,:\, x \in X,\ \lambda \in L\} \subset X \times X.$$

A comparison with (3.ii) yields that for n-a.e. $y \in Y$, the measure μ_y is supported on a single L-orbit and moreover for μ_y-a.e. x

$$\int_L f(\lambda x) \, d\sigma(\lambda) = \int_X f(x') \, d\mu_y(x') \qquad (f \in C_c(X)).$$

Since the roles of x and x' are symmetric, σ is a symmetric measure, i.e., $d\sigma(\lambda) = d\sigma(\lambda^{-1})$. Moreover, for any $f \in C_c(X)$ and μ_y-a.e. x we have

$$\begin{aligned} \int_L \int_L f(\lambda_2 \lambda_1 x) \, d\sigma(\lambda_1) \, d\sigma(\lambda_2) &= \int_L \left(\int_X f(\lambda_2 x_1) \, d\mu_y(x_1) \right) d\sigma(\lambda_2) \\ &= \int_X \left(\int_L f(\lambda_2 x_1) \, d\sigma(\lambda_2) \right) d\mu_y(x_1) = \int_X f \, d\mu_y = \int_L f(\lambda x) \, d\sigma(\lambda) \end{aligned}$$

which implies that $\sigma * \sigma = \sigma$.

Lemma 3.11. *A symmetric probability measure σ on a locally compact group L satisfies $\sigma * \sigma = \sigma$ if and only if it is the Haar measure $\sigma = m_K$ on a compact subgroup $K < L$.*

Proof. The "if" part is evident. For the "only if" part assume σ is symmetric and $\sigma * \sigma = \sigma$ and let $K = \mathrm{supp}(\sigma)$. Then K is a closed subset of L satisfying $K^{-1} = K$ and $K \cdot K \subseteq K$, i.e. K is a closed subgroup of L. To see the latter, given $k_1, k_2 \in \mathrm{supp}(\sigma)$ and a neighborhood U of $k_1 \cdot k_2$, choose open neighborhoods V_i of k_i so that $V_1 \cdot V_2 \subset U$, and note that

$$\sigma(U) = \sigma * \sigma(U) \geq \sigma(V_1) \cdot \sigma(V_2) > 0.$$

As U was arbitrary, it follows that $k_1 \cdot k_2 \in K$.

Now let P_σ be the Markov operator

$$(P_\sigma f)(k) = \int_K f(kk')\, d\sigma(k')$$

which is defined on $C_0(K, \mathbf{R})$ and takes values in $C_0(K, \mathbf{R})$. It is a projection because $P_\sigma^2 = P_{\sigma*\sigma} = P_\sigma$. If $g \in C_0(K, \mathbf{R})$ is a P_σ-invariant function, then the closed set $A_g = \{k \in K : g(k) = \max g\}$ satisfies $A_g k' = A_g$ for σ-a.e. $k' \in K$, and $K = \mathrm{supp}(\sigma)$ yields $A_g = K$ and so $g = \mathit{const}$. This implies that K is compact for $C_0(K, \mathbf{R})$ contains a non-trivial constant function. Hence, for $f \in C(K)$ and $k_0 \in K$ we have

$$\int_K f(k)\, d\sigma(k) = (P_\sigma f)(e) = (P_\sigma f)(k_0) = \int_K f(k_0 k)\, d\sigma(k)$$

which means that the probability measure σ is left invariant on the compact group K. □

Remark 3.12. The assumption that σ is symmetric in Lemma 3.11 is redundant: from $\sigma = \sigma * \sigma$ we deduced that $K = \mathrm{supp}(\sigma)$ is a closed subsemigroup of L; but compact subsemigroups in topological groups are known to form subgroups.

Returning to the description of the quotient (Y, n) of (X, m), observe that for n-a.e. $y \in Y$ the measure ν_y is supported and equidistributed on a single K-orbit in X. Hence (Y, n) can be identified with $(X/K, m/K)$. The Γ-action on X descends to an action on X/K because Γ and $K < L$ commute. This completes the proof of Theorem 3.9. □

Next we consider isomorphisms and, more generally, joinings of two ergodic infinite measure actions (X_i, m_i, Γ) $(i = 1, 2)$ of the same group Γ. We assume that the systems are principal L_i-bundles with alignment. The isomorphism case follows from Corollary 3.10: any (τ-twisted) isomorphism $T : (X_1, m_1) \to (X_2, m_2)$ identifies the structure groups $L_1 \cong L_2$, provides a measurable (τ-twisted) Γ-equivariant isomorphism $\phi : (B_1, [\nu_1]) \to (B_2, [\nu_2])$. Similar phenomena hold for general joinings:

Theorem 3.13 (Joinings and boundary maps). *Let (X_i, m_i, Γ), $i = 1, 2$, be two ergodic infinite measure preserving actions of some countable group Γ, which are principal L_i-bundles with alignment $\pi_i : X_i \to B_i$. Suppose the systems (X_i, m_i, Γ) admit a (τ-twisted) ergodic joining $\bar{m}$. Then the following hold.*

(1) *There exists a unique (τ-twisted) Γ-equivariant measure class preserving isomorphism $\phi : (B_1, [\nu_1]) \xrightarrow{\cong} (B_2, [\nu_2])$, where $[\nu_i] = (\pi_i)_*[m_i]$; the joining $\bar{m}$ is supported on*

$$\mathcal{F}_\phi = \{(x_1, x_2) \in X_1 \times X_2 : \phi(\pi_1(x_1)) = \pi_2(x_2)\}.$$

(2) *Structure groups L_i contain compact subgroups K_i, so that $\bar{m}$ is $K_1 \times K_2$-invariant; $\bar{m}$ projects to the joining $\bar{m}'$ of the quotient systems $(X_i', m_i') = (X_i, m_i)/K_i$ with $\bar{m}'$ being the graph of a (τ-twisted) isomorphism*

$$T' : (X_1', m_1', \Gamma) \xrightarrow{\cong} (X_2', m_2', \Gamma).$$

Furthermore, if K_i are normal in L_i, then we have the following.

(3) *The alignment systems $\pi_i' : X_i' \longrightarrow B_i$ are principal $\Lambda_i = L_i/K_i$-bundles and*

$$T'(\lambda_1 x) = \rho(\lambda_1) T'(x)$$

for some continuous group isomorphism $\rho : \Lambda_1 \xrightarrow{\cong} \Lambda_2$.

The following commutative diagram schematically summarizes these statements:

$$\begin{array}{ccc}
(X_1, m_1) & \xrightarrow{\bar{m}} & (X_2, m_2) \\
\Big\downarrow {\scriptstyle /K_1} & & \Big\downarrow {\scriptstyle /K_2} \\
(X_1', m_1') & \xrightarrow{T'} & (X_2', m_2') \\
{\scriptstyle \pi_1'} \Big\downarrow {\scriptstyle \circlearrowleft \Lambda_1} & & {\scriptstyle \cong \Lambda_2 \circlearrowright} \Big\downarrow {\scriptstyle \pi_2'} \\
(B_1, [\nu_1]) & \xrightarrow{\phi} & (B_2, [\nu_2])
\end{array}$$

Proof of Theorem 3.13. Hereafter, we shall use the standard decompositions of $\bar{m}$:

$$\bar{m} = c_1 \cdot \int_{X_1} \delta_x \otimes \mu_x \, dm_1(x) = c_2 \cdot \int_{X_2} \nu_y \, dm_2(y)$$

where $X_1 \to \mathrm{Prob}(X_2)$, $x \mapsto \mu_x$, and $X_2 \to \mathrm{Prob}(X_1)$, $y \mapsto \nu_y$, are, respectively, τ- and τ^{-1}-twisted Γ-equivariant measurable maps.

Claim 3.14. There exists a unique measurable τ-twisted Γ-equivariant map

$$p : X_1 \longrightarrow B_2.$$

We start with the existence claim. Consider the amalgamation $\tilde{m} = \check{\bar{m}} \times_{X_1} \bar{m}$, which is an untwisted (not necessarily ergodic) self-joining of (X_2, m_2, Γ) given by

$$\tilde{m} = \int_{X_1} \mu_x \otimes \mu_x \, dm_1(x). \tag{3.iii}$$

Using Theorem 3.9 (2) and the ergodic decomposition of $\tilde{m}$, we obtain

$$\tilde{m} = const \cdot \int_{L_2} \delta_y \otimes \delta_{\lambda y} \, d\sigma(\lambda)$$

for some probability measure σ on L_2. In particular, $\tilde{m}$ is supported on

$$\mathcal{F}_2 = \{(y, y') \in X_2 \times X_2 \mid \pi_2(y) = \pi_2(y')\}.$$

It follows from the construction (3.iii) that for m_1-a.e. $x \in X_1$ for $\mu_x \times \mu_x$-a.e. (y, y') one has $\pi_2(y) = \pi_2(y')$. Thus $p : X_1 \to B_2$ can be defined by

$$p(x) = \mathrm{supp}((\pi_2)_* \mu_x) \in B_2.$$

For the uniqueness, observe that any τ-twisted Γ-equivariant map $P : X_1 \to \mathrm{Prob}(B_2)$ takes values in Dirac measures. Indeed, the map

$$X_2 \xrightarrow{\nu_*} \mathrm{Prob}(X_1) \xrightarrow{P_*} \mathrm{Prob}(B_2), \qquad y \mapsto P_* \nu_y$$

(using Lemma 2.3) is an untwisted Γ-equivariant measurable map. By the alignment property, $P_*\nu_y = \delta_{\pi_2(y)}$. In particular, $P(x)$ is a Dirac measure on B_2. This observation, in particular, gives the uniqueness part of the claim: if $p, q : X_1 \to B_2$ are τ-twisted Γ-equivariant maps, then

$$P(x) = \frac{1}{2}(\delta_{p(x)} + \delta_{q(x)})$$

should take values in Dirac measures, which is possible only if $p(x) = q(x)$ a.e. Claim 3.14 is proved.

(1) We shall use both the fact that $\pi_1 : X_1 \to B_1$ is a principle L_1-bundle and the uniqueness of the τ-twisted Γ-equivariant map $p : X_1 \to B_2$ to construct $\phi : B_1 \to B_2$. Indeed, for any $\lambda \in L_1$ the map $p_\lambda : X_1 \to B_2$ defined by

$$p_\lambda(x) = p(\lambda x)$$

is a τ-twisted Γ-equivariant measurable map. Thus by Claim 3.14, for every $\lambda \in L_1$ for m_1-a.e. $x \in X_1$,

$$p(\lambda x) = p_\lambda(x) = p(x).$$

Using Fubini, the map $p : X_1 \to B_2$ descends to a τ-twisted Γ-equivariant measurable map $\phi : B_1 = X_1 /\!\!/ L_1 \to B_2$.

It follows that for a.e. $x, x' \in X_1$ with $\pi_1(x) = \pi_1(x') = b_1$ the measures μ_x, $\mu_{x'}$ are supported on the π_2-fiber of $\phi(b_1) \in B_2$. In other words, the original joining $\bar{m}$ is supported on

$$\mathcal{F}_\phi = \{(x, y) \in X_1 \times X_2 \mid \phi(\pi_1(x)) = \pi_2(y)\}.$$

Since $\bar{m}$ projects to $const_i \cdot m_i$ in the factors and since $[\nu_i] = (\pi_i)_*[m_i]$, it follows that $\phi_*[\nu_1] = [\nu_2]$. A symmetric argument provides a τ^{-1}-twisted Γ-equivariant map $\psi : B_2 \to B_1$ with $\psi_*[\nu_2] = [\nu_1]$. Lemma 3.2 shows $\phi = \psi^{-1}$. This completes the proof of part (1).

(2) For m_1-a.e. $x \in X_1$ and μ_x-a.e. $y \in X_2$ let $\eta^{(2)}_{(x,y)} \in \mathrm{Prob}(L_2)$ be the probability measure defined by

$$\eta^{(2)}_{(x,y)}(E) = \mu_x\{\lambda_2 y \mid \lambda_2 \in E \subset L_2\}, \qquad E \in \mathcal{B}(L_2).$$

Similarly, there is a measurable map $(x, y) \mapsto \eta^{(1)}_{(x,y)} \in \mathrm{Prob}(L_1)$, given by

$$\eta^{(1)}_{(x,y)}(E) = \nu_y\{\lambda_1 x \mid \lambda_1 \in E \subset L_1\}, \qquad E \in \mathcal{B}(L_1).$$

The Γ-action $\gamma : (x, y) \mapsto (\gamma x, \gamma^\tau y)$ commutes with the $L_1 \times L_2$-action. Hence, for $i = 1, 2$, we have $\eta^{(i)}_{\gamma(x,y)} = \eta^{(i)}_{(x,y)}$ for every $\gamma \in \Gamma$ and $\bar{m}$-a.e. (x, y). Hence, $\eta^{(i)}_{(x,y)}$ are $\bar{m}$-a.e. equal to fixed probability measures $\eta^{(i)}_{(x,y)} = \eta^{(i)}$ on L_i $(i = 1, 2)$.

For both $i = 1, 2$, the probability measures $\eta^{(i)}$ on L_i are symmetric and satisfy $\eta^{(i)} = \eta^{(i)} * \eta^{(i)}$. This follows from the fact that for m_1-a.e. $x \in X_1$, choosing y and y' in X_2 independently according to μ_x, we will have $y' = \lambda y$ where the distribution of $\lambda \in L_2$ is $\eta^{(2)}$, and similarly for $\eta^{(1)}$ (this is analogous to the argument in the proof of part (3) of Theorem 3.9). Thus, Lemma 3.11 yields that $\eta^{(i)}$ is the normalized Haar measure on a *compact subgroup* $K_i < L_i$.

Next consider the natural Γ-equivariant quotients

$$p_i : (X_i, m_i) \to (X_i', m_i') = (X_i, m_i)/K_i \qquad (i = 1, 2)$$

as in Lemma 2.2. Consider $\mathcal{F}_\phi$ with the joining measure $\bar{m}$ as an ergodic infinite measure preserving action of Γ. The measure $\bar{m}$ is invariant under the action of the compact group $K = K_1 \times K_2$ which commutes with the Γ-action. The quotient system $(\mathcal{F}, \bar{m})/K$ is a subset of $X_1' \times X_2'$ with the measure $\bar{m}' = \bar{m}/K$ having one-to-one projections on X_i'. Therefore, it is supported on a graph of a measurable map $T' : X_1' \to X_2'$, and since $\bar{m}'$ is invariant under the τ-twisted diagonal Γ-action, T' is a τ-twisted isomorphism.

(3) We assume that K_i **are normal in** L_i for both $i = 1, 2$.

Let $\Lambda_i = L_i/K_i$ and observe that $\pi_i : X_i \to B_i$ is a principal Λ_i-bundle which still has the alignemnt property (Lemma 3.4).

Claim 3.15. The groups $\Lambda_i = L_i/K_i$ are continuously isomorphic.

The graph of T' is supported on

$$\mathcal{F}_\phi' = \{(x_1', x_2') \in X_1' \times X_2' \,:\, \phi(\pi_1'(x_1')) = \pi_2'(x_2')\} = \mathcal{F}_\phi/(K_1 \times K_2).$$

This allows us to define a Borel map $\rho : \Lambda_1 \times X_1' \to \Lambda_2$ by

$$T'(\lambda x) = \rho(\lambda, x) T'(x) \qquad (\lambda \in \Lambda_1,\ x \in X_1'). \tag{3.iv}$$

It is an (a.e.) cocycle, i.e., $\rho(\lambda'\lambda, x) = \rho(\lambda', \lambda x)\rho(\lambda, x)$ for m_1'-a.e. $x \in X_1'$ and a.e. $\lambda \in \Lambda_1$. Another a.e. identity is

$$\rho(\lambda, \gamma x) = \rho(\lambda, x)$$

for $\gamma \in \Gamma$ whose action commutes with both Λ_1 on X_1' and Λ_2 on X_2'. Ergodicity implies that for a.e. $\lambda \in \Lambda_1$ the value $\rho(\lambda, x)$ is a.e. constant $\rho(\lambda)$. The a.e. cocycle property of $\rho(\lambda, x)$ means that $\rho : \Lambda_1 \to \Lambda_2$ is a measurable a.e. homomorphism. It is well known that a.e. homomorphisms between locally compact groups coincide a.e. with a continuous homomorphism, which we continue to denote by ρ. Thus, (3.iv) translates into (3) of Theorem 3.13, and the fact that $T_*'m_1 = m_2$ implies that $\rho : \Lambda_1 \to \Lambda_2$ is one-to-one onto.

This completes the proof of Theorem 3.13. □

4. Main examples of the alignment property

Let us focus on two examples of principal bundles, which will prove to have the alignment property under some mild assumptions.

Example 4.1 (Homogeneous spaces)**.** Let G be a locally compact group and $H \triangleleft Q$ closed subgroups. Set

$$X = G/H, \quad B = G/Q, \quad \pi : X \to B, \quad \pi(gH) = gQ.$$

Observe that Q acts on G/H from the right by $q : gH \mapsto gq^{-1}H$. This action is transitive on the π-fibers with H being the stabilizer of every point gH. Thus $L = Q/H$ acts freely on X producing the π-fibers as its orbits. Thus $\pi : G/H \to G/Q$ is a principal L-bundle. In this setup the group G and its subgroups act by bundle automorphisms.

Example 4.2 (Space of horospheres)**.** Let N be a complete simply connected Riemannian manifold of pinched negative curvature, and let ∂N denote the boundary of N. For $p, q \in N$ and $\xi \in \partial N$ the Busemann function is defined as

$$\beta_\xi(p, q) = \lim_{z \to \xi} \left[d(p, z) - d(q, z) \right]. \tag{4.i}$$

The *horospheres* in N are the level sets of the Busemann function:

$$\mathrm{hor}_\xi(t) = \{p \in N \,:\, \beta_\xi(p, o) = t\}$$

where $o \in N$ is some reference point. We denote by

$$\mathrm{Hor}(N) = \{\mathrm{hor}_\xi(t) \,:\, \xi \in \partial N,\ t \in \mathbf{R}\}$$

the space of horospheres.

$\mathrm{Hor}(N)$ fibers over ∂N via $\mathrm{hor}_\xi(t) \mapsto \xi$. This is a principal $\mathbf{R}$-bundle over ∂N, where $\mathbf{R}$ acts by $s : \mathrm{hor}_\xi(t) \mapsto \mathrm{hor}_\xi(t+s)$. Since $\beta_\xi(p, q) + \beta_\xi(q, 0) = \beta_\xi(p, o)$, different choices of $o \in N$ change only the trivialization of this bundle.

The group of isometries of N acts also on the boundary $B = \partial N$ and on the space $\mathrm{Hor}(N)$ of horospheres because $\beta_{\gamma\xi}(\gamma p, \gamma o) = \beta_\xi(p, o)$ for $\gamma \in \mathrm{Isom}(N)$. In the above parametrization this action takes the form

$$\gamma : \mathrm{hor}_\xi(t) \mapsto \mathrm{hor}_{\gamma\xi}(t + c(\gamma, \xi)), \qquad \text{where} \qquad c(\gamma, \xi) = \beta_\xi(\gamma o, o).$$

Note that $c : \Gamma \times \partial N \to \mathbf{R}$ is an additive cocycle, that is,

$$(\gamma'\gamma, \xi) = c(\gamma', \gamma \cdot \xi) + c(\gamma, \xi).$$

Theorem 4.3 (Space of horospheres)**.** *Let N be a complete simply connected Riemannian manifold of pinched negative curvature, $X = \mathrm{Hor}(N)$ be the space of horoshperes, $B = \partial N$ be the boundary, $\pi : X \to B$ be the projection $\mathrm{hor}_\xi(-) \mapsto \xi$. Let $\Gamma < \mathrm{Isom}(N)$ be a discrete group and m be some Borel regular Γ-invariant measure with full support on X.*

Then $\pi : (X, m) \to B$ has the alignment property with respect to Γ if and only if the Γ-action on (X, m) is conservative.

Proof. Lemma 3.3 provides the "only if" direction. The content of this theorem is the "if" direction. We assume that the Γ-action on (X, m) is conservative and $x \in X \mapsto \mu_x \in \mathrm{Prob}(B)$ is a measurable Γ-equivariant map, which should be proven to coincide m-a.e. with $\delta_{\pi(x)}$. Assuming the contrary, the set

$$A = \{x \in X \,:\, \mu_x(\{\pi(x)\}) < 1\} \qquad \text{has} \qquad m(A) > 0.$$

Note that A is Γ-invariant and we may assume that $\mu_{\gamma x} = \gamma_*\mu_x$ for *all* $x \in A$ and all $\gamma \in \Gamma$. For $x \in A$ denote by ν_x the normalized restriction of μ_x to $B \setminus \{\pi(x)\}$. Then also $\{\nu_x\}$, $x \in A$, is Γ-equivariant: $\nu_{\gamma x} = \gamma_*\nu_x$.

Fix some (say piecewise linear) continuous function $\phi : [0, \infty) \to [0, 1]$ with $\phi|_{[0,1]} \equiv 0$ and $\phi|_{[2,\infty)} \equiv 1$, and choose some metric ρ on $B = \partial N$, e.g., the visual metric from the base $o \in N$. Let $f_{x,r}(\xi) = \phi(r \cdot \rho(\pi(x), \xi))$ and let $U_{\xi,r} = \{\eta \in B \,:\, \rho(\xi, \eta) < r\}$ be small neighborhoods of ξ. Then $f_{x,r} \in C(B)$ and

$$f_{x,r}|_{U_{\xi,r}} \equiv 0, \qquad f_{x,r}|_{B \setminus U_{\xi,2r}} \equiv 1.$$

Consider the set $A_r = \{x \in A \,:\, \nu_x(f_{x,r}) > 1/2\}$. Then $m(A_\epsilon) > 0$ for some sufficiently small $\epsilon > 0$. By Luzin's theorem there exists a (compact) subset $C \subseteq A_\epsilon$ with $m(C) > 0$ so that the map $x \in C \mapsto \nu_x \in \mathrm{Prob}(B)$ is *continuous* on C. Since Γ is conservative on (X, m) and m is positive on non-empty open sets, for m-a.e. $x \in C$ there exists an infinite sequence of elements $\gamma_n \in \Gamma$ so that

$$\gamma_n x \to x \qquad \text{and} \qquad \gamma_n x \in C \subseteq A_\epsilon,\ n \in \mathbf{N}.$$

Let us fix such an x and the corresponding infinite sequence $\{\gamma_n\}$; denote $f = f_{x,\epsilon} \in C(B)$ and $K = \mathrm{supp}(f) \subset B \setminus U_{x_0,\epsilon}$. We shall show below that for $n \geq n_0$

the functions $f \circ \gamma_n$ and f have disjoint supports. This will lead to a contradiction because then

$$\nu_x(f) + \nu_x(f \circ \gamma_n) \le 1 \qquad (n \ge n_0)$$

while

$$\nu_x(f \circ \gamma_n) = \nu_{\gamma_n.x}(f) \to \nu_x(f) > \frac{1}{2}.$$

In order to show that $K = \mathrm{supp}(f)$ and $\mathrm{supp}(f \circ \gamma_n) = \gamma_n^{-1}K$ are disjoint for large n, we look at the unit tangent bundle SN of N. It is homeomorphic to

$$\{(\eta, \xi, t) \,:\, \eta \ne \xi\} \quad \text{via} \quad v \in SN \mapsto (v^+, v^-, t(v))$$

were $v^+ \ne v^- \in \partial N$ denote the forward and backward end points of the geodesic in N determined by v and $t(v) = \beta_{v^+}(p(v), o) \in \mathbf{R}$, where $p(v) \in N$ is the base point of the unit tangent vector v.

Let $\xi_0 = \pi(x_0)$. The horosphere $x_0 = \mathrm{hor}_{\xi_0}(t_0)$ corresponds in a one-to-one fashion to $\{(\xi_0, \eta, t_0) \,:\, \eta \ne \xi_0\}$—the set of unit vectors based at points of $\mathrm{hor}_{\xi_0}(t_0)$ and pointing towards $\xi_0 \in \partial N$ (this is the usual identification between the horosphere as a subset of N and as the stable leaf in SN). The assumption that $\gamma_n x_0 \to x_0$ in X means that $\gamma\xi_0 \to \xi_0$ and $t_n \to t_0$ in $\mathbf{R}$, where $\gamma_n : \mathrm{hor}_{\xi_0}(t_0) \mapsto \mathrm{hor}_{\gamma_n\xi_0}(t_n)$.

Let $n_0 \in \mathbf{N}$ be such that $\rho(\gamma_n\xi_0, \xi_0) < \epsilon/2$ and $|t_n - t_0| < 1$ for all $n \ge n_0$. The set

$$Q = \left\{ u \in SN \mid \rho(u^+, u^-) \ge \frac{\epsilon}{2},\ t(u) \in [t_0 - 1, t_0 + 1] \right\}$$

is compact in SN. Note that $\mathrm{Isom}(N)$ acts properly on N and on SN. Thus, the infinite sequence $\{\gamma_n\}$ in the discrete subgroup $\Gamma < \mathrm{Isom}(N)$ eventually moves Q away from itself. In other words, there exists n_1 so that

$$\gamma_n Q \cap Q = \emptyset, \qquad \forall n \ge n_1.$$

We claim that $\gamma_n^{-1}K \cap K = \emptyset$ for all $n \ge \max\{n_0, n_1\}$. Indeed suppose that $\eta \in \gamma_n^{-1}K \cap K$ for some $n \ge n_0$, and let $v \in SN$ be (the unique) vector with $(v^+, v^-, t(v)) = (\xi_0, \eta, t_0)$. Then $v \in Q$ and $\gamma_n v$, corresponding to $(\gamma_n\xi_0, \gamma_n\eta, t_n)$, also lies in Q. This is possible only if $n < n_1$. This completes the proof of the theorem. □

For the proof of the alignment property for maps between homogeneous spaces we shall need the following general result.

Theorem 4.4 (Homogeneous spaces). *Let G be a locally compact group, $H < Q < G$ be closed subgroups and $\Gamma < G$ be a discrete subgroup. Suppose that there exists an open cover*

$$G/Q \setminus \{eQ\} = \bigcup V_i$$

and closed subgroups $T_i < H$ so that for each i

1. *V_i is a T_i-invariant set and T_i acts properly on V_i,*
2. *the Γ-action on $(G/T_i, m_{G/T_i})$ is conservative.*

Let $(X, m) = (G/H, m_{G/H})$, $B = G/Q$ and $\pi : X \to B$ be the natural projection. Then $\pi : (X, m) \to B$ has the alignment property with respect to the Γ-action.

Proof. Let $X \to \mathrm{Prob}(G/Q)$, $x \mapsto \mu_x$, be a fixed Borel map satisfying

$$\mu_{\gamma x} = \gamma_* \mu_x$$

for all $\gamma \in \Gamma$ and m-a.e. $x \in X$. We shall prove that $\mu_x = \delta_{\pi(x)}$ for m-a.e. $x \in X$ by reaching a contradiction starting from the assumption that $m(A) > 0$ where

$$A = \{x \in X \,:\, \mu_x(B \setminus \{\pi(x)\}) > 0\}\,.$$

Let $x \mapsto g_x$ be some Borel cross section of the projection $G \to G/H$. As $B\setminus\{\pi(x)\} = g_x(G/Q \setminus \{eQ\}) = g_x\left(\bigcup V_i\right)$, we have

$$A = \left\{x \in X \,:\, \mu_x(\bigcup_i g_x V_i) > 0\right\} \subseteq \bigcup_i \{x \in X \,:\, \mu_x(g_x V_i) > 0\}\,.$$

The set of indices i in the assumption of the theorem can always be taken to be finite or countable because all the homogeneous spaces in question are separable (in the applications below, the set is actually finite). Thus $m(A) > 0$ implies that for some i, the set

$$A_i = \{x \in X \,:\, \mu_x(g_x V_i) > 0\}$$

has positive m-measure. Let $x \in A \mapsto \nu_x \in \mathrm{Prob}(G/Q)$ denote the normalized restrictions of μ_x to $g_x V_i$:

$$\nu_x(E) = \mu_x(E \cap g_x V_i)/\mu_x(g_x V_i) \qquad (E \subset G/Q \text{ measurable}).$$

We still have $\nu_{\gamma x} = \gamma_* \nu_x$ for a.e. $x \in A_i$ and all $\gamma \in \Gamma$. Given a continuous function $f : V_i \to [0,1]$ with compact support, define

$$A_{i,f} = \left\{x \in A_i \,:\, \nu_x(f \circ g_x^{-1}) > \frac{1}{2}\right\}. \tag{4.ii}$$

Let $0 \le f_1 \le f_2 \le \cdots \nearrow 1$ be an increasing sequence of compactly supported continuous functions $f_n : V_i \to [0,1]$ pointwise converging to 1. For each $x \in A_i$ we have $\nu_x(f_n \circ g_x^{-1}) \nearrow 1$; so we can choose some $f = f_{n_0}$ with $m(A_{i,f}) > 0$.

By Luzin's theorem, there exists a (compact) subset $C \subseteq A_{i,f}$ with $m(C) > 0$, so that both $g_x \in G$ and $\nu_x \in \mathrm{Prob}(G/L)$ vary *continuously* on $x \in C$ (we use the weak-* topology on $\mathrm{Prob}(G/L) \subset C_c(G/L)^*$). We can also assume that $\nu_{\gamma x} = \gamma_* \nu_x$ for *all* $x \in C$ and all $\gamma \in \Gamma$.

We shall now use the recurrence of the Γ-action on $(X', m') = (G/T_i, m_{G/T_i})$ to obtain the following

Lemma 4.5. *For m-a.e. $x \in C$ there exist sequences $\gamma_n \to \infty$ in Γ, $u_n, v_n \to e$ in G, and $t_n \to \infty$ in T_i, so that*

$$\gamma_n = g_x v_n t_n u_n^{-1} g_x^{-1}, \qquad \gamma_n x \in C, \qquad \gamma_n x \to x.$$

Proof. Let U be a neigborhood of $e \in G$. Choose a smaller neighborhood W for which $e \in W = W^{-1}$ and $W^2 \subset U$. Since $x \in C \mapsto g_x \in G$ is continuous, the compact set $C \subset X$ can be covered by (finitely many) subsets $C = \bigcup B_j$ of small enough size to ensure that $g_x^{-1} g_y \in W$ whenever x and y lie in the same B_j. Consider the subsets $E_j \subset X'$ defined by

$$E_j = \{g_x w T_i \,:\, x \in B_j,\ w \in W \cap H\}\,.$$

Then $m'(E_j) > 0$ whenever $m(B_j) > 0$. For each such j apply the following argument: Γ is conservative on (X', m'), i.e., m'-a.e. point of E_j is recurrent. This implies that for m-a.e. $x \in B_j$ and m_H-a.e. $w \in W \cap H$, there exists a non-trivial $\gamma \in \Gamma$ such that $\gamma g_x w T_i = g_y w' T_i$ where $y = \gamma x \in B_j$ and $w' \in W \cap H$. Hence for some $t \in T_i$ we have $\gamma g_x w = g_y w' t$ and

$$\gamma = g_x(g_x^{-1}g_y)w'tw^{-1}g_x^{-1} = g_x utv g_x^{-1}$$

where $u = (g_x^{-1}g_y)w' \in W^2 \subset U$ and $v = w^{-1} \in W^{-1} \subset U$.

This shows (by applying these arguments to all B_j of positive measure) that for m-a.e. $x \in C \subset X$ there exists $\gamma \in \Gamma$ with $\gamma x \in C$ and $\gamma = g_x utv g_x^{-1}$ where $u, v \in U$ and $t \in T_i$. By passing to a sequence $\{U_n\}$ of neighborhoods shrinking to identity in G, we obtain the sequence $\gamma_n = g_x v_n t_n u_n^{-1} g_x^{-1}$ with $u_n, v_n \to e$. Here Γ is discrete in G. Thus $\gamma_n \to \infty$ in G which yields $t_n \to \infty$ in T_i. By construction, $\gamma_n x \in E$, and $\gamma_n x \to x$ from the above form of γ_n. □

We return to the proof of the theorem, where a function of compact support $f : V_i \to [0,1]$ was chosen so that the set $A_{i,f}$ as in (4.ii) has $m(A_{i,f}) > 0$. Let $K \subset V_i$ be a compact set containing $\mathrm{supp}(f)$ in its interior, and let U be a symmetric neighborhood U of $e \in G$ small enough to ensure $U\mathrm{supp}(f) \subset K$. For a.e. $x \in C$ let γ_n, u_n, v_n and t_n be as in Lemma 4.5. For n large enough, $u_n, v_n \in U$ and $t_n K \cap K = \emptyset$; hence

$$u_n\mathrm{supp}(f) \ \cap \ t_n v_n \mathrm{supp}(f) = \emptyset$$

and so, pointwise on V_i, $0 \le f + f \circ v_n t_n u_n^{-1} \le 1$, giving

$$\nu_x(f \circ g_x^{-1}) + \nu_x(f \circ v_n t_n u_n^{-1} g_x^{-1}) \le 1.$$

Since $g_x v_n t_n u_n^{-1} g_x^{-1} x = \gamma_n x \to x$ in C, we have

$$\begin{aligned} \nu_x(f \circ v_n t_n u_n^{-1} g_x^{-1}) &= \nu_x(f \circ g_x^{-1}\gamma_n) = \gamma_n \nu_x(f \circ g_x^{-1}) \\ &= \nu_{\gamma_n x}(f \circ g_x^{-1}) \quad \to \quad \nu_x(f \circ g_x^{-1}). \end{aligned}$$

This leads to a contradiction because $2 \cdot \nu_x(f \circ g_x^{-1}) > 1$. □

Let us illustrate this general result in two concrete examples and then in a more general situation.

Corollary 4.6. *Let k be a local field, $G = \mathbf{SL}_2(k)$, and $\Gamma < G$ be a discrete subgroup acting conservatively on k^2 w.r.t. the Haar measure. Then the projection $\pi : k^2 \setminus \{0\} \to kP^1$ has the alignment property with respect to the Γ-action.*

Proof. Denote by H the stabilizer in G of $e_1 = (1,0) \in k^2$ and by Q the stabilizer in G of the projective point $[e_1] = ke_1 \in kP^1$. Then $G/Q \cong kP^1$ and $(G/H, m_{G/H}) \cong (k^2 \setminus \{0\}, \mathrm{Haar})$, and the H-action on G/Q acts properly discontinuously (and transitively) on the complement $V = G/Q \setminus \{eQ\}$ of the fixed point $\{eQ\}$. The assumptions of Theorem 4.4 are satisfied. □

Corollary 4.7. *Let k be a local field, $G = \mathbf{SL}_n(k)$ and $\Gamma < G$ be a lattice. Then the projection $\pi : k^n \setminus \{0\} \to kP^{n-1}$ has the alignment property with respect to the Γ-action and the Haar measure on k^n.*

Proof. Let H be the stabilizer in G of $e_1 = (1, 0, \dots, 0) \in k^n$, and let Q be the stabilizer of the $[e_1] = ke_1 \in kP^{n-1}$. Then $(G/H, m_{G/H}) \cong (k^n \setminus \{0\}, \mathrm{Haar})$ and $G/Q \cong kP^1$. For $i = 2, \dots, n$ let

$$V_i = \left\{[(x_1, \dots, x_n)] \in kP^{n-1} \, : \, x_i \neq 0\right\}, \qquad T_i = \{I + tE_{1,i} \, : \, t \in k\}$$

where I denotes the identity matrix and $E_{j,k}$ the elementary matrix with 1 in the j,k-place and zeros elsewhere. This system satisfies the assumptions of Theorem 4.4. Indeed the only non-elementary condition is conservativity of the Γ-action on G/T_i. By Moore's ergodicity theorem, T_i acts ergodically on G/Γ, which is equivalent to the ergodicity of the Γ-action on G/T_i. □

The above are particular cases of the following more general theorem.

Theorem 4.8. *Let $G = \prod \mathbf{G}_\alpha(k_\alpha)$ be a semi-simple group and $H < G$ be a "super-spherical" subgroup as in Definition 1.7, i.e., H is pinched $\check{H} \lhd H < \hat{H}$ between certain unimodular subgroups $\check{H}, \hat{H} \lhd Q$ associated to a parabolic $Q < G$. Let $\Gamma < G$ be a lattice (not necessarily irreducible). Denote $(X, m) = (G/H, m_{G/H})$, $(\check{X}, \check{m}) = (G/\check{H}, m_{G/\check{H}})$, $(\hat{X}, \hat{m}) = (G/\hat{H}, m_{G/\hat{H}})$ and $B = G/Q$. Then*

1. *the natural projections*
$$\check{\pi} : (\check{X}, \check{m}) \to B, \quad \pi : (X, m) \to B, \quad \hat{\pi} : (\hat{X}, \hat{m}) \to B$$
have the alignment property with respect to the action of Γ,
2. *the systems $\check{\pi} : \check{X} \to B$, $\hat{\pi} : \hat{X} \to B$ are principal bundles with structure groups $\check{L} = Q/\check{H}$ and $\hat{L} = Q/\hat{H}$.*

Proof of Theorem 4.8. The natural projections are nested:
$$\check{X} \to X \to \hat{X} \longrightarrow B, \qquad g\check{H} \mapsto gH \mapsto g\hat{H} \mapsto gQ.$$
So by Lemma 3.4 it suffices to prove the alignment property for the Γ-action on $\check{\pi} : (\check{X}, \check{m}) \to B$.

The group G, as well as $\check{H} \lhd Q$, is a product group:
$$G = \prod_{\alpha \in A} G_\alpha, \qquad \check{H} = \prod_{\alpha \in A} \check{H}_\alpha, \qquad Q = \prod_{\alpha \in A} Q_\alpha,$$
formed by the k_α-points of the corresponding k_α-groups. So $\check{\pi} : \check{X} \to B$ splits as a product of projections
$$\check{\pi}_\alpha : \check{X}_\alpha = G_\alpha/H_\alpha \longrightarrow B_\alpha = G_\alpha/Q_\alpha.$$
If Γ is reducible, then some subgroup of finite index Γ' splits as a product of irreducible lattices $\Gamma_i < \prod_{\alpha \in A_i} G_\alpha$, where $A = \bigcup_i A_i$ is some non-trivial partition. Thus Lemmas 3.6 and 3.7 allow us to transfer the alignment property from $\Gamma^{(i)}$-actions on $\check{X}^{(i)} = \prod_{\alpha \in A_i} \check{X}_\alpha \to B^{(i)} = \prod_{\alpha \in A_i} B_\alpha$ to that of Γ on $\check{X} \to B$.

Therefore, we may assume that Γ itself is irreducible in G. We now point out two important properties of $\check{H}$:

(i) $\check{H}$ is generated by unipotent elements in its factors $\check{H}_\alpha$.
(ii) If $g \in G$ satisfies $g^{-1}\check{H}g < Q$, then necessarily $g \in Q$.

The first property follows from the construction; in fact, $\check{H}_\alpha$ is generated by all the roots of G_α contained in Q_α. This also explains the second statement (see [3, Corollary 4.5]).

Next, let $\{T_i\}$ be a (finite) family of one parameter unipotent subgroups (collected from different factors) in $\check{H}$ generating $\check{H}$. The T_i-action on the projective variety $B = G/Q$ is algebraic. Hence G/Q decomposes as $F_i \sqcup V_i$ where F_i is the set of T_i-fixed points and V_i is the union of free T_i-orbits ($T_i \cong k$ has no proper algebraic subgroups). The intersection $\bigcap F_i$ consists of $\check{H}$-fixed points, i.e., points gQ such that $\check{H}gQ = gQ$. Property (ii) above yields
$$\bigcap F_i = \{eQ\}, \qquad \text{so that} \qquad \bigcup V_i = G/Q \setminus \{eQ\}.$$

By Moore's ergodicity theorem, T_i acts ergodically on G/Γ and so the Γ-action on G/T_i is ergodic and hence conservative with respect to the Haar measure m_{G/T_i}. The T_i-action on V_i is properly discontinuous. It therefore follows from Theorem 4.4 that $\pi : G/\check{H} \to G/Q$ has the alignment property. $\square$

5. Rigidity for spaces of horospheres

In this section we consider the geometric framework of pinched negatively curved manifolds. The analysis of this geometric situation, in particular, implies the rigidity results for rank one symmetric spaces: Theorems A and B (see section 6 below).

Let N be a a complete simply connected Riemannian manifold with pinched negative curvature, let ∂N denote its boundary at infinity (homeomorphic to a sphere $S^{\dim N-1}$), and let $\Gamma < \mathrm{Isom}(N)$ be some non-elementary discrete group of isometries.

Recall some fundamental objects associated with this setup. The critical exponent $\delta = \delta(\Gamma)$ of Γ is

$$\delta(\Gamma) = \lim_{R\to\infty} \frac{1}{R} \log |\{\gamma \in \Gamma \mid d(\gamma p, q) < R\}|$$

(the limit exists and is independent of $p, q \in N$). The Poincaré series of Γ is

$$P_s(p,q) = \sum_{\gamma\in\Gamma} e^{-s\cdot d(\gamma p,q)}.$$

It converges for all $s > \delta(\Gamma)$ and diverges for all $s < \delta(\Gamma)$ regardless of the location of $p, q \in N$. If the series diverges at the critical exponent $s = \delta(\Gamma)$, the group Γ is said to be of *divergent type.*

The Patterson-Sullivan measure(s) is a measurable family $\{\nu_p\}_{p\in N}$ of mutually equivalent finite non-atomic measures supported on the limit set $L(\Gamma) \subseteq \partial N$ of Γ, satisfying

$$\text{(5.i)} \qquad \frac{d\nu_p}{d\nu_q}(\xi) = e^{-\delta\cdot\beta_\xi(p,q)} \qquad \text{and} \qquad \nu_{\gamma p} = \gamma_* \nu_p$$

where $\beta_\xi(p,q)$ is the Busemann cocycle $\beta_\xi(p,q) = \lim_{z\to\xi} [d(p,z) - d(q,z)]$ (the limit exists and is well defined for any $p, q \in N$ and $\xi \in \partial N$). Patterson-Sullivan measures exist, and for Γ of divergent type, the family $\{\nu_p\}_{p\in N}$ is defined by the above properties uniquely, up to a scalar multiple.

The space $\mathrm{Hor}(N)$ of horospheres is a principal $\mathbf{R}$-bundle over ∂N (see Example 4.2). In the parametrization $\mathrm{Hor}(N) \cong \partial N \times \mathbf{R}$ defined by a base point $o \in N$, the Γ-action on $\mathrm{Hor}(N)$ is given by

$$\gamma : (\xi, t) \mapsto (\gamma\xi, t + c(\gamma, \xi)) \qquad \text{where} \qquad c(\gamma, \xi) = \beta_\xi(\gamma o, o).$$

Define an infinite measure m on $\mathrm{Hor}(N)$ by

$$\text{(5.ii)} \qquad dm(\xi, t) = e^{-\delta\cdot t}\, d\nu_o(\xi)\, dt$$

where ν_o is the Patterson-Sullivan measure. Then m is Γ-invariant.

Remark 5.1. Measure-theoretically, the Γ-action on $(\mathrm{Hor}(N), m)$ can be viewed as the standard measure preserving extension of the measure class preserving Γ-action on $(\partial N, \nu_*)$, where ν_* is any representative of the measure class of the Patterson-Sullivan measures.

Theorem 5.2 (Rigidity for actions of spaces of horospheres)**.** *Let N_1 and N_2 be complete simply connected Riemannian manifolds of pinched negative curvature and*

$$\mathrm{Isom}(N_1) > \Gamma_1 \xrightarrow{\cong} \Gamma_2 < \mathrm{Isom}(N_2)$$

be two abstractly isomorphic $\Gamma_1 \xrightarrow{\tau} \Gamma_2$ discrete non-elementary groups of isometries. Fix some base points $o_i \in N_i$. Let ν_i denote the Patterson-Sullivan measures on $L(\Gamma_i) \subseteq \partial N_i$, and let m_i be the corresponding measures on the extensions on $\mathrm{Hor}(N_i)$. Assume that $(\mathrm{Hor}(N_i), m_i, \Gamma_i)$ are ergodic.

Then the following are equivalent:

(1) *The actions $(\mathrm{Hor}(N_i), m_i, \Gamma_i)$ $(i = 1, 2)$ admit an ergodic τ-twisted joining.*
(2) *The actions $(\mathrm{Hor}(N_i), m_i, \Gamma_i)$ $(i = 1, 2)$ admit a measurable τ-twisted isomorphism*

$$T : (\mathrm{Hor}(N_1), m_1) \longrightarrow (\mathrm{Hor}(N_2), m_2).$$

(3) *There exists a τ-twisted measure class preserving isomorphism*

$$\phi : (L(\Gamma_1), \nu_1) \longrightarrow (L(\Gamma_2), \nu_2).$$

Under these (equivalent) conditions, all ergodic joinings are graphs of isomorphisms, the map ϕ is uniquely defined (up to null sets), and every measurable isomorphism $\mathrm{Hor}(N_1) \to \mathrm{Hor}(N_2)$ has the form

$$\mathrm{hor}_\xi(t) \in \mathrm{Hor}(N_1) \quad \mapsto \quad \mathrm{hor}_{\phi(\xi)}(\frac{\delta_2}{\delta_1} \cdot t + s_\xi).$$

Furthermore, if Γ_i are of divergent type, then the above conditions imply that $\phi : L(\Gamma_1) \to L(\Gamma_2)$ is a homeomorphism and the groups Γ_i have proportional length spectra:

$$\delta_2 \cdot \ell_2(\gamma^\tau) = \delta_1 \cdot \ell_1(\gamma) \qquad (\gamma \in \Gamma_1),$$

where $\ell_i(\gamma) = \inf\{d_i(\gamma p, p) \mid p \in N_i\}$ is the translation length of $\gamma \in \mathrm{Isom}(N_i)$.

Remark 5.3. In some cases, the last statement implies that N_1 is equivariantly *isometric* to N_2, after a rescaling by δ_1/δ_2. This is known as the Marked Length Spectrum Rigidity (Conjecture). It has been proved, for example, for surface groups ([14]) and in the case that N_1 is a symmetric space and N_1/Γ_1 is compact ([9]). See also [5] for some related results.

Proof of Theorem 5.2. "(1) ⇒ (2) and (3)". By Theorem 4.3 the $(\mathrm{Hor}(N_i), m_i) \to \partial N_i$ are $\mathbf{R}$-principal with alignment with respect to Γ_i $(i = 1, 2)$. Let $\bar{m}$ be an ergodic τ-twisted joining on $\mathrm{Hor}(N_1) \times \mathrm{Hor}(N_2)$. Applying Theorem 3.13, we get the desired (unique) measure class preserving τ-twisted equivariant map

$$\phi : (\partial N_1, \nu_1) \longrightarrow (\partial N_2, \nu_2).$$

As $(\mathbf{R}, +)$ has no compact subgroups, it follows that $\bar{m}$ is a graph of an isomorphism. As the only automorphisms of $\mathbf{R}$ are $t \mapsto c \cdot t$, we get the general form of such an isomorphism defining s_ξ by $\mathrm{hor}_\xi(0) \mapsto \mathrm{hor}_{\phi(\xi)}(s_\xi)$ as stated.

"(1) ⇒ (2)" and "(3) or (2) ⇒ (1)" being trivial, we are left with "(3) ⇒ (2)". But this follows from Remark 5.3.

We are now left with the proof of the geometric conclusions in the case of divergent groups. The arguments below are probably known to experts, but we could not find a good reference in the existing literature.

Recall some general facts from Patterson-Sullivan theory. Let $\Gamma < \mathrm{Isom}(N)$ be a discrete group of isometries of a connected, simply connected manifold N of pinched negative curvature. Let

$$\partial^2 N = \{(\xi,\eta) \in \partial N \times \partial N \,:\, \xi \neq \eta\}$$

denote the space of pairs of distinct points at infinity of N (this is the space of oriented but unparametrized geodesic lines in N). Another Busemann cocycle (or Gromov product) can be defined for $\xi \neq \eta \in \partial N$ and $p \in N$ by

$$B_p(\xi,\eta) = \lim_{x\to\xi, y\to\eta} \frac{1}{2}\left[d(p,x) + d(p,y) - d(x,y)\right].$$

It can also be written as $B_p(\xi,\eta) = \beta_\xi(p,q) + \beta_\eta(p,q)$, where q is an arbitrary point on the geodesic line $(\xi,\eta) \subset N$. For any fixed $p \in N$, the function $B_p(\xi,\eta)$ is continuous and proper on $\partial^2 N$, i.e., tends to ∞ as ξ and η approach each other.

We have $\beta_{\gamma\xi}(\gamma p, \gamma q) = \beta_\xi(p,q)$ and $B_{\gamma p}(\gamma\xi,\gamma\eta) = B_p(\xi,\eta)$ for any isometry $\gamma \in \mathrm{Isom}(N)$. This implies that

$$B_p(\gamma\xi,\gamma\eta) - B_p(\xi,\eta) = \frac{1}{2}\left[\beta_\xi(\gamma p, p) + \beta_\eta(\gamma p, p)\right]. \tag{5.iii}$$

In view of (5.i) the measure $\overline{\mu}$ on $\partial^2 N \subset \partial N \times \partial N$, defined by

$$d\overline{\mu}(\xi,\eta) = e^{2\delta B_p(\xi,\eta)}\, d\nu_p(\xi)\, d\nu_p(\eta), \tag{5.iv}$$

is Γ-invariant. This definition is independent of $p \in N$. One of the basic facts in Patterson-Sullivan theory states that Γ is of divergent type iff its action on $(\partial^2 N, \overline{\mu})$ is ergodic ([22], [23]).

The function $B_o(\cdot,\cdot)$ can be used to define a *cross-ratio* on ∂N by

$$[\xi_1,\xi_2,\eta_1,\eta_2] = e^{2\delta\cdot[B_o(\xi_1,\eta_1)+B_o(\xi_2,\eta_2)-B_o(\xi_1,\eta_2)-B_o(\xi_2,\eta_1)]} \tag{5.v}$$

where $o \in N$ is some reference point. This cross-ratio is independent of the choice of $o \in N$ and is invariant under $\mathrm{Isom}(N) > \Gamma$ and satisfies the usual identities.

Returning to the given pair $\Gamma_i < \mathrm{Isom}(N_i)$ $(i = 1, 2)$, we have

Claim 5.4. The measurable τ-twisted Γ-equivariant map

$$\phi : (L(\Gamma_1), \nu_1) \to (L(\Gamma_2), \nu_2)$$

is a τ-twisted Γ-equivariant homeomorphism (possibly after an adjustment on null sets). Moreover, for all distinct $\xi_1, \xi_2, \eta_1, \eta_2 \in L(\Gamma_1) \subseteq \partial N_1$:

$$[\phi(\xi_1), \phi(\xi_2), \phi(\eta_1), \phi(\eta_2)]_2 = [\xi_1, \xi_2, \eta_1, \eta_2]_1. \tag{5.vi}$$

This is a consequence of property (i) and the ergodicity of Γ_i on $\overline{\mu}_i$ (the following argument is a version of Sullivan's argument for Kleinian groups in [22] but can also be traced back to Mostow in the context of quasi-Fuchsian groups).

The idea is that $\phi_*\nu_1 \sim \nu_2$ and $\overline{\mu}_i \sim \nu_i \otimes \nu_i$ imply that the pushforward measure $(\phi\times\phi)_*\overline{\mu}_1$ is absolutely continuous with respect to $\overline{\mu}_2$. Since $\overline{\mu}_1$ is Γ_1-invariant while ϕ is equivariant, it follows that $(\phi\times\phi)_*\overline{\mu}_1$ is Γ_2-invariant. Hence, its Radon-Nikodym derivative with respect to $\overline{\mu}_2$ is a.e. a constant. In view of (5.i), (5.iv) this amounts to a $\overline{\mu}_1$-a.e. relation

$$\begin{aligned} 2\delta_2 \cdot B_2(\phi(\xi),\phi(\eta)) &= 2\delta_1 \cdot B_1(\xi,\eta) + f(\xi) + f(\eta) + C, \\ \text{where} \qquad & f(\xi) = 2\delta_1 \cdot \log \frac{d\phi_*\nu_1}{d\nu_2}(\phi(\xi)). \end{aligned}$$

Substituting these into the definition of the cross-ratios, one observes that the f-terms and the constant C cancel out. It follows that (5.vi) holds ν_1-almost everywhere.

For any fixed distinct ξ_2, ξ_3, ξ_4 we have

$$[\xi, \xi_2; \xi_3, \xi_4]_1 \to 0 \qquad \text{iff} \qquad \xi \to \xi_3.$$

This allows us, using Fubini's theorem and the a.e. identity (5.vi), to conclude that ϕ agrees ν_1-a.e. with a *continuous* function ϕ_0 defined on $\operatorname{supp}(\nu_1) = L(\Gamma_1)$. Since all the data are symmetric, it follows that ϕ_0 is a homeomorphism, and the relation (5.vi) extends from a.e. to everywhere on $\operatorname{supp}(\nu_1) = L(\Gamma_1)$ by continuity.

The cross-ratio determines the marked length spectrum. More precisely,

Lemma 5.5. *Let N be a simply connected Riemannian manifold of pinched negative curvature, $\Gamma < \operatorname{Isom}(N)$ be a non-elementary discrete group of isometries, $\delta = \delta(\Gamma)$ is the growth exponent, and $[,;,]$ is the corresponding cross-ratio as in* (5.v). *If $\gamma \in \Gamma$ is a hyperbolic element with attracting, repelling points $\gamma_+, \gamma_- \in \partial N$, then*

$$2\delta \cdot \ell(\gamma) = \log[\gamma_+, \gamma_-; \xi, \gamma\xi]$$

for all $\xi \in \partial N \setminus \{\gamma_-, \gamma_+\}$.

Proof. It is well known that $B_p(\xi, \eta)$ is within a constant (depending only on N) from $\operatorname{dist}(p, (\xi, \eta)) = \inf\{d(p, x) \mid x \in (\xi, \eta)\}$. For a fixed $\xi \neq \gamma_\pm$ and $p \in N$ we can estimate (with an error depending on p and ξ but independent of $n \in \mathbf{N}$)

$$\operatorname{dist}(p, (\gamma_+, \gamma^n\xi)) = \operatorname{dist}(\gamma^{-n}p, (\gamma_+, \xi)) \asymp d(\gamma^{-n}p, p) \asymp n \cdot \ell(\gamma).$$

Hence

$$\frac{1}{n} B_p(\gamma_+, \gamma^n\xi) \longrightarrow \ell(\gamma).$$

At the same time, $\operatorname{dist}(p, (\gamma_-, \gamma^n\xi)) \longrightarrow \operatorname{dist}(p, (\gamma_-, \gamma_+))$, and so

$$\frac{1}{n} B_p(\gamma_-, \gamma^n\xi) \longrightarrow 0.$$

Since γ fixes the points $\gamma_-, \gamma_+ \in \partial N$ and preserves the cross-ratio we have for each n

$$\begin{aligned}
\log[\gamma_+, \gamma_-; \xi, \gamma\xi] &= \frac{1}{n} \sum_{k=0}^{n-1} \log[\gamma_+, \gamma_- \,;\, \gamma^k\xi, \gamma^{k+1}\xi] \\
&= \frac{2\delta}{n} \cdot \sum_{k=0}^{n-1} \left(B_o(\gamma_+, \gamma^k\xi) - B_o(\gamma_+, \gamma^{k+1}\xi) + B_o(\gamma_-, \gamma^{k+1}\xi) - B_o(\gamma_-, \gamma^k\xi) \right) \\
&= \frac{2\delta}{n} \cdot \left(B_o(\gamma_+, \xi) - B_o(\gamma_-, \xi) + B_o(\gamma_+, \gamma^n\xi) - B_o(\gamma_-, \gamma^n\xi) \right) \\
&\longrightarrow \quad \ell(\gamma) \qquad \text{as} \quad n \to \infty.
\end{aligned}$$

□

We return to $\Gamma_i < \operatorname{Isom}_+(N_i)$ $(i = 1, 2)$, related by an abstract isomorphism $\tau : \Gamma_1 \to \Gamma_2$ and an equivariant homeomorphism $\phi : L(\Gamma_1) \to L(\Gamma_2)$. The classification of elements of $\operatorname{Isom}_+(N_i)$ into elliptic, parabolic and hyperbolic can be done in terms of the dynamics on the boundaries, e.g., a hyperbolic isometry g has two fixed points g_-, g_+ and source/sink dynamics. Thus, the topological conjugacy ϕ of the Γ_i-actions on $L(\Gamma_i) \subset \partial N_i$ shows that τ preserves the types of the elements.

If $\gamma \in \Gamma_1$ is hyperbolic, then so is $\gamma^\tau \in \Gamma_2$, and ϕ maps the corresponding repelling contracting points $\gamma_\pm$ of γ to those of $\gamma^\tau \in \Gamma_2$, because

$$\phi(\gamma_\pm) = \phi(\lim_{n\to\pm\infty} \gamma^n \xi) = \lim_{n\to\pm\infty} (\gamma^\tau)^n \phi(\xi) = \gamma^\tau_\pm$$

for any $\xi \in \partial N_1 \setminus \{\gamma_-, \gamma_+\}$. Thus, using the previous lemma we arrive at

$$\begin{aligned} \delta_1 \ell_1(\gamma) &= \log[\gamma_+, \gamma_-; \xi, \gamma\xi]_1 = \log[\phi(\gamma_+), \phi(\gamma_-); \phi(\xi), \phi(\gamma\xi)]_2 \\ &= \log[\gamma^\tau_+, \gamma^\tau_-; \phi(\xi), \gamma^\tau \phi(\xi)]_2 = \delta_2 \ell_2(\gamma^\tau). \end{aligned}$$

Hence $\delta_1 \ell_1(\gamma) = \delta_2 \ell_2(\gamma^\tau)$ for all hyperbolic elements $\gamma \in \Gamma$ and the same formula (in the trivial form of $0 = 0$) applies to parabolic and elliptic $\gamma \in \Gamma_1$. This completes the proof of Theorem 5.2. ☐

6. Proofs of the rigidity results

In this section we put all the ingredients developed in Sections 3–5 together in order to deduce the results stated in the Introduction. We shall need some auxiliary facts, some of which, e.g., Theorem 6.3, may be of independent interest. We start with the following general lemma.

Lemma 6.1. *Let Γ be some discrete group with II_∞-actions on six infinite measure spaces linked into two sequences as follows:*

$$(\check{X}_i, \check{m}_i) \xrightarrow{p_i} (X_i, m_i) \xrightarrow{q_i} (\hat{X}_i, \hat{m}_i) \qquad (i = 1, 2).$$

Suppose $\bar{m}$ is an ergodic (possibly τ-twisted) joining of the Γ-actions on (X_1, m_1) with (X_2, m_2). Then there exist ergodic (τ-twisted) joinings $\overline{\check{m}}$ of $(\check{X}_1, \check{m}_1)$ with $(\check{X}_2, \check{m}_2)$ and $\overline{\hat{m}}$ of $(\hat{X}_1, \hat{m}_1)$ with $(\hat{X}_2, \hat{m}_2)$ so that

$$(\check{X}_1 \times \check{X}_2, \overline{\check{m}}) \overset{p_1\times p_2}{\longrightarrow} (X_1 \times X_2, \overline{m}) \overset{q_1 \times q_2}{\longrightarrow} (\hat{X}_1 \times \hat{X}_2, \overline{\hat{m}}) \tag{6.i}$$

are quotient maps for the II_∞ diagonal (τ-twisted) Γ-actions.

Proof. The measure $\overline{\hat{m}}$ of $\overline{m}$ is defined by $\overline{\hat{m}}(E_1 \times E_2) = \overline{m}(q_1^{-1} E_1 \times q_2^{-1} E_2)$ and it is straightforward to verify that it is a joining of $\hat{m}_1$ with $\hat{m}_2$; its ergodicity follows from the ergodicity of $\overline{m}$.

To construct $\overline{\check{m}}$, first consider the disintegration of $\check{m}_i$ with respect to m_i:

$$\check{m}_i = \int_{X_i} \mu^{(i)}_x \, dm_i(x) \qquad (i = 1, 2).$$

Consider the measure $\overline{m}^*$ on $\check{X}_1 \times \check{X}_2$ defined by

$$\overline{m}^* = \int_{X_1 \times X_2} \mu^{(1)}_x \otimes \mu^{(2)}_y \, dm(x, y).$$

This measure forms a (τ-twisted) joining of the Γ-actions on $(\check{X}_i, \check{m}_i)$ for $i = 1, 2$ and also projects to $\overline{m}$ under $p_1 \times p_2$. Let $\overline{m}^* = \int \overline{\check{m}}_t \, d\eta(t)$ denote the ergodic decomposition of $\overline{m}^*$ into ergodic joinings. Then $\overline{m}$ is an average of ergodic joinings $(p_1 \times p_2)_* \overline{\check{m}}_t$. Since $\overline{m}$ is ergodic, the η-a.e. ergodic joining $\overline{\check{m}}_t$ projects to a multiple of $\overline{m}$ and can serve as $\overline{\check{m}}$ in the lemma. ☐

Proof of Theorem A. Let G be a real, connected, simple, non-compact, center-free, rank one group, and let $G = KP$ and $P = MAN$ be the Iwasawa decompositions. Denote by $\mathbf{H} = G/K$ the associated symmetric space and by $\partial\mathbf{H} = G/P = G/MAN$ its boundary. The unit tangent bundle is $S\mathbf{H} = G/M$, and the space of horospheres $\mathrm{Hor}(\mathbf{H})$ can be identified with G/MN.

Let $H < G$ be a closed, unimodular, proper subgroup containing N, and denote $\hat{H} = MN$, $\check{X} = G/N$, $X = G/H$, $\hat{X} = G/\hat{H} = \mathrm{Hor}(\mathbf{H})$ and let $\check{m}$, m, $\hat{m}$ denote the corresponding Haar measures. We assume that Γ acts ergodically on $(\check{X}, \check{m}) = (G/N, m_{G/N})$ and hence on (X, m). The projection

$$X = G/H \longrightarrow G/P, \qquad gH \mapsto gP$$

has the alignment property by Theorem 4.3 and Lemma 3.5. If H is normal in P, for example if $H = \check{H} = N$ or $H = \hat{H} = MN$, then $(X, m) \to B$ is a principal bundle with an alignment property (Theorem 4.3), and therefore Theorem A is a direct corollary of Theorem 3.9.

In the general case, the argument for algebraicity of quotients is the simplest: any quotient $q : (X, m) \to (Y, n)$ defines a quotient of $(\check{X}, \check{m}) \xrightarrow{p} (X, m) \xrightarrow{q} (Y, n)$. As mentioned, Theorem 3.9 applies to $\check{X} = G/\check{H}$, which gives that (Y, n) can be identified with $(G/H', m_{G/H'})$ where $\check{H} < H'$ with $H'/\check{H}$ compact. Since the factor map $q \circ p : g\check{H} \mapsto gH'$ factors through G/H, it follows that $H < H'$ and $p(gH) = gH'$.

To analyze the centralizers of the Γ-action on (X, m), we first consider general ergodic self-joinings $\overline{m}$ of (X, m). Let $\overline{\check{m}}$ and $\overline{\hat{m}}$ be ergodic self-joinings of $(\check{X}, \check{m})$ and $(\hat{X}, \hat{m})$ as provided by Lemma 6.1. Applying Theorem 3.9 to $(\check{X}, \check{m})$ and $(\hat{X}, \hat{m})$, we deduce that there exist $\lambda \in \check{\Lambda} = N_G(N)/N = P/N = MA$ and $a \in \hat{\Lambda} = N_G(MN) = P/MN = A$ so that

$$\overline{\check{m}} = const \cdot \int_{\check{X}} \delta_x \otimes \delta_{\lambda x} \, d\check{m}(x) \qquad \text{and} \qquad \overline{\hat{m}} = const \cdot \int_{\hat{X}} \delta_x \otimes \delta_{ax} \, d\hat{m}(x).$$

Since $(X \times X, \overline{m})$ is an intermediate quotient as in (6.i), it follows that $a \in A$ is the image of $\lambda \in MA$ under the natural epimorphism $\check{\Lambda} = MA \to \hat{\Lambda} = A$. This completes the description of self-joinings in the theorem.

Finally, let $T : X \to X$ be a measurable centralizer of the Γ-action on (X, m). Applying the above arguments to the corresponding self-joining

$$\overline{m} = \int_X \delta_x \otimes \delta_{T(x)} \, dm(x),$$

we deduce, in particular, that T is "covered" by an algebraic automorphisms $\check{T}$ of $\check{X} = G/N$, i.e., for some $q \in P = MAN$ the map $\check{T} : gN \mapsto gqN$. The fact that the graph of $\check{T}$ covers that of T means that for a.e. $gH \in X$, if $T(gH) = g'H$, then the map $\hat{T}$ takes the preimage

$$\check{T}\left(p^{-1}(gH)\right) = p^{-1}(g'H), \qquad \text{where} \qquad p^{-1}(gH) = \{ghN \in \hat{X} \mid h \in H\}.$$

This implies that $q \in N_G(H)$ and $T(gH) = gqH$ is algebraic as in Example 1.3. Theorem A is proved.

Proof of Theorem B. Now consider a discrete subgroup $\Gamma < G$ which satisfies property (E2) (defined before the statement of Theorem B). Such groups Γ have the full limit set $L(\Gamma) = \partial\mathbf{H}$, the maximal critical exponent $\delta(\Gamma) = \delta(\mathbf{H})$ at which its Poincare series diverges (so they are of divergent type), and the associated Patterson-Sullivan measures are in the Haar class on $\partial\mathbf{H}$. The Γ-invariant measure m on $S\mathbf{H} \cong G/H_1$ as in (5.ii) is a scalar multiple of $m_{G/H'}$. It is well known that $G = \mathrm{Isom}_+(\mathbf{H})$ can be identified with the *conformal group* on the boundary, and the latter can be defined using *the cross-ratio*

$$G = \mathrm{Isom}_+(\mathbf{H}) \cong \{\psi \in \mathrm{Homeo}_+(\partial\mathbf{H}) \mid [,;,] \circ \psi = [,;,]\}\,. \tag{6.ii}$$

Consider the framework of Theorem B, in which two rank one groups G_i $(i = 1, 2)$ as above contain abstractly isomorphic discrete subgroups $\Gamma_i < G_i$, $\tau : \Gamma_1 \xrightarrow{\cong} \Gamma_2$, and the homogeneous spaces $X_i = G_i/H_i$ admit a τ-twisted joining $\overline{m}$ with respect to the Γ_i-actions. Denote by $\mathbf{H}_i$, $\partial\mathbf{H}_i$, $[,,,]_i$, etc., the corresponding symmetric spaces, their boundaries, cross-ratios, etc. Set $\check{H}_i = N_i < H_i < \hat{H}_i = M_iN_i < G_i$ and

$$\check{X}_i = G_i/\check{H}_i \longrightarrow X_i = G_i/H_i \longrightarrow \hat{X}_i = G_i/\hat{H}_i \qquad (i = 1, 2).$$

Let $\overline{\hat{m}}$ on $\hat{X}_1 \times \hat{X}_2$ denote the quotient joining of $\overline{m}$ on $X_1 \times X_2$ as in Lemma 6.1. Note that $\hat{X}_i = \mathrm{Hor}(\mathbf{H}_i)$. Applying Theorems 4.3 and 5.2, we conclude that there exists a homeomorphism $\phi : \partial\mathbf{H}_1 \to \partial\mathbf{H}_2$ such that

(i) $[,,,]_2 \circ \phi = [,,,]_1$,
(ii) $\phi(\gamma\xi) = \gamma^\tau\phi(\xi)$ for all $\xi \in \partial\mathbf{H}_1$, $\gamma \in \Gamma_1$.

In view of (6.ii), property (i) yields an isomorphism $G_1 \xrightarrow{\cong} G_2$ for which ϕ serves as the *boundary map*. It follows from (ii) that this isomorphism extends $\tau : \Gamma_1 \to \Gamma_2$. Thus the result essentially reduces to that of Theorem A (see the proof of Theorem D for full details).

Proof of Theorem C. The proof of Theorem A applies almost verbatim to that of Theorem C with the appeal to Theorem 4.3 replaced by Theorem 4.8.

For the proof of Theorem D, we need some preparations, which are of independent interest.

Let (B, ν) be a standard probability space and Γ a group acting by measure class preserving transformations on (B, ν). Such an action is called "strongly almost transitive" if

$$\forall A \subset B \quad \text{with} \quad \nu(A) > 0, \qquad \exists \gamma_n \in \Gamma : \quad \nu(\gamma_n^{-1}A) \to 1. \tag{SAT}$$

Lemma 6.2. *Let Γ be a group with a measure class preserving* (SAT) *action on a standard probability space (B, ν), let C be a standard Borel space with a measurable Γ-action, and let $\pi_1, \pi_2 : B \to C$ be two measurable maps such that*

$$\pi_i(\gamma x) = \gamma\pi_i(x) \quad \textit{for} \qquad \nu\textit{-a.e. } x \in B \qquad (\gamma \in \Gamma).$$

Then $\pi_1(x) = \pi_2(x)$ for ν-a.e. $x \in B$, unless the measures $(\pi_1)_*\nu$, $(\pi_2)_*\nu$ are mutually singular.

Proof. Suppose that $\nu(\{x \in B \mid \pi_1(x) \neq \pi_2(x)\}) > 0$. In this case, there exists a measurable set $E \subset C$ so that the symmetric difference $\pi_1^{-1}(E)\triangle\pi_2^{-1}(E)$ has positive ν-measure. Upon possibly replacing E by its complement, we may assume that the set $A = \pi_1^{-1}(E) \setminus \pi_2^{-1}(E)$ has $\nu(A) > 0$. Set $F_i = \pi_i(A) \subset C$. Then F_1 and F_2 are disjoint. By the (SAT) property there exists a sequence $\{\gamma_n\}$ in Γ so that

$$\sum_{n=1}^{\infty} \nu(\gamma_n^{-1}(B \setminus A)) < \infty.$$

Then for ν-a.e. $x \in B$, we have $\gamma_n x \in A$ for all $n \geq n_0(x) \in \mathbf{N}$. For $i = 1, 2$, let $C_i \subset C$ denote the set of points $y \in C$ for which $\{n \mid \gamma_n y \notin F_i\}$ is finite, i.e., $C_i = \liminf \gamma_n^{-1} F_i$. Hence $C_1 \cap C_2 = \emptyset$ because $F_1 \cap F_2 = \emptyset$. In view of ν-a.e. equivariance of π_i we get that the measure $\eta_i = (\pi_i)_* \nu$ is supported on C_i. Hence, $\eta_1 \perp \eta_2$. □

Theorem 6.3. *Let G be a semi-simple group, $\Gamma < G$ a Zariski dense subgroup with full limit set, e.g., a lattice in G (irreducible or not), $P < G$ a minimal parabolic and ν a probability measure on G/P in the Haar measure class.*

(1) *Let $Q < G$ be some parabolic subgroup containing P and $\pi : G/P \to G/Q$ be a measurable map, s.t. $\pi(\gamma x) = \gamma \pi(x)$ a.e. on G/P for all $\gamma \in \Gamma$. Then ν-a.e. $\pi(gP) = gQ$.*
(2) *Let $Q_1, Q_2 < G$ be two parabolic subgroups containing P, ν_i be probability measures on G/Q_i in the Haar measure class, and $\phi : G/Q_1 \to G/Q_2$ be a measurable bijection with $\phi_* \nu_1 \sim \nu_2$ and s.t. $\phi(\gamma x) = \gamma \phi(x)$ a.e. on G/Q_1 for all $\gamma \in \Gamma$. Then $Q_1 = Q_2$ and $\phi(x) = x$ a.e. on G/Q.*

In particular, the second statement with parabolic subgroups $Q_1 = Q_2$ immediately gives

Corollary 6.4. *Let $\Gamma < G$ be as in Theorem* 6.3*, e.g., a lattice; let $Q < G$ be a parabolic subgroup and ν be a probability measure on G/Q in the Haar measure class. Then the measurable centralizer of the Γ-action on G/Q is trivial.*

Remark 6.5. If G is of higher rank and $\Gamma < G$ is an irreducible lattice, then the only measurable quotients of the Γ-action on G/Q are algebraic, i.e., they are given by $G/Q \to G/Q'$ with $Q < Q'$ and are given by $gQ \mapsto gQ'$. This is the content of Margulis's Factor Theorem (see [13]).

Proof of Theorem 6.3. (1) The natural projection $\pi_0 : G/P \to G/Q$, $\pi_0(gP) = gQ$ is G-equivariant. It is well known that an action of a Zariski dense subgroup Γ on $(G/P, \nu)$ is (SAT). The argument is then completed by Lemma 6.2.

(2) Denote by $\pi_i : G/P \to G/Q_i$ the natural projections $\pi_i(gP) = gQ_i$. Lemma 6.2 shows that the maps $\phi \circ \pi_1$ and π_2 agree ν-a.e. on G/P. This in particular implies that for a.e. gP and all $q \in Q_1$,

$$gQ_2 = \pi_2(gP) = \phi(\pi_1(gP)) = \phi(gQ_1) = \phi(gqQ_1) = \pi_2(gqP) = gqQ_2.$$

This means that $Q_1 < Q_2$ and that $\phi(gQ_1) = gQ_2$ a.e. The same reasoning applies to $\phi^{-1} \circ \pi_2$ and π_1 as maps $G/P \to G/Q_1$, giving $Q_2 < Q_1$ and ϕ being a.e. the identity. □

Proof of Theorem D. Let G be a semi-simple group and $H_i < G$ $(i = 1, 2)$ be two super-spherical subgroups as in Definition 1.7. We consider the action of a lattice $\Gamma < G$ on the two homogeneous spaces $X_i = G/H_i$ equipped with the Haar measures $m_i = m_{G/H_i}$.

Each of the groups H_i $(i = 1, 2)$ is pinched between $\check{H}_i \lhd \hat{H}_i$ associated to some parabolic subgroup $Q_i < G$. The corresponding homogeneous spaces are linked by the natural G-equivariant projections

$$\check{X}_i = G/\check{H}_i \xrightarrow{p_i} X_i = G/H_i \xrightarrow{q_i} \hat{X}_i = G/\hat{H}_i.$$

These homogeneous spaces naturally project to $B_i = G/Q_i$, which is a compact space with a continuous action of G. The G-equivariant projections

$$\check{\pi}_i : \check{X}_i \longrightarrow B_i, \qquad \pi_i : X_i \longrightarrow B_i, \qquad \hat{\pi}_i : \hat{X}_i \longrightarrow B_i \qquad (i = 1, 2)$$

have the alignment property with respect to the corresponding Haar measures and the Γ-action (Theorem 4.8). We also note that $\check{\pi}_i : \check{X}_i \to B_i$ and $\hat{\pi}_i : \hat{X}_i \to B_i$ are principle bundles with amenable structure groups $\check{L}_i = Q_i/\check{H}_i$ and $\hat{L}_i = Q_i/\hat{H}_i$.

Let $\overline{m}$ be an ergodic joining of the Γ-actions on (X_i, m_i), and let $\overline{\check{m}}$ and $\overline{\hat{m}}$ be ergodic joinings of $\check{X}_1 \times \check{X}_2$ and $\hat{X}_1 \times \hat{X}_2$ as in Lemma 6.1. Applying Theorem 3.13 to $\overline{\check{m}}$ (or $\overline{\hat{m}}$), we deduce that there exists a Γ-equivariant measurable bijection $\phi : B_1 \to B_2$ mapping the Haar measure class $[\nu_1]$ on B_1 to the measure class $[\nu_2]$ on B_2. Then part (2) of Theorem 6.3 shows that under these circumstances $Q_1 = Q_2$. Hence, simplifying the notations, we have a single parabolic $Q < G$ and

$$B < \check{H} < H_1, H_2 < \hat{H} \triangleleft Q \qquad \text{and} \qquad \check{L} = Q/\check{H}, \qquad \hat{L} = Q/\hat{H}$$

and $\overline{\check{m}}$ and $\overline{\hat{m}}$ are ergodic self-joinings of the Γ-actions on $\check{X} = G/\check{H}$ and $\hat{X} = G/\hat{H}$, which are principal bundles over B with the alignment property with respect to the Γ-action. By Theorem 3.9, they have the form

$$\text{(6.iii)} \quad \overline{\check{m}} = const\cdot \int_{G/\check{H}} \delta_{g\check{H}} \otimes \delta_{gq\check{H}}\, d\check{m}(g\check{H}), \quad \overline{\hat{m}} = const\cdot \int_{G/\hat{H}} \delta_{g\hat{H}} \otimes \delta_{gq\hat{H}}\, d\check{m}(g\hat{H})$$

for some fixed $q \in Q$.

To complete the proof of the theorem, it remains to consider in more detail the case of an isomorphisms $T : X_1 \to X_2$; we shall prove that in this case the above $q \in Q$ conjugates H_1 to H_2. So let $\overline{m}$ be the joining coming from the graph of T, and let $\overline{\check{m}}$ be the ergodic joining supported on the graph of the map $\check{T}(g\check{H}) = gq\check{H}$ where $q\check{H} = \lambda \in \check{L}$. Then $\check{T}$ maps the preimage

$$p_1^{-1}(\{gH_1\}) = \{gh_1\check{H} \in \check{X} \mid h_1 \in H_1\}$$

of a typical point $gH_1 \in X_1$ to the preimage

$$p_2^{-1}(\{g'H_2\}) = \{g'h_2\check{H} \in \check{X} \mid h_2 \in H_2\}$$

of the point $g'H_2 = T(gH_1) \in X_2$. With $q \in Q$ as in (6.iii), we have $gH_1q = g'H_2 = T(gH_1)$. This implies that $H_2 = q^{-1}H_1q$ and $T(gH_1) = gqH_2$ a.e. This completes the proof of Theorem D.

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