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On the ergodic properties of Cartan flows in ergodic actions of $\mathbf{SL_2(R)}$ and $\mathbf{SO}(n, 1)$

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Abstract. Let $G = \mathrm{SL}_2(\mathbf{R})$ (or $G = \mathrm{SO}(n, 1)$) act ergodically on a probability space (X, m). We consider the ergodic properties of the flow $(X, m, \{g_t\})$, where $\{g_t\}$ is a Cartan subgroup of G. The geodesic flow on a compact Riemann surface is an example of such a flow; here $X = G/\Gamma$ is a transitive G-space, $G = \mathrm{SL}_2(\mathbf{R})$ and $\Gamma \subset G$ is a lattice. In this case the flow is Bernoullian.

For the general ergodic G-action, the flow $(X, m, \{g_t\})$ is always a K-flow, however there are examples in which it is not Bernoullian.

1. *Introduction*

Let G be a semisimple Lie group, acting measurably on a probability measure space $(X, \mathcal{B}, \mu)$, where the measure μ is G-invariant and ergodic. Let $\{g_t\}_{t\in\mathbf{R}}$ be some one-parameter subgroup of a Cartan subgroup of G. We are interested in the ergodic properties of the flow $(X, \mathcal{B}, \mu, \{g_t\})$. In this paper we shall consider rank-one groups $G = \mathrm{SO}(n, 1)$, $n \geq 2$, and, in particular, the case $G = \mathrm{SO}(2, 1)$, which is essentially $\mathrm{SL}_2(\mathbf{R})$. In the latter case

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

The most studied example of an ergodic G-action is the action on the homogeneous space G/Γ, where $\Gamma \subset G$ is a lattice. In the case of a cocompact, torsion free lattice Γ in $G = \mathrm{SL}_2(\mathbf{R})$, the space G/Γ is naturally identified with the unit tangent bundle SM of a compact Riemann surface M:

$$M = \mathbf{H}/\Gamma = K\backslash G/\Gamma,$$

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where $\mathbf{H}$ is the hyperbolic space, and $SO(2) \cong K \subset G$ is a maximal compact subgroup. With these identifications, the action of $\{g_t\}$ on G/Γ corresponds (up to time scaling) to the *geodesic flow* on SM.

Now consider the group $G = SO(n, 1)$ with a cocompact torsion free lattice Γ, maximal compact subgroup $K \cong SO(n)$, and a Cartan subgroup $\{g_t\}$. Then the flow $(G/\Gamma, \lambda, \{g_t\})$ can be naturally identified with the geodesic flow on the frame bundle FM to a compact n-dimensional manifold of constant negative curvature -1:

$$M = \mathbf{H}^{(n)}/\Gamma = K\backslash G/\Gamma,$$

where $\mathbf{H}^{(n)} = K\backslash G$ denotes the n-dimensional hyperbolic space. The group $\{g_t\}$ commutes with a compact subgroup $SO(n-1) \cong K_0 \subseteq K \cong SO(n)$ (K_0 preserves the direction of the flow) and the flow $(K_0\backslash G/\Gamma, \{g_t\})$ is identified with the geodesic flow on the unit tangent bundle SM. In fact, the geodesic flow on any compact manifold of a *constant* negative curvature arises in this way.

Ergodic properties of the geodesic flow on negatively curved compact manifolds have been extensively studied since the late 1930s, when Hopf [**8**] and Hedlund [**7**] proved ergodicity of the geodesic flow. Gelfand and Fomin [**4**] proved that the geodesic flow has countable Lebesgue spectrum, and Sinai [**17**] obtained the K-property of the flow. Ornstein and Weiss [**13**] have shown that the geodesic flow on a negatively curved compact surface is Bernoullian, determining its measure theoretic properties uniquely (up to time scaling). Using their technique, Dani [**3**] has proved the Bernoulli property for a class of transformations on homogeneous spaces, including the case $(SO(n, 1)/\Gamma, \{g_t\})$.

In the more general setup of an *ergodic*, rather than a *transitive*, action of $G = SL_2(\mathbf{R})$ or $G = SO(n, 1)$ on a Lebesgue probability space $(X, \mathcal{B}, \mu)$, Howe–Moore's theorem on vanishing of matrix coefficients [**9**] implies, that the flow $(X, \mathcal{B}, \mu, \{g_t\})$ is always ergodic and mixing. It follows from the general result of Dani [**2**] that $(X, \mathcal{B}, \mu, \{g_t\})$ is always a K-flow. For completeness, we shall sketch the proof for the case $G = SL_2(\mathbf{R})$ in §4.

Our goal is to exhibit a class of natural examples of ergodic actions of $G = SL_2(\mathbf{R})$ and $G = SO(n, 1)$, for which the flow $(X, \mathcal{B}, \mu, \{g_t\})$ is not Bernoullian (though satisfies the K-property). These examples arise naturally among induced actions: given an ergodic measure preserving action of a lattice Γ in a locally compact group G on a probability space (Y, ν), one constructs an *induced* G-action on the product space $(G/\Gamma \times Y, \lambda \times \nu)$ (see below). It is known that the induced G-action is ergodic if the action of the lattice Γ is ergodic.

MAIN THEOREM. *Let $G = SL_2(\mathbf{R})$ or $SO(n, 1)$, $\Gamma \subset G$ be a cocompact torsion free lattice, and $\tau : \Gamma \to \mathbf{Z}$ be an epimorphism. Let (Y, ν, T) be an invertible ergodic system with positive entropy, that is, $h(Y, T) > 0$, and consider the ergodic Γ-action on (Y, ν), defined by $\gamma \cdot y = T^{\tau(\gamma)}y$. Then the induced G-action on $(X, \mu) = (G/\Gamma \times Y, \lambda \times \nu)$ is ergodic, while the flow $(X, \mu, \{g_t\})$ is not Bernoullian.*

Remark 1. It is well known that any surface group $\Gamma \subset G = SL_2(\mathbf{R})$ has epimorphisms $\tau : \Gamma \to \mathbf{Z}$. For $G = SO(n, 1)$ with $n > 2$, there exist lattices which admit epimorphisms onto $\mathbf{Z}$; and, in fact, all arithmetic lattices in $SO(n, 1)$ with $n \neq 3, 7$ have this property (see [**11**]).

Remark 2. There exists a *smooth* ergodic action of $G = \mathrm{SL}_2(\mathbf{R})$ or $G = \mathrm{SO}(n,1)$ on a compact manifold X with a smooth measure μ, so that the flow $(X,\mu,\{g_t\})$ is not Bernoullian (though it is a K-flow). This follows from the Main Theorem, with (Y,ν,T) being a *smooth* system with positive entropy, and the fact that an induced G-action, constructed from a smooth action of a lattice $\Gamma \subset G$, can be realized in a smooth form (see Remark 4).

One can consider ergodic actions of other simple groups G, and their one-parameter subgroups $\{g_t\}$, generated by a semi-simple element of a Cartan subgroup. The result of Dani [**2**] applies to these cases too, so in any ergodic G-action on (X,μ), the flow $\{g_t\}$ is a K-flow.

As for the Bernoulli property, it seems possible that examples like that in the Main Theorem can also be constructed for $\mathrm{SU}(n,1)$. However, $\mathrm{Sp}(n)$ and higher rank simple Lie groups satisfy Kazhdan's property T, and thus do not have lattices which map onto $\mathbf{Z}$. On the other hand, property T implies, that the group $\{g_t\}$ has very strong mixing properties. It is possible that these mixing properties already imply that $\{g_t\}$ is Bernoullian.

2. *Preliminaries*

2.1. *Induced actions.* In this section we briefly discuss the construction of group actions, induced from lattice actions.

Let Γ be a lattice in a locally compact group G, and assume that Γ acts ergodically on a probability space (Y,ν). Let $\Omega \subset G$ be a measurable fundamental domain for G/Γ, i.e. $\{\Omega\gamma\}_{\gamma\in\Gamma}$ are disjoint, and $\Omega\Gamma = \bigcup_{\gamma\in\Gamma}\Omega\gamma$ is a conull set in G with respect to the Haar measure λ_G. Let λ be the restriction of λ_G to Ω, normalized to 1. Then (Ω,λ) is a model for G/Γ, where the G-action on G/Γ corresponds to the action $g : \omega \mapsto g\cdot x$, given by the following rule. For $g \in G$ and a.e. $\omega \in \Omega \subset G$, consider $g\omega \in G$, and let $\gamma \in \Gamma$ be the unique element, satisfying $g\omega \in \Omega\gamma \subset G$. Set $g\cdot\omega = g\omega\gamma^{-1}$.

Denoting the above γ by $\alpha_\Omega(g,\omega)$, we observe that the measurable function $\alpha = \alpha_\Omega : G \times G/\Gamma \to \Gamma$ is a cocycle, i.e. given $g_1, g_2 \in G$,

$$\alpha_\Omega(g_1g_2,\omega) = \alpha_\Omega(g_1, g_2\cdot\omega)\alpha_\Omega(g_2,\omega) \quad \text{for a.e. } \omega \in \Omega. \tag{1}$$

It is known (cf. [**18**, Theorem B9, p. 200]) that such an α_Ω is a.e. equal to a *strict* cocycle, i.e. a measurable function, satisfying (1) *for all* $g_1, g_2 \in G$ and all ω from a conull set $\Omega_0 \subset \Omega$. Therefore, one can define the *induced* G-action as the skew-product action on $\Omega \times Y$ given by

$$g\cdot(\omega,y) = (g\cdot\omega, \alpha_\Omega(g,\omega)y), \quad \omega\in\Omega,\ y\in Y.$$

This G-action preserves the probability measure $\lambda\times\nu$.

The construction depends on the choice of the fundamental domain Ω. However, different domains give rise to measurably cohomologous cocycles, and thus to measure-theoretically isomorphic G-actions.

There exists another, *invariant* construction of the induced action: consider the G-action on $G\times Y$, given by $g(g',y) = (g\,g',y)$. This action preserves the infinite measure

$\lambda_G \times \nu$, and commutes with the Γ-action, defined by $\gamma \cdot (g, y) = (g\gamma^{-1}, \gamma \cdot y)$. The induced G-action is the quotient $G \times Y/\Gamma$ of $G \times Y$ by this Γ-action. Using this description one can establish the following.

Remark 3. The G-action induced from an *ergodic* action of a lattice in G is *ergodic* (see [**18**, p. 75]).

Remark 4. Given a smooth action of a cocompact, torsion free lattice $\Gamma \subset G$ on a compact manifold Y with a smooth measure ν, the induced G-action can be realized as a smooth action on a compact manifold $G \times Y/\Gamma$ with a smooth measure $\lambda \times \nu$.

2.2. *Geometric construction of cocycles α and $\tau \circ \alpha$.* Now let us consider the case of a cocompact, torsion free lattice Γ in $G = \mathrm{SL}_2(\mathbf{R})$ (or in $G = \mathrm{SO}(n, 1)$). We shall focus on the cocycle α_Ω, restricted to the one-parameter subgroup $\{g_t\}$. Removing some finite collection of codimension 1 submanifolds from the compact manifold $M = \mathbf{H}^{(n)}/\Gamma$, one can obtain an open, connected, simply connected subset D of M. Let $\tilde{D} \subset \tilde{M} = \mathbf{H}^{(n)}$ be a connected homeomorphic preimage of D under the natural projection $\pi : \tilde{M} \to M$. Then $\tilde{D}$ forms a fundamental domain for $\mathbf{H}^{(n)}/\Gamma$. Fix some point $p \in D \subset M$ and define a map

$$\beta_D : \mathbf{R} \times SM \to \pi_1(M, p)$$

by the following rule. Let $\beta_D(t, (v, x))$ be the homotopy class of the closed curve $[p, y] \circ \gamma_t(v, x) \circ [x, p]$ on M, obtained by concatenating three paths: (1) $[x, p]$, some path connecting p with x within D; (2) $\gamma_t(v, x)$, the geodesic of length t emerging from x in direction v (let y denote its endpoint); (3) $[p, y]$, some path connecting y with p within D. Observe that $\beta_D(t, (v, x))$ is defined as soon as $x, y \in D$, and since D is simply connected, its value does not depend on the choice of $[p, y]$ and $[x, p]$ in D. One easily checks that given t, s for a.e. $(v, x) \in SM$ the map β_D satisfies the cocycle equation, like (1), and thus coincides a.e. with a strict cocycle.

PROPOSITION 5. *Let $G = \mathrm{SO}(n, 1)$ (or $G = \mathrm{SL}_2(\mathbf{R})$) and let $\Gamma \subset G, M, D, \tilde{D}, \beta_D$ be as above. Then taking $\Omega = K \cdot \tilde{D} \subset G$ as a fundamental domain for G/Γ, one has $\alpha_\Omega(g_t, K_0\cdot) = \beta_D(t, \cdot)$. More precisely, consider the natural isomorphism $j : \pi_1(M, p) \to \Gamma$ and let $\theta : \Omega \to SM$ correspond to the projection $G/\Gamma \to K_0\backslash G/\Gamma$. Then*

$$\alpha_\Omega(g_t, \omega) = j \circ \beta_D(t, \theta(\omega)) \tag{2}$$

for a.e. $\omega \in \Omega$ and a.e. $t \in \mathbf{R}$.

Proof. Fix $t \in \mathbf{R}$ and consider a generic $\omega \in \Omega$ as an element of G, and let $h = g_t\omega \in \Omega\gamma_\alpha \subseteq G$ with $\gamma_\alpha \in \Gamma$ and $\gamma_\alpha = \alpha_\Omega(g_t, \omega)$ by definition of α_Ω. Let $(v, x) = \theta(\omega)$ be the corresponding element in SM. Then $\beta_D(t, (v, x))$ is well defined with probability 1. Consider the closed curve

$$c = [p, y] \circ \gamma_t(v, x) \circ [x, p]$$

on M, and its lifting $\tilde{c}$ to the universal covering $\tilde{M} = \mathbf{H}^{(n)}$:

$$\tilde{c} = [\tilde{q}, \tilde{y}] \circ \tilde{\gamma}_t(v, \tilde{x}) \circ [\tilde{x}, \tilde{p}].$$

Assume that $\tilde{p} \in \tilde{D}$, $[\tilde{x}, \tilde{p}]$ projects onto $[x, p]$, $\tilde{\gamma}_t(v, \tilde{x})$ onto $\gamma_t(v, x)$, and $[\tilde{y}, \tilde{q}]$ onto $[y, p]$. Denote $\gamma_\beta = j \circ \beta_D(t, (v, x)) \in \Gamma$. We have to show that $\gamma_\alpha = \gamma_\beta$.

Observe that $\tilde{q} = \tilde{p}\gamma_\beta$ and $\tilde{y} \in \tilde{D}\gamma_\beta$, while $\tilde{x} \in \tilde{D}$. On the other hand, under the projection $\tilde{\theta} : G \to S\tilde{M}$, $\omega \in G$ is mapped onto the tangent vector $(v, \tilde{x}) \in S\tilde{M}$ starting the geodesic $\tilde{\gamma}_t(v, \tilde{x})$, while $g_t\omega \in G$ is mapped onto $(u, \tilde{y})$—the end of $\tilde{\gamma}_t(v, \tilde{x})$. Since $\tilde{y} \in \tilde{D}\gamma_\beta$, we get $g_t\omega \in K \cdot D\gamma_\beta = \Omega\gamma_\beta$, and since $g_t\omega \in \Omega\gamma_\alpha$, we conclude $\gamma_\alpha = \gamma_\beta$. □

The homomorphism $\tau \circ j : \pi_1(M, p) \to \Gamma \to \mathbf{Z}$ factors through the homology group $H_1(M) \cong \pi_1(M, p)/[\pi_1(M, p), \pi_1(M, p)]$. Any homomorphism $H_1(M) \to \mathbf{Z}$ can be realized as an integration of some closed 1-form ξ on M. Thus, choosing the paths $[x, p]$ and $[y, p]$ in the definition of the cocycle β_D to be smooth, we obtain that for $\omega \in \Omega$ and $(v, x) = \theta(\omega)$,

$$\tau \circ \alpha_\Omega(g_t, \omega) = \tau \circ j \circ \beta_D(t, (v, x)) = \int_{[p,y]} \xi + \int_{\gamma_t(v,x)} \xi + \int_{[x,p]} \xi.$$

Introducing a function $\psi : K \cdot D \subset SM \to \mathbf{R}$ defined by

$$\psi(v, x) = \int_{[x,p]} \xi,$$

we note that $\|\psi\|_\infty \le \operatorname{diam}(D) \cdot \|\xi\|_\infty$, and obtain the following.

PROPOSITION 6. *Let G, Γ, M, θ and $\Omega \subset G$ be as in Proposition 5. Then for any homomorphism $\tau : \Gamma \to \mathbf{Z}$, there exists a closed 1-form ξ, defined on M, so that the cocycle*

$$\tau \circ \alpha_\Omega : \{g_t\} \times G/\Gamma \to \Gamma \to \mathbf{Z}$$

and the cocycle $\phi_\xi : \mathbf{R} \times SM \to \mathbf{R}$, defined by

$$\phi_\xi(t, (v, x)) = \int_{\gamma_t(v,x)} \xi,$$

are L^∞-cohomologous in the sense that there exists $\psi \in L^\infty(SM)$ s.t.

$$\tau \circ \alpha_\Omega(g_t, \omega) = \phi_\xi(t, \theta(\omega)) + \psi(\theta(\omega)) - \psi(g_t\theta(\omega)).$$

In what follows, we shall fix such $D, \tilde{D}, \Omega = K \cdot \tilde{D}$ and α will refer to the cocycle α_Ω. We shall think of the G-action on (X, μ) as a $(\tau \circ \alpha)$-defined skew-product over (Ω, λ). We shall also identify the measure space G/Γ with Ω and $K_0\backslash G/\Gamma$ with SM, and use either of the above notations according to the context.

2.3. *Asymptotically Brownian processes.* In the following we shall use some notions from the theory of stochastic processes. Let (Z, η, S) be an invertible ergodic system. Given a measurable function $f : Z \to \mathbf{R}$, let us define $F : \mathbf{Z} \times Z \to \mathbf{R}$ by

$$F(n, z) = \begin{cases} f(z) + f(Sz) + \cdots + f(S^{n-1}z) & n > 0 \\ 0 & n = 0 \\ -f(S^n z) - \cdots - f(S^{-1}z) & n < 0. \end{cases} \tag{3}$$

Then F is a $\mathbf{Z}$-cocycle, i.e. $F(n + m, z) = F(n, z) + F(m, S^n z)$.

Definition 1. A function $f : Z \to \mathbf{R}$ is said to be *asymptotically Brownian* if there exists a joining of (Z, η, S) with a Brownian motion $W(t, \cdot)$, $t \in \mathbf{R}$ (i.e. $W(s, \cdot)$ and $W(-s, \cdot)$, $s \geq 0$, are independent standard Brownian motions with $W(0, \cdot) = 0$), and constants $\sigma > 0$, $\delta > 0$, so that almost everywhere

$$\lim_{|n| \to \infty} \frac{F(n, \cdot) - \sigma W(n, \cdot)}{|n|^{1/2-\delta}} = 0. \tag{4}$$

Theorem 7 below, proved by Phillips and Stout [**14**], gives a sufficient condition for a function to be asymptotically Brownian, using the following notions.

Definition 2. A measurable partition P of an ergodic system (Z, η, S) is said to be *exponentially mixing* if there exist C and $\lambda < 1$, s.t. for any $n \geq 1$, any $A \in \bigvee_{-\infty}^{0} S^i P$ and any $B \in \bigvee_{n}^{\infty} S^i P$,

$$|\eta(A \cap B) - \eta(A)\eta(B)| \leq C\lambda^n.$$

Definition 3. Define a semi-metric $d_P : Z \to \mathbf{R}_+$ on Z, associated with the process (P, S), by $d_P(z, w) = \inf_n e^{-n}$, where the inf is taken over all indices $n \geq 0$, for which z and w belong to the same atom of $P_{-n}^{n} = \bigvee_{i=-n}^{n} S^i P$. A measurable function $f : Z \to \mathbf{R}$ is said to satisfy a *Hölder condition* with respect to the partition P if there exists C and $\kappa > 0$ so that

$$|f(z) - f(w)| \leq C \cdot d_P(z, w)^{\kappa}.$$

THEOREM 7. ([**14**]) *Let (Z, η, S) be some ergodic system, and $f : Z \to \mathbf{R}$ a measurable function, satisfying the following conditions:*

(i) $\int_Z f(z)\, d\eta(z) = 0$*;*

(ii) *$f(z)$ is not an L^2-coboundary, i.e. $f \neq g - g \circ S$, for $g \in L^2(\eta)$;*

(iii) *f satisfies a Hölder condition with respect to an exponentially mixing finite partition P of Z.*

Then $f : Z \to \mathbf{R}$ is asymptotically Brownian.

The main tool in the proof of the Main Theorem is the construction, due to Rudolph [**16**], of non loosely Bernoulli skew-products. This construction was motivated by Kalikow's solution of the T, T^{-1} problem [**10**].

THEOREM 8. ([**16**]) *Let (Z, η, S) be an invertible ergodic system with an asymptotically Brownian function $f : Z \to \mathbf{Z}$, and let (Y, ν, T) be an invertible ergodic system with positive entropy: $h(Y, T) > 0$. Then the skew-product*

$$(Z \times_f Y, \eta \times \nu, \hat{S}) \quad \text{with } \hat{S}(z, y) = (Sz, T^{f(z)}y)$$

is not a loosely Bernoulli system, i.e. it is not isomorphic to a Poincaré cross-section of a Bernoulli flow.

Remark 9. It is unclear whether the assumption that f is asymptotically Brownian cannot be weakened. In particular, is it true that properties (i) and (ii) from Theorem 7 suffice to show that a skew-product $(Z \times_f Y, \eta \times \nu, \hat{S})$ is not *Bernoullian* for any (Y, ν, T) with $h(Y, T) > 0$?

3. *Proof of the Main Theorem*

Consider the induced G-action on $(X, \mu) = (\Omega \times Y, \lambda \times \nu)$. Since (Y, ν, T) is ergodic, and $\tau : \Gamma \to \mathbf{Z}$ is onto, Γ acts ergodically on (Y, ν), and hence, by Remark 3, G acts ergodically on (X, μ).

Viewing the G-action on (X, μ) as a skew-product extension of the G-action on (Ω, λ), we observe that the relevant cocycle is $\tau \circ \alpha : G \times \Omega \to \Gamma \to \mathbf{Z}$, and the action is given by

$$g(\omega, y) = (g \cdot \omega, T^{\tau\circ\alpha(g,\omega)} y).$$

In particular, the flow $(X, \mu, \{g_t\})$ is a $(\tau \circ \alpha)$-defined skew-product over the geodesic flow $(\Omega, \lambda, \{g_t\})$, with (Y, ν, T) being the fiber. From now on we shall focus on the flow $(\Omega, \lambda, \{g_t\})$ and its extension $(X, \mu, \{g_t\})$.

Consider a Poincaré cross-section (Z, η, S) of the flow $(\Omega, \lambda, \{g_t\})$ and let $l(z)$ denote the return time to the cross-section Z. Then $S z = g_{l(z)} z$, and the probability measure η on Z refers to the invariant probability measure λ on Ω by the formula

$$\int_{SM} F(u, x)\lambda(u, x) = \left(\int_Z l(z)\, d\eta(z) \right)^{-1} \cdot \int_Z \int_0^{l(z)} F(g_t\, z)\, dt\, d\eta(z).$$

Let us define a function $f : Z \to \mathbf{Z}$ by

$$f(z) = \tau \circ \alpha(g_{l(z)},\, z) \tag{5}$$

and form a skew-product $(Z \times_f Y, \eta \times \nu, \hat{S})$ with $\hat{S}(z, y) = (S z, T^{f(z)} y)$. Note that the system $(Z \times_f Y, \eta \times \nu, \hat{S})$ is a Poincaré cross-section for the flow $(X, \mu, \{g_t\}) = (\Omega \times Y, \lambda \times \nu, \{g_t\})$. Indeed, $l(z)$ is the return time for $(z, y) \in Z \times Y$:

$$g_{l(z)}(z, y) = (g_{l(z)} z, T^{\tau\circ\alpha(g_{l(z)}, z)} y) = (S\, z, T^{f(z)} y) = \hat{S}(z, y).$$

The idea of the proof is to show that for an appropriate choice of the cross-section (Z, η, S) the function $f : Z \to \mathbf{Z}$ in (5) is asymptotically Brownian. As soon as this is established, Theorem 8 implies that the transformation $(Z \times_f Y, \eta \times \nu, \hat{S})$ is not *loosely Bernoullian* and, therefore, the flow $(X, \mu, \{g_t\})$ is not Bernoullian, proving the theorem.

We are left with the proof that for some Poincaré cross-section (Z, η, S) of $(\Omega, \lambda, \{g_t\})$, the function $f : Z \to \mathbf{Z}$, given by (5), is asymptotically Brownian. In the two-dimensional case Proposition 6 allows us to substitute $\tau \circ \alpha$ by ϕ_ξ on SM which is isomorphic to Ω, and thus, in this case, it is enough to show that the function $h : Z \to \mathbf{R}$ defined by

$$h(z) = \phi_\xi(l(z), z) = \int_{\gamma_{l(z)}(z)} \xi \tag{6}$$

is asymptotically Brownian. In the general case, where SM is a factor of Ω, we shall assume that the cross-section (Z, η, S) of $(\Omega, \lambda, \{g_t\})$ arises from a cross-section (Z', η', S') of the geodesic flow on SM: $Z = \theta^{-1}(Z')$, and by Proposition 6 it is enough to show that the function $h : Z' \to \mathbf{R}$, defined by (6) with $z \in Z'$, is asymptotically Brownian.

Since from now on we shall be dealing only with the geodesic flow on SM (and not with it's compact extension to Ω), we shall, with some abuse of notation, denote by (Z, η, S) the cross-section of the geodesic flow $(SM, \lambda, \{g_t\})$.

Thus we shall prove that $h : Z \subset SM \to \mathbf{R}$ given by (6) is asymptotically Brownian, verifying conditions (i)–(iii) of Theorem 7. In fact, conditions (i) and (ii) are satisfied by any cross-section of the geodesic flow. The following lemma appears essentially in [**5**].

LEMMA 10. *Let (Z, η, S) be a Poincaré cross-section of the geodesic flow on SM, ξ a closed 1-form on M, and $h : Z \to \mathbf{R}$ be given by (6). Then*

$$\int_Z h(z)\, d\eta(z) = 0.$$

Proof. Denote by $\xi(v, x)$ the value of ξ on the tangent vector $(v, x) \in SM$. Then

$$\int_Z h(z)\, d\eta(z) = \int_Z \int_0^{l(z)} \xi(g_t z)\, dt\, d\eta(z) = \left(\int_Z l(z)\, d\eta(z) \right) \cdot \int_{SM} \xi(u, x)\, d\lambda(u, x).$$

The invariant probability measure $\lambda(u, x)$ on SM is the product measure of the normalized Riemann volume on M and the uniform distribution on the directions u. Thus $d\lambda(u, x) = d\lambda(-u, x)$, while $\xi(-u, x) = -\xi(u, x)$ for any fixed $x \in M$. Therefore the last integral has to be 0. □

LEMMA 11. *Let (Z, η, S) be any Poincaré cross-section of the geodesic flow $(SM, \lambda, \{g_t\})$. Then the function $h : Z \to \mathbf{R}$, defined by (6), is not a coboundary, i.e. $h(z) \neq k - k \circ S$ for any measurable $k : Z \to \mathbf{R}$.*

Proof. Suppose h is a coboundary, then so is f. One easily concludes that the f-defined skew-product $(Z \times_f Y, \eta \times \nu, \hat{S})$ is not ergodic, and thus the flow $(X, \mu, \{g_t\})$ is also not ergodic. Therefore, by Moore's theorem [**12**], the G-action on $(X, \mu) = (G/\Gamma \times Y, \lambda \times \nu)$ induced from the Γ-action on (Y, ν) is not ergodic. But that is a contradiction, since (Y, ν, T) was ergodic and $\tau : \Gamma \to \mathbf{Z}$ is an epimorphism. □

To show that h satisfies condition (iii) of Theorem 7, we use *Markov partitions*, which can be constructed for general Anosov flows (see [**15**] for details). A Markov partition for the flow $(SM, \lambda, \{g_t\})$ consists of a Poincaré cross-section (Z, η, S), and a finite measurable partition P of (Z, η, S), so that (P, S) is a topological Markov chain, and the probability measure ν on Z is a Gibbs measure (with respect to the return time function l). The return time function $l : Z \to \mathbf{R}_+$ is bounded: $0 < l_0 \leq l(z) \leq l_1 < \infty$, and Hölder continuous (see [**15**, Theorem 1]). Moreover, Markov partitions of arbitrarily small size can be constructed, where size of the partition is defined to be

$$\max\{\mathrm{diam}(P_1), \ldots, \mathrm{diam}(P_p), \|l\|_\infty\}.$$

For a Markov partition the process (P, S) is exponentially mixing with respect to η. In fact, Gibbs measures satisfy an even stronger mixing property (see [**1**]): there exists $\lambda < 1$, s.t. for any $n \geq 1$, any $A \in \bigvee_{-\infty}^{0} S^i P$ and any $B \in \bigvee_{n}^{\infty} S^i P$,

$$|\eta(A \cap B) - \eta(A)\eta(B)| < \eta(A) \cdot \eta(B) \cdot \lambda^n.$$

To show the Hölder property of the function h with respect to (P, S), we shall use a general property of Markov partitions of an Anosov flow: the Riemannian metric on $Z \subset SM$ induced from the metric on SM is Hölder continuous with respect to the metric d_P on Z. (In fact, d_P and ρ are Hölder equivalent.) We shall prove this in our special setup.

LEMMA 12. *Let (Z, η, S) with a finite partition P be a Markov partition of the geodesic flow $(SM, \lambda, \{g_t\})$ on a compact manifold M with a constant negative curvature. Let d_P denote the metric on Z as in Definition 3, and let ρ denote the Riemannian metric, induced from SM. Assume that the Markov partition is sufficiently small. Then ρ, restricted to an open dense subset $Z_0 \subset Z$ of full η-measure, is Hölder continuous with respect to d_P, i.e. there exist C and $\kappa > 0$, s.t.*

$$\forall z, w \in Z_0, \quad \rho(z, w) \leq C \cdot d_P(z, w)^{\kappa}.$$

Proof. We have to show that there exist C and $\lambda < 1$ so that any two points $z, w \in Z \subset SM$, with the property that for any $|i| \leq n$ both points $S^i z, S^i w$ belong to the same P-atom P_{j_i}, satisfy the estimation $\rho(z, w) \leq C \cdot \lambda^n$.

Consider the covering map $S\tilde{M} \to SM$. First we shall show that the geodesics $\{g_t\, z\}$ and $\{g_t\, w\}$ can be lifted to $S\tilde{M}$, so that they stay close for $|t| \leq c \cdot n$.

If the Markov partition is sufficiently small, we can assume that the preimage of each of the (connected) sets

$$\hat{P}_j = \bigcup_{x \in P_j} \{g_t x \mid 0 \leq t \leq l(x)\} \subset SM, \quad P_j \subset Z,$$

is a disjoint union of connected sets $\tilde{P}_j \cdot \gamma \subset S\tilde{M}$ with $\gamma \in \Gamma$, each of which is homeomorphically projected onto $\hat{P}_j$. Let us define times $\{t_i\}_{-n}^n$, $\{s_i\}_{-n}^n$ by

$$t_0 = s_0 = 0, \quad t_{i+1} = t_i + l(S^i z), \quad s_{i+1} = s_i + l(S^i w), \quad -n \leq i < n.$$

From the assumption on z, w, we conclude that

$$g_t z, g_s w \in \hat{P}_{j_i} \quad \text{for } t_i \leq t \leq t_{i+1}, s_i \leq s \leq s_{i+1}, -n \leq i < n.$$

Choose preimages $\tilde{z}$ and $\tilde{w}$ of z, w, lying in the same connected $\tilde{P}_{j_0} \cdot \gamma_0$, which maps homeomorphically onto $\hat{P}_{j_0}$. Then lifting $\hat{P}_{j_i}$ to $S\tilde{M}$, one by one starting from $i = 0$, we obtain connected homeomorphic preimages $\tilde{P}_{j_i} \cdot \gamma_i \subset S\tilde{M}$, with $\gamma_i \in \Gamma$, so that

$$g_t \tilde{z}, g_s \tilde{w} \in \tilde{P}_{j_i} \cdot \gamma_i, \quad \text{for } t_i \leq t \leq t_{i+1}, s_i \leq s \leq s_{i+1}, -n \leq i < n.$$

Let a denote the size of the partition P. Then $\rho(g_{t_i}\tilde{z},\ g_{s_i}\tilde{w}) < a$ in $S\tilde{M}$, where $|t_i| \geq |i| \cdot l_0$ and $|s_i| \geq |i| \cdot l_0$, for all $|i| \leq n$. Since $\{g_t\}$-orbits are global geodesics in $S\tilde{M}$, we have $|t_i - s_i| < a$, for $|i| \leq n$, and hence, denoting $T = n \cdot l_0$, we obtain

$$\rho(g_t \tilde{z}, g_t \tilde{w}) < 3a \quad \text{for } |t| \leq T. \tag{7}$$

The hyperbolic structure of $S\tilde{M}$ implies that any two geodesic lines diverge exponentially fast in (at least) one of the two directions. More precisely, there exist C_1 and $k > 0$ so that (7) implies that there exists t_0 with $|t_0| \leq 3a$, s.t. for $|t| \leq T - 3a$,

$$\rho(g_{t+t_0}\tilde{z}, g_t \tilde{w}) \leq C_1 \cdot e^{-kT} \cdot \min\{\rho_1, \rho_2\} \leq C_1 \cdot 3a \cdot e^{-kl_0 n}, \tag{8}$$

where $\rho_1 = \rho(g_{T+t_0}\tilde{z},\ g_T \tilde{w})$ and $\rho_2 = \rho(g_{-T+t_0}\tilde{z},\ g_{-T}\tilde{w})$. In other words, the geodesic lines $\{g_t \tilde{z}\}$ and $\{g_t \tilde{w}\}$ in $S\tilde{M}$ (and hence $\{g_t z\}$ and $\{g_t w\}$ in SM) are *exponentially* close *as sets* in terms of n. We claim that, in fact, the *points* z, w are exponentially close.

By the construction of Markov partitions (see [**15**]), the sets P_j have a special description as unions of leaves of contracting (or expanding) foliations. In the case of a constant negative curvature, each of the sets P_j is a compact subset of a codimension 1 *smooth* submanifold in SM, and P_j is the closure of its relative interior $\mathrm{Int}\, P_j$. We take Z_0 to be the union of all $\mathrm{Int}\, P_j$.

Since z, w are points of intersection of the geodesics $\{g_t z\}$ and $\{g_t w\}$ with $\mathrm{Int}\, P_{j_0}$, and since the geodesic flow is uniformly transversal to P_{j_0}, we conclude from (8) that for some C_2,

$$\rho(z, w) \leq C_2 \cdot e^{-kl_0 n} = C_2 \cdot \lambda^n, \quad \text{where } \lambda = e^{-kl_0} < 1,$$

completing the proof of the lemma. □

Remark 13. The above argument also shows that the return time function $l : Z_0 \to \mathbf{R}_+$ is Hölder continuous with respect to d_P. Indeed, $l(z)$ and $l(w)$ are the lengths of two exponentially close geodesic segments, cut by transvere smooth sets $\mathrm{Int}\, P_{j_0}$, $\mathrm{Int}\, P_{j_1}$. Finally, we remark that in the case of general C^2-smooth Anosov flows, the sets P_j are not necessarily smooth, but are still Hölder.

We can now complete the proof of the theorem. The function $h : Z \to \mathbf{R}$ is given by

$$h(z) = \int_0^{l(z)} \xi(g_t z)\, dt,$$

where the function $\xi(\cdot)$ is Lipschitz on SM and bounded, while $l(z)$ is Hölder. We conclude that h is Hölder with respect to the Riemann metric, and therefore, by Lemma 12, h satisfies the Hölder condition with respect to (P, S). This completes the proof of the Main Theorem. □

Remark 14. Guivarc'h in [**6**] describes a method for exhibiting K-flows that are not Bernoullian that is somewhat different from that of [**16**]. He also follows the proof of [**10**], but avoids the machinery of asymptotically Brownian processes. One can also carry out a proof of our result based on his approach, but we preferred to base ours on the more explicit statements in [**16**].

4. *The K-property*

The following theorem is a special, but typical, case of a result due to Dani [**2**].

THEOREM 16. *Let $G = \mathrm{SL}_2(\mathbf{R})$ act ergodically on the probability space $(X, \mathcal{B}, \mu)$. Then the flow $(X, \mu, \{g_t\})$ is a K-flow.*

Proof. We will show that for every $t_0 \neq 0$, the transformation $T = g_{t_0}$ is a K-automorphism of (X, μ). It is enough to construct an increasing sequence of finite measurable partitions $P^{(n)}$ of $(X, \mathcal{B}, \mu)$, s.t. $\mathcal{B} = \bigvee_n P^{(n)}$, but all the $P^{(n)}$ have a trivial tail, that is any measurable set in

$$P_\infty^{(n)} = \bigwedge_{k=1}^{\infty} \bigvee_{i=k}^{\infty} T^i P^{(n)}$$

has μ-measure 0 or 1. Partitions $P^{(n)}$ are constructed using the horocyclic flow

$$h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}. \tag{9}$$

By the classical theorem of Ambrose–Kakutani, the flow $(X, \mu, \{h_s\})$ can be represented as a special flow under a function $l : X_0 \to \mathbf{R}_+$, with a base $(X_0, \mathcal{B}_0, \mu_0, T_0)$. Let $Q^{(n)}$ be some increasing sequence of finite measurable partitions of X_0, generating $\mathcal{B}_0$. Points

$$(x, s),\ (x', s') \in \{(x, s) \mid x \in X_0, 0 \le s < l(x)\} = X$$

are defined to belong to the same atom of $P^{(n)}$ if x, x' lie in the same atom of $Q^{(n)}$, and $k2^{-n} \le s, s' < (k+1)2^{-n}$.

We claim that for μ-a.e. $x \in X$, the pair of points x and $h_1 x$ is not separated by $P^{(n)}_\infty$. This will complete the proof, since by Moore's theorem [**12**] h_1 is ergodic on $(X, \mathcal{B}, \mu)$, implying that the measurable partition $P^{(n)}_\infty$ is trivial.

The relation $g_{-t} h_s g_t = h_{e^{-2t}s}$ in $\mathrm{SL}_2(\mathbf{R})$ implies that

$$T^j(h_1 x) = h_{\lambda^j}(T^j x) \quad \text{with } \lambda = e^{-2t_0} < 1,$$

and thus:

(i) the points $T^j x$, $T^j(h_1 x)$ belong to the same $\{h_s\}$ orbit for all j; and

(ii) the $\{h_s\}$-distance between $T^j x$ and $T^j(h_1 x)$ decays exponentially with $j \ge 0$.

Therefore, the Borel–Cantelli lemma implies that for any $x_0 \in X_0$ and a.e. $0 \le s < l(x)$ the point $x = h_s x_0$ has the property that the pair of points $T^j x$, $T^j(h_1 x)$ lies in the same atom of $P^{(n)}$ for all, but finitely many, $j \ge 0$. Since $d\mu = d\mu_0\, ds$, we conclude that for μ-a.e. $x \in X$ the points x and $h_1 x$ are not separated by $P^{(n)}_\infty$. □

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