RANDOM WALKS ON GROUPS AND RANDOM TRANSFORMATIONS

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Contents

Introduction

The study of asymptotic behavior of sums $S_n = X_n + \cdots + X_1$ of independent identically distributed real valued random variables is a well established part of the classical probability theory. Under suitable boundness conditions the phenomena of Recurrence, Law of Large Numbers, Functional Central Limit Theorem, Iterated Logarithm Law and further limit theorems have been well understood already in the first half of 20-th century. In the recent decades the study of more general non-commutative situation became an active area of research. The general setup that will be discussed in this paper consists of a group G with a probability measure μ on it, and the questions concern the statistic behavior of the *products* $S_n = X_n \cdots X_1$ of independent G-valued random variables $\{X_n\}$ which have a common distribution μ. A sequence $\{S_n\}$ of such products is referred to as a path of the μ-random walk on G.

In this paper we discuss three aspects of random walks. Chapter 1 is devoted to random walks on matrix groups $G = \mathrm{SL}_k(\mathbb{R})$. This theory is mainly concerned with Laws of Large Numbers and other limit theorems. The techniques involved include Markov processes, the dynamics of the projective G-action on flag varieties, elements of the structure theory of $\mathrm{SL}_k(\mathbb{R})$ as an algebraic group and some considerations with unitary representations. Although the presentation is restricted to $\mathrm{SL}_k(\mathbb{R})$, most of the results can be formulated and proved in the more general context of semisimple real Lie groups.

In chapter 2 random walks on general (locally compact or discrete) groups are discussed. The focus is on the connections between the properties of the group (such as nilpotency, amenaibility, growth etc.) and the behavior of the random walks on it. After a discussion of recurrence properties of random walks, we discuss bounded μ-harmonic functions, the concept of the Poisson boundary and related notions of boundary entropy and random walk entropy.

Chapter 3 is about random walks on groups of transformations of measure spaces and manifolds. The chapter starts with Random Ergodic Theorem and related results, and then turns to the random-walk-based notion of entropy. This notion is discussed in the context of diffeomorphisms of manifolds and in the general measurable setting. The discussion closes with a connection between entropy of random volume-preserving diffeomorphisms and the starting topic of the paper - random walks on matrix groups.

Obviously, the choice of the presented material was dictated by time, space and expertise limitations of the author as well as the personal taste. In particular, several important topics are not represented in this paper at all, most notably: random walks on graphs, Martin boundary, random perturbations in smooth dynamics and applications to stochastic differential equations. References to these topics in the present context should include: Woess [69], Guivarc'h, Ji, Taylor [25] and Kaimanovich [33], and works of Y. Kifer [41], [43].

The results in the paper are often stated not in the most general form known. For general statements and detailed proofs the reader is referred to the original papers, references to which are included. Obviously, any mistakes in the statements or proofs are in the full responsibility of the present author. Proofs of the stated results are included in the paper only if they are relatively short and exhibit important ideas. In many cases the proofs are just outlined with some details left out, however an effort was made to achieve a consistent and "relatively self contained" presentation. Included proofs appear either in the main text, or in the last sections 4, in which case the statements are marked by a star.

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1. Products of random matrices

1.1. General Overview. One of the most natural and important examples of transformation groups are linear transformations of finite dimensional real linear spaces, i.e. subgroups of $\mathrm{GL}_k(\mathbb{R})$. In this chapter we shall focus on the behavior of random products of such transformations, namely the products of random matrices. If $\{X_n, n \geq 1\}$ is a sequence in $\mathrm{GL}_k(\mathbb{R})$ consider the real numbers $x_n = k^{-1} \log |\det X_n|$ and the sequence $X'_n = e^{-x_n} X_n$ of $k \times k$-matrices, for which $\det X'_n = \pm 1$, and

$$X_n \cdots X_1 = e^{x_n + \cdots + x_1} X'_n \cdots X'_1$$

The study of sums of i.i.d. real variables $x_n + \cdots + x_1$ belongs to the classical probability theory, while the new aspects of the matrix setup arise from the absence of commutativity in matrix multiplication of the X'_n part. Therefore, it is customary to discuss random products of matrices in the group

$$\mathrm{SL}'_k(\mathbb{R}) = \{g \in \mathrm{GL}_k(\mathbb{R}) \mid \det g = \pm 1\}$$

or in $\mathrm{SL}_k(\mathbb{R})$ which is a subgroup of index two in $\mathrm{SL}'_k(\mathbb{R})$.

Let μ be a Borel probability measure on $G = \mathrm{SL}'_k(\mathbb{R})$, and let $S_n(\omega) = X_n(\omega) \cdots X_1(\omega)$ be a sequence of random products, where $\{X_n(\omega), n \geq 1\}$ are independent μ-distributed random variables. X_n can be viewed as coordinate projections of the product probability space

$$(\Omega, \mathsf{P}) = (G^{\mathbb{N}}, \mu^{\mathbb{N}})$$

related by $X_n = X_k \circ \theta^{n-k}$ for $n > k$, where $\theta : \Omega \to \Omega$ is the shift $(\theta\omega)_i = \omega_{i+1}$ on (Ω, P). Often we shall omit the point $\omega \in \Omega$ from the notation, and denote the random variables by just $X_1, X_2, \ldots, X_n, \ldots$ and $S_n = X_n \cdots X_1$.

We shall be interested in the qualitative and quantitative properties of the random products $S_n \in G$ and in their actions on the vector space $\mathbb{R}^k$, the projective space $\mathbb{P}^{k-1}$, Grassmannians and the flag varieties. In the context of products of random matrices one can study a non-commutative analogue of the *Law of Large Numbers* which describes the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1\|$$

where $\|\cdot\|$ is some norm on the matrix algebra $M_{n\times n}(\mathbb{R})$. Since any two such norms $\|\cdot\|$, $\|\cdot\|'$ are equivalent (in the sense that $\|\cdot\|/\|\cdot\|'$ is bounded from 0 and ∞) the existence and the value of the above limit does not depend on the choice of the norm. Hereafter it will be convenient to use the operator norm $\|A\| = \sup_{\|v\|=1} \|Av\|$ with respect to the l^2-norm $\|v\| = (v_1^2 + \cdots + v_k^2)^{1/2}$ on $\mathbb{R}^k$.

Laws of large numbers require some, typically L^1, boundness condition on the variables. We shall say that μ has *finite first moment* if

$$\int_G \log \|g\| \, d\mu(g) < \infty \tag{1.1}$$

Proposition 1.1. *Let μ on $\mathrm{SL}'_k(\mathbb{R})$ be a probability measure with finite first moment, and let $\{X_n, n \geq 1\}$ denote the sequence of independent μ-distributed random variables. Then with* P*-probability one the limit*

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1\|$$

exists, its value $\lambda_1(\mu)$ is P*-a.e. constant and can also be expressed as*

$$\lambda_1(\mu) = \lim_{n\to\infty} \frac{1}{n} \int_G \log \|g\| \, d\mu^n(g) = \inf_{n>0} \frac{1}{n} \int_G \log \|g\| \, d\mu^n(g)$$

where μ^n denotes the n-th convolution power of μ.

Proof. This fact was originally proved by Furstenberg and Kesten [18]. Now it can be easily deduced from the subadditive ergodic theorem of Kingman as follows: the functions $h_n(\omega) = \log \|S_n(\omega)\|$ on $(\Omega, \mathsf{P}, \theta)$ form a subadditive cocycle

$$\begin{aligned} h_{n+m}(\omega) &= \log \|S_{n+m}(\omega)\| = \log \|S_n(\theta^m\omega) S_m(\omega)\| \\ &\leq \log \|S_n(\theta^m\omega)\| + \log \|S_m(\omega)\| = h_n(\theta^m\omega) + h_m(\omega) \end{aligned}$$

and condition (1.1) is $h_1 \in L^1(\Omega, \mathsf{P})$. Since the the shift θ is ergodic on the space (Ω, P), there exists a P-a.e. constant limit $\lambda_1(\mu) = \lim_{n\to\infty} n^{-1} \cdot h_n(\omega)$ which can also be expressed as

$$\lambda_1(\mu) = \lim_{n\to\infty} \frac{1}{n} \int h_n \, d\mathsf{P} = \inf_n \frac{1}{n} \int h_n \, d\mathsf{P}$$

Since all matrices in $G = \mathrm{SL}'_k(\mathbb{R})$ have norm of at least one, $h_n \geq 0$ and therefore $\lambda_1 \geq 0$. □

The number $\lambda_1(\mu)$ is the *top Lyapunov exponent* of the random matrices. It is the first in a sequence

$$\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_k(\mu)$$

of k constants referred to as the *Lyapunov exponents* or the *Lyapunov spectrum* of random matrices with the law μ. The exponents $\lambda_p(\mu)$, $p > 1$, are defined inductively, via the top Lyapunov exponents of the exterior powers $\wedge^p S_n = \wedge^p X_n \cdots \wedge^p X_1$ by the relations

$$\lambda_1(\mu) + \cdots + \lambda_p(\mu) = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^p S_n\| \qquad p = 1, 2, \ldots, k.$$

For any $A \in \mathrm{SL}'_k(\mathbb{R})$ one has $\wedge^k A = \det A = \pm 1$ and consequently $\lambda_1 + \cdots + \lambda_k = 0$.

This formal definition of λ_p, $1 \leq p \leq k$ has a more transparent interpretation. Recall that any matrix $M \in \mathrm{SL}'_k(\mathbb{R})$ can be written in the *polar* form $M = VDU$, where $D = \mathrm{diag}\,[e^{a_1}, \ldots, e^{a_k}]$ denotes the diagonal matrix with $a_1 \geq a_2 \geq \cdots \geq a_k$ and U, V are orthogonal matrices. Due to our choice of the norm one has

$$\|M\| = e^{a_1}, \qquad \| \wedge^p M\| = e^{a_1 + \cdots + a_p} \qquad \text{for} \qquad 1 \leq p \leq k. \tag{1.2}$$

which in particular shows that $a_1 \geq \cdots \geq a_k$ are uniquely determined by M. Proposition 1.1 asserts that writing the random products S_n in the polar form

$$S_n = X_n \cdots X_1 = V_n \operatorname{diag}\left[e^{a_1(n)}, \ldots, e^{a_k(n)}\right] U_n \tag{1.3}$$

with $a_1(n) \geq \cdots \geq a_k(n)$ and $U_n, V_n \in \mathrm{O}(k)$, one has with P-probability one: $\lim_{n\to\infty} a_1(n)/n = \lambda_1$, and (1.2) implies that $\lim_{n\to\infty} a_p(n)/n = \lambda_p$ for all $p = 1, \ldots, k$.

This argument constitutes a part of the Oseledec theorem, which applies to general (ergodic) *stationary* processes $\{X_n, n \geq 1\}$ in G. In what follows we shall specialize to the case of *independent* random variables, obtaining much more detailed information.

1.2. **Preliminaries on Markov processes.** Consider the following general setup: let M be a compact metric space and denote by $\mathcal{P}(M)$ the space of all probability measures on M, which is a convex *compact* metric space in the weak-* topology induced by $\mathcal{C}(M)$. Given a continuous map $M \to \mathcal{P}(M)$, $x \mapsto \mu_x$, one can define a *Markov operator* P acting on $\mathcal{C}(M)$ by

$$Pf(x) = \int_M f(y)\, d\mu_x(y)$$

The measures $\mu_x, x \in X$, are called *transition probabilities* of P. The dual operator P^* acts on the space of measures on M preserving the convex compact set $\mathcal{P}(M)$, and the set of P^*-fixed measures in $\mathcal{P}(M)$ is a convex compact subset. By a standard fixed point theorem it is *non-empty* - indeed for any $\eta_0 \in \mathcal{P}(M)$ any accumulation point of $n^{-1} \cdot \sum_1^n (P^*)^j \eta_0$ is fixed by P^*. A probability measure $\eta \in \mathcal{P}(M)$ with $P^*\eta = \eta$ is called *P-stationary.*

A Markov operator P as above and an arbitrary measure $\theta \in \mathcal{P}(M)$ gives rise to the corresponding Markov process $\{Z_n, n \geq 0\}$ on M, where Z_0 is taken with distribution θ on M, and the conditional distribution $E(Z_{n+1} \mid Z_n, \ldots, Z_0)$ is μ_{Z_n}. Such a Markov process $\{Z_n, n \geq 0\}$ is *stationary* if and only if the initial distribution θ of Z_0 is P-stationary. In this general context we shall use the following fact:

Theorem 1.2 (Furstenberg [13], see also [19])**.** *Let P be a Markov operator on a compact metric space M and σ be a continuous function on M. Take an arbitrary (not necessarily P-stationary) measure $\theta \in \mathcal{P}(M)$, and let $\{Z_n, n \geq 0\}$ denote the corresponding Markov process on M. Then with probability one*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sigma(Z_k) \leq \sup\left\{ \int_M \sigma\, d\eta \;\middle|\; \eta \in \mathcal{P}(M) \text{ s.t. } P^*\eta = \eta \right\}$$

If, moreover, $\int \sigma\, d\eta$ takes the same value α for all P-stationary probability measures η on M, then with probability one $n^{-1} \sum_{k=1}^{n} \sigma(Z_k) \to \alpha$.

Next consider the situation where a locally compact group (or a semigroup) G acts on a compact metric space M so that the action map $G \times M \to M$, $(g, x) \mapsto g \cdot x$, is continuous. Hereafter such M with the G-action is called a *G-space.* Given a probability measure μ on G and a probability measure ν on M one can define the *convolution* probability measure $\mu * \nu$ on M as the image of $\mu \times \nu$ under the action map $G \times M \to M$. In other words

$$\int_M f\, d(\mu * \nu) = \int_G \int_M f(g \cdot x)\, d\nu(x)\, d\mu(g) \qquad \text{for} \qquad f \in \mathcal{C}(M)$$

In particular, taking Dirac measures $\nu = \delta_x$, $x \in M$, one obtains a continuous assignment $x \mapsto \mu_x = \mu * \delta_x$, which defines a Markov operator P on $\mathcal{C}(M)$ by

$$(Pf)(x) = \int_G f(g \cdot x)\, d\mu(g) \qquad \text{for} \qquad f \in \mathcal{C}(M),\ x \in M$$

Probability measures $\nu \in \mathcal{P}(M)$ satisfying $\mu * \nu = \nu$ (equivalently $P^*\nu = \nu$) are called *μ-stationary* measures. As before, one notes that the set of all μ-stationary measures in $\mathcal{P}(M)$ is non-empty convex compact set, and it is easy to see that each μ-stationary measure ν gives rise to a stationary M-valued Markov process $\{U_n^\nu, n \geq 0\}$. It is a general fact about Markov processes that $\{U_n^\nu, n \geq 0\}$ fails to be *ergodic* iff there exists a P-invariant measurable set $E \subset M$ with $0 < \nu(E) < 1$ (see Theorem 3.1 below). In the present context of a compact G-space M one can check that a μ-stationary measure ν admits P-invariant measurable $E \subset M$ with $0 < \nu(E) < 1$ iff ν is not an extremal point of the set of all μ-stationary measures. This discussion can be summarized in terms of ergodic theory as follows:

Proposition 1.3. *Let G be a locally compact group, μ a probability measure on G, M a compact G-space, and ν - a probability measure on M. Consider the one-sided Bernoulli shift $(\Omega, \mathsf{P}, \theta)$ where $(\Omega, \mathsf{P}) = (G^{\mathbb{N}}, \mu^{\mathbb{N}})$, $(\theta\omega)_i = \omega_{i+1}$, $i \in \mathbb{N}$, and the transformation T of the product space $\Omega \times M$ defined by $T(\omega, x) = (\theta\omega, \omega_1 \cdot x)$. Then*

(a) *The product measure $\mathsf{P} \times \nu$ is T-invariant if and only if ν is a μ-*stationary *measure.*

(b) *The product measure $\mathsf{P} \times \nu$ is an ergodic T-invariant measure iff ν is an extremal point of the compact convex set of all μ-stationary measures on M.*

Given a probability measure μ on G and a G-space M, one can also consider the product space $\tilde{M} = G \times M$ with the Markov operator $\tilde{P}$ defined by

$$\tilde{P} f(g, x) = \int_G f(g', g' \cdot x)\, d\mu(g') \tag{1.4}$$

The corresponding $\tilde{M}$-valued Markov process Z_n^ν can be described as follows: let U_0 be a M-valued random variable with distribution ν, $\{X_n,\ n \geq 0\}$ a sequence of G-valued random variables with distribution μ which are independent from each other and from U_0; and let

$$Z_n^\nu = (X_n, U_n) \qquad \text{where} \qquad U_n = X_n \cdots X_1 \cdot U_0 = S_n \cdot U_0 \tag{1.5}$$

Notice that this construction does not quite fit into the framework of Markov processes on *compact* spaces if G is not compact. If μ has a compact support $S \subset G$, then one may simply replace the non-compact space $G \times M$ by the compact one

$\tilde{M} = S \times M$. In general, one can consider the compact space $\tilde{M} = \hat{G} \times M$ where $\hat{G} = G \cup \{*\}$ is a one point compactification of G, taking special care while applying Theorem 1.2 to functions σ on G which do not extend continuously to $\hat{G}$.

One can easily check that for any μ-stationary measure $\nu \in \mathcal{P}(M)$, the measure $\mu \times \nu$ on $\tilde{M}$ is $\tilde{P}$-stationary, and so is the Markov process $\{Z_n^\nu, n \geq 0\}$ given by (1.5). Moreover

Lemma 1.4 (cf. [14]). *The set of $\tilde{P}$-stationary probability measures on $\tilde{M}$ is*

$$\{\mu \times \nu \mid \nu \in \mathcal{P}(M) \text{ s.t. } \mu * \nu = \nu\}$$

Hence every stationary Markov processes on $\tilde{M}$ has the form (1.5). Moreover, the stationary Markov processes $\{Z_n^\nu, n \geq 0\}$ is ergodic if and only if the μ-stationary measure ν is an extremal point of the convex compact consisting of all μ-stationary probability measures on M.

1.3. **A formula for the top Lyapunov exponent.** Consider the natural projective action of $G = \mathrm{SL'}_k(\mathbb{R})$ on the projective space $M = \mathbb{P}^{k-1}$, take $\tilde{M} = G \times M$ and let $\sigma_1 : G \times \mathbb{P}^{k-1} \to \mathbb{R}$ be defined by

$$\sigma_1(g, \bar{u}) = \log \frac{\|gu\|}{\|u\|} \tag{1.6}$$

where $u \in \mathbb{R}^k$ is a non-zero vector in the line $\bar{u}$. Note that σ_1 is a cocycle with respect to the G-action on $\mathbb{P}^{k-1}$, i.e. $\sigma_1(g'g, \bar{u}) = \sigma_1(g', g \cdot \bar{u}) + \sigma_1(g, \bar{u})$, and for any non-zero vector $u \in \mathbb{R}^k$ and any $g_0, g_1, \dots, g_n \in G$ one has

$$\log \|g_n \cdots g_0 u\| - \log \|u\| = \sigma_1(g_n, \bar{u}_n) + \cdots + \sigma_1(g_0, \bar{u}_0) \tag{1.7}$$

where $\bar{u}_0$ is the line spanned by u and $\bar{u}_{j+1} = g_j \cdot \bar{u}_j$, $j = 0, \dots, n-1$.

Let μ be a probability measure on G, and let us assume for the moment that μ has compact support $S \subset G$. Taking $\tilde{M} = S \times \mathbb{P}^{k-1}$ one obtains a continuous function $\sigma_1 : \tilde{M} = S \times \mathbb{P}^{k-1} \to \mathbb{R}$ on a compact space, so that Theorem 1.2 applies to any Markov process $Z_n^\nu = (X_n, U_n) = (X_n, X_n \cdots X_1 U_0)$ where the distribution ν of U_0 is arbitrary (not necessarily μ-stationary), while $\{X_n,\ n \geq 0\}$, are independent μ-distributed random variables. In particular, taking $\nu = \delta_{\bar{u}}$ where u is any non-zero vector $u \in \mathbb{R}^k$, one has with P-probability one

$$\limsup_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1 u\| = \sup\left\{\alpha_{\mu,\nu} \mid \ \nu \in \mathcal{P}(\mathbb{P}^{k-1}) \text{ s.t. } \mu * \nu = \nu\right\} \tag{1.8}$$

where $\alpha_{\mu,\nu}$ denotes the integral

$$\alpha_{\mu,\nu} = \int_G \int_{\mathbb{P}^{k-1}} \sigma_1 \, d\mu \times \nu = \int_G \int_{\mathbb{P}^{k-1}} \log \frac{\|gu\|}{\|u\|} \, d\nu(\bar{u}) \, d\mu(g) \tag{1.9}$$

The equality (1.8) holds not only for compactly supported μ, but for any μ with finite first moment. To show this, one needs to consider the space $\tilde{M} = \hat{G} \times \mathbb{P}^{k-1}$ and a sequence of *continuous* functions $\sigma_1^{(T)} : \tilde{M} \to \mathbb{R}$, where $\sigma_1^{(T)}$ are certain truncations of σ_1. We refer to [19] for the details of this argument. Let us also point out that often there is only one μ-stationary measure ν on $\mathbb{P}^{k-1}$ (see Theorem 1.34).

Remark 1.5. The definition of σ_1 involved the choice of a norm $\|\cdot\|$ on $\mathbb{R}^k$, which has no influence on the limit in the left hand side of (1.9). In order to see that the right hand side is not sensitive to such a choice either, consider a cocycle σ_1' and $\alpha'_{\mu,\nu}$ which are defined similarly to σ_1 and $\alpha_{\mu,\nu}$, with $\|\cdot\|$ been replaced by another norm $\|\cdot\|'$. Then the cocycles σ_1 and σ_1' are *cohomologous* in the sense that

$$\sigma_1'(g,\bar{u}) - \sigma_1(g,\bar{u}) = \phi(g\cdot\bar{u}) - \phi(\bar{u})$$

where $\phi(\bar{u}) = \log(\|u\|'/\|u\|)$ is a bounded function of $\bar{u}$. Hence

$$\begin{aligned}\alpha'_{\mu,\nu} - \alpha_{\mu,\nu} &= \int_G\int_{\mathbb{P}^{k-1}} \phi(g\cdot\bar{u})\,d\mu(g)\,d\nu(\bar{u}) - \int_{\mathbb{P}^{k-1}} \phi(\bar{u})\,d\nu(\bar{u}) \\ &= \int_{\mathbb{P}^{k-1}} \phi\,d(\mu*\nu) - \int_{\mathbb{P}^{k-1}} \phi\,d\nu = 0\end{aligned}$$

Definitions 1.6. Hereafter the following definitions are frequently used

(a) Given a probability measure μ on $G = \mathrm{SL}'_k(\mathbb{R})$, or more generally on an arbitrary locally compact second countable group, denote by $\mathsf{grp}(\mu)$ (resp. $\mathsf{sgr}(\mu)$) the smallest closed subgroup (resp. semi-group) of G with full μ measure. Equivalently, $\mathsf{grp}(\mu)$ (resp. $\mathsf{sgr}(\mu)$) is the closed subgroup (resp. semi-group) generated by $\mathrm{supp}(\mu)$. If $G = \mathsf{grp}(\mu)$ we shall say that μ is *generating*.

(b) A closed (semi)group T of $G = \mathrm{SL}'_k(\mathbb{R})$ is said to be *strongly irreducible*, if there does not exist a finite union $W = V_1 \cup \cdots \cup V_r$ of proper linear subspaces of $\mathbb{R}^k$ with $TW \subseteq W$. If T is a group strong irreducibility is equivalent to the condition that all finite index subgroups of T act irreducibly on $\mathbb{R}^k$.

(c) The notion of strong irreducibility of a (semi)group $T \subseteq G = \mathrm{SL}'_k(\mathbb{R})$ extends in an obvious way to any linear representation of G. For $p \in \{1,\ldots,k\}$ we shall say that a (semi)group T is *strongly p-irreducible* if it is strongly irreducible in the natural G-action on the exterior power $\wedge^p\mathbb{R}^k$. A (semi)group T which is strongly p-irreducible for all $p = 1,\ldots,k$, will be called *totally irreducible.*

(d) A probability measure μ on $G = \mathrm{SL}'_k(\mathbb{R})$ is called *strongly irreducible/strongly p-irreducible/ totally irreducible* if $G_\mu = \mathsf{grp}(\mu)$ (equivalently $T_\mu = \mathsf{sgr}(\mu)$) has the corresponding property.

(e) A probability measure ν on the projective space $\mathbb{P}^{k-1}$ is called *proper*, if $\nu(\bar{L}) = 0$ for any proper linear subspace $L \subset \mathbb{R}^k$.

The key property of strongly irreducible measures is the following

Lemma (*) 1.7 (Furstenberg, [13]). *A probability measure μ on $G = \mathrm{SL}'_k(\mathbb{R})$ is strongly irreducible if and only if all μ-stationary measures ν on $\mathbb{P}^{k-1}$ are proper.*

Let μ be a fixed strongly irreducible probability measure μ on $G = \mathrm{SL'}_k(\mathbb{R})$ with finite first moment. Choose an extremal μ-stationary measure ν on $\mathbb{P}^{k-1}$. Since such ν is necessarily proper (Lemma 1.7), taking k unit vectors $u_1, \ldots, u_k$ with the directions $\bar{u}_1, \ldots, \bar{u}_k \in \mathbb{P}^{k-1}$ being chosen independently with distribution ν, one obtains with probability one a *basis* for $\mathbb{R}^k$. For any such choice $u_1, \ldots, u_k$ there exists a constant $C = C(u_1, \ldots, u_k)$ such that for any matrix $X \in \mathrm{SL'}_k(\mathbb{R})$ the norm $\|X\|$ can be estimated by

$$\max_{1\le i\le k} \|Xu_i\| \le \|X\| \le C \cdot \max_{1\le i\le k} \|Xu_i\|$$

Since ν was chosen to be extremal, the process Z_n^ν is ergodic, so that for a.e. choice of $\bar{u}_i$, $i = 1, \ldots, k$, Birkhoff's Ergodic Theorem gives

$$\frac{1}{n}\log \|X_n \cdots X_1 u_i\| = \frac{1}{n}\sum_1^n \sigma(Z_j^\nu) \to \int_G \int_{\mathbb{P}^{k-1}} \sigma_1 \, d\mu \, d\nu = \alpha_{\mu,\nu}$$

while

$$\frac{1}{n}\log \|X_n \cdots X_1 u_i\| \le \frac{1}{n}\log \|X_n \cdots X_1\| \le \max_{1\le i\le m} \frac{1}{n}\log \|X_n \cdots X_1 u_i\| + \frac{\log C}{n}$$

Therefore, for strongly irreducible μ with finite first moment one has $\alpha_{\mu,\nu} = \lambda_1(\mu)$ for all *extremal* μ-stationary measures ν. Since $\alpha_{\mu,\nu}$ is affine in ν (as well as in μ), it takes the same value $\lambda_1(\mu)$ for all μ-stationary measures ν - which form a convex closure of the extremal (i.e. ergodic) ones. Applying the second part of Theorem 1.2, (and taking special care of non-compactness of G, by considering appropriately truncated functions $\pm\sigma^{(T)}$), one deduces the following result:

Theorem 1.8 (Furstenberg, [13]). *Let μ on $G = \mathrm{SL'}_k(\mathbb{R})$ be a strongly irreducible probability measure with finite first moment. Then*

$$\lambda_1(\mu) = \alpha_{\mu,\nu} = \int_G \int_{\mathbb{P}^{k-1}} \log \frac{\|gu\|}{\|u\|} \, d\nu(\bar{u}) \, d\mu(g)$$

for any probability measure $\nu \in \mathcal{P}(\mathbb{P}^{k-1})$ satisfying $\mu * \nu = \nu$. Moreover, for every non-zero vector $u \in \mathbb{R}^k$ with P-probability one the random products $X_n \cdots X_1$ satisfy

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1 u\| = \lim_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1\| = \lambda_1(\mu)$$

Remark 1.9. Note that the statement about vector growth $n^{-1} \cdot \log \|S_n u\| \to \lambda_1(\mu)$ in the theorem differs, in general, from a similar assertion in Oseledec theorem. For example, if μ is strongly irreducible with $\lambda_2(\mu) < \lambda_1(\mu)$, which as we shall see is a typical situation, Oseledec theorem states that there is a measurable family of *random* (i.e. depending on ω) codimension one subspaces $E_2(\omega)$ such that for P-a.e. ω

$$\forall\ u \in \mathbb{R}^k \setminus E_2(\omega) : \qquad \lim_{n\to\infty} n^{-1} \cdot \log \|S_n(\omega)u\| = \lambda_1(\mu)$$

$$\forall\ u \in E_2(\omega) \setminus \{0\} : \qquad \lim_{n\to\infty} n^{-1} \cdot \log \|S_n(\omega)u\| \le \lambda_2(\mu) < \lambda_1(\omega)$$

Yet Theorem 1.8 asserts that for *every fixed* no-zero vector u, for P-a.e. ω one has

$$\lim_{n\to\infty} n^{-1}\cdot\log\|S_n(\omega)u\| \to \lambda_1(\mu)$$

This just means that for every non-zero vector u the event $\{\omega \mid u \in E_2(\omega)\}$ has P-probability zero.

The formula of Theorem 1.8 for $\lambda_1(\mu)$ has many important applications, but in general it does not allow explicit computations of $\lambda_1(\mu)$ in terms of μ, for it involves auxiliary μ-stationary measures ν on the boundary, and these are generally hard to identify. However, an explicit computation of $\lambda_1(\mu)$ is available for some special family of measures μ on G:

Theorem 1.10 (Furstenberg, [13] 7.3)**.** *Let μ be a probability measure on $G = \mathrm{SL}_k(\mathbb{R})$ of the form $\mu = \mu_1 * m_K * \mu_2$, where m_K denotes the Haar measure on the maximal compact $K = \mathrm{SO}(k)$. Then μ has a unique stationary measure $\nu = \mu_1 * \nu_0$ where ν_0 is the unique K-invariant probability measure on $\mathbb{P}^{k-1}$, and if μ has finite first moment then*

$$\lambda_1(\mu) = \int_K\int_K\int_G\int_G\int_G \log\frac{\|g_1 k g_2 g_1' k' u\|}{\|g_1' k' u\|}\, d\mu_1(g_1)\, d\mu_1(g_1')\, d\mu_2(g_2)\, dk\, dk' \tag{1.10}$$

for any non zero vector $u \in \mathbb{R}^k$. For bi-K-invariant measures $\mu = m_K * \mu_0 * m_K$ with finite first moment (1.10) reduces to

$$\lambda_1(\mu) = \int_G\int_{\mathbb{P}^{k-1}} \log\|gu\|\, d\nu_0(u)\, d\mu_0(g)$$

Proof. Observe that for any probability measure η on $\mathbb{P}^{k-1}$ one has $m_K * \eta = \nu_0$, and therefore $(\mu_1 * m_K * \mu_2) * \eta = \mu_1 * \nu_0$. Thus $\mu_1 * \nu_0$ is the unique μ-stationary measure for $\mu = \mu_1 * m_K * \mu_2$, and (1.10) follows from Theorem 1.8. If μ is bi-K-invariant, then ν_0 is the unique μ-stationary measure and the second formula follows. □

1.4. **Non-random filtration associated to μ.** The probabilistic approach based on Markov processes on $G \times \mathbb{P}^{k-1}$, briefly described above, allows further analysis of vector growth under products of random matrices. In particular, *strong irreducibility* condition in Theorem 1.8 can be relaxed to just *irreducibility*, and furthermore, in the general case (of possibly reducible $\mathsf{grp}(\mu)$) one has the following:

Theorem 1.11 (Furstenberg-Kifer [19])**.** *Let μ on $G = \mathrm{SL}'_k(\mathbb{R})$ be a probability measure with finite first moment, and let $G_\mu = \mathsf{grp}(\mu)$. Then there exists an integer r with $1 \le r \le k$, a sequence of G_μ-invariant subspaces*

$$\{0\} = L_{r+1} \subset L_r \subset \cdots \subset L_2 \subset L_1 = \mathbb{R}^k$$

and a sequence of real numbers $\tilde{\lambda}_1(\mu) > \tilde{\lambda}_2(\mu) > \cdots > \tilde{\lambda}_r(\mu)$, with $\tilde{\lambda}_1(\mu) = \lambda_1(\mu)$, such that for every vector $u \in L_i \setminus L_{i+1}$ with P*-probability one*

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1 u\| = \tilde{\lambda}_i(\mu)$$

Moreover the set $\tilde{\Lambda} = \{\tilde{\lambda}_i \mid 1 \le i \le r\}$ coincides with the set of values of of $\alpha_{\mu,\nu}$, where ν ranges over all extremal μ-stationary probability measures on $\mathbb{P}^{k-1}$. For an extremal μ-stationary measure ν, $\alpha_{\mu,\nu} = \tilde{\lambda}_i(\mu)$ if and only if $\nu(\overline{L}_i \setminus \overline{L}_{i+1}) = 1$.

Corollary 1.12. *There exists a μ-stationary measure $\nu \in \mathcal{P}(\mathbb{P}^{k-1})$ with $\nu(\overline{L}_2) = 0$. For any such measure ν, one has $\alpha_{\mu,\nu} = \tilde{\lambda}_1(\mu) = \lambda_1(\mu)$. If G_μ is irreducible then $\alpha_{\mu,\nu} = \lambda_1(\mu)$ for all μ-stationary ν.*

Remark 1.13. Note that the theorem describes a *non-random* filtration $L_r \subset \cdots \subset L_2 \subset L_1 = \mathbb{R}^k$ which differs, in general, from the random (i.e. depending on $\omega \in \Omega$) filtration provided by the Oseledec theorem. The set of the corresponding exponents $\tilde{\Lambda} = \{\tilde{\lambda}_i(\mu) \mid 1 \le i \le r\}$ is contained in the Lyapunov spectrum $\Lambda = \{\lambda_j(\mu) \mid 1 \le j \le k\}$, but is, typically, smaller. In particular if G_μ is irreducible then the filtration is trivial: $\{0\} = L_2 \subset L_1 = \mathbb{R}^k$ and $\tilde{\Lambda} = \{\tilde{\lambda}_1(\mu) = \lambda_1(\mu)\}$.

The key ingredients of the proof of Theorem 1.11 are the following observations:

(i) For a fixed non-zero vector $v \in \mathbb{R}^k$ the upper limit

$$a(v) = \limsup_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1 v\|$$

is P-a.e. constant. This is basically a $0-1$ law.

(ii) For any real a the collection of vectors $E(a) = \{v \in \mathbb{R}^k \mid v = 0 \text{ or } a(v) \le a\}$ is a *linear subspace* of $\mathbb{R}^k$, which is invariant under μ-a.e. g. Hence $E(a)$ is a G_μ-invariant subspace.

(iii) Inspecting the discrete drops in $\dim E(a)$ as a varies from λ_1 to $-\infty$, one recognizes the exponents $\tilde{\lambda}_i$ and the spaces L_i.

1.5. **Furstenberg's condition for positive growth.** The mere existence of the top Lyapunov exponent $\lambda_1(\mu) \ge 0$, or even the formulae in Theorems 1.8 and 1.11, do not give a clear indication whether the growth of the random products $S_n = X_n \cdots X_1$ is actually exponential ($\lambda_1(\mu) > 0$) or sub-exponential ($\lambda_1(\mu) = 0$). If μ happens to be supported by a compact subgroup of $G = \mathrm{SL}'_k(\mathbb{R})$, then the random products $X_n \cdots X_1$ are bounded and clearly $\lambda_1(\mu) = 0$. In his fundamental work [13] Furstenberg proved that for strongly irreducible μ, this obvious obstacle to exponential growth is the only one. More precisely

Theorem 1.14 (Furstenberg, [13])**.** *Let μ on $G = \mathrm{SL}'_k(\mathbb{R})$ be a probability measure with finite first moment. Then $\lambda_1(\mu) > 0$ unless, $G_\mu = \mathsf{grp}(\mu)$ is not strongly irreducible or compact.*

We postpone the proof of Theorem 1.14 to the next section.

Remark 1.15. A remarkable feature of Furstenberg's condition for $\lambda_1(\mu) > 0$ is that it is not given in terms of μ itself, but rather in terms of the closed subgroup $\mathsf{grp}(\mu)$ generated by μ. In fact, since both (strong) irreducibility and compactness are *algebraic* properties (i.e. can be described by polynomial equations in the entries of the matrices) Furstenberg's condition can be formulated in terms of the Zariski closure $H_\mu = \overline{\mathsf{grp}(\mu)}^Z$ of $\mathsf{grp}(\mu)$ (compare this with Theorem 1.25 below).

To illustrate the (strong) irreducibility condition, consider a measure μ with finite first moment on $G = \mathrm{SL}_k(\mathbb{R})$ such that G_μ is reducible. Then G_μ is conjugate into

$$\begin{pmatrix} A'_{11} & A'_{12} \\ 0 & A'_{22} \end{pmatrix}, \qquad A'_{11} \in \mathrm{GL}_{k_1}(\mathbb{R}),\ A'_{22} \in \mathrm{GL}_{k_2}(\mathbb{R}),\ A'_{12} \in M_{k_1 \times k_2}(\mathbb{R})$$

where $k = k_1 + k_2$ and $0 < k_1, k_2 < k$. Let μ'_i denote the distribution of the A'_{ii} part on $\mathrm{GL}_{k_i}(\mathbb{R})$. It is not hard to show (cf. [59] or [19]) that in this case

$$\lambda_1(\mu) = \max\{\lambda_1(\mu'_1),\ \lambda_1(\mu'_2)\}$$

that is to say that the growth of the A'_{12}-part is dominated by the maximum of the growths of the A'_{11} and A'_{22} parts. Separate the scalar parts $a_i = |\det A'_{ii}|$ from the truly non-commutative components

$$A_{ii} = a_i^{-1/k_i} \cdot A'_{ii} \in \mathrm{SL'}_{k_i}(\mathbb{R})$$

and let μ_i denote the distribution of A_{ii} on $\mathrm{SL'}_{k_i}(\mathbb{R})$, $i = 1, 2$. Since $\log a_1 + \log a_2 = 0$ one obtains

$$\lambda_1(\mu) = \left| \int \log a_1 \, d\mu'_1 \right| + \max\{\lambda_1(\mu_1),\ \lambda_1(\mu_2)\}$$

If G_μ is contained in (a conjugate of) the upper triangular group P then applying the above argument inductively one concludes that

$$\lambda_1(\mu) = \max_{1 \le i \le k} \left| \int \log |g_{ii}| \, d\mu(g) \right|$$

where g_{ij} denote the ij-entry of a matrix $g \in \mathrm{SL}_k(\mathbb{R})$. Observe that the map $g \mapsto (\log|g_{11}|, \ldots, \log|g_{kk}|)$ is a *homomorphism*

$$\rho : P \to \mathbb{R}^k_0 = \{x \in \mathbb{R}^k \mid x_1 + \cdots + x_k = 0\}$$

and $\lambda_1(\mu)$ is the $\|\cdot\|_\infty$ norm of the barycenter of the push-forward measure $\rho_*\mu$ on $\mathbb{R}^k_0$.

1.6. **Unitary representation approach I.** Positivity of the top Lyapunov exponent $\lambda_1(\mu) > 0$ describes the exponential growth of products of μ-distributed independent random matrices. In the projective action on $\mathbb{P}^{k-1}$ such growth corresponds to certain (exponential) contraction, which can further be related to a "spectral gap" of a Markov operator acting on certain space of functions on $\mathbb{P}^{k-1}$. This idea was made precise by Virtzer, who deduced Theorem 1.8 from a spectral gap in the quasi-regular unitary representation on $L^2(\mathbb{P}^{k-1})$. In fact, the result of Virtzer [67] is more general, in the sense that it gives a sufficient condition for $\lambda_1(\mu) > 0$ not only for products of *independent* μ-distributed random matrices, but also for products of random matrices satisfying certain condition on their correlations. Here we shall discuss the independent case only:

Theorem 1.16 (Virtzer [67])**.** *Let μ be a probability measure on $G = \mathrm{SL}'_k(\mathbb{R})$, such that $G_\mu = \mathsf{grp}(\mu)$ does not have an invariant probability measure on $\mathbb{P}^{k-1}$. Then there exists a positive constant $\gamma(\mu)$, so that with P-probability one*

$$\liminf_{n\to\infty} \frac{1}{n} \log \|X_n \cdots X_1\| \geq \gamma(\mu) > 0 \tag{1.11}$$

In particular, if μ has a finite first moment then $\lambda_1(\mu) \geq \gamma(\mu) > 0$.

Proof. Denote by ν_0 the $\mathrm{SO}(k)$-invariant probability measure on $\mathbb{P}^{k-1}$. One can verify by a direct computation that for $g \in G = \mathrm{SL}'_k(\mathbb{R})$ and every non-zero $u \in \mathbb{R}^k$

$$\frac{dg^{-1}\nu_0}{d\nu_0}(\bar{u}) = \left(\frac{\|gu\|}{\|u\|}\right)^{-k} \tag{1.12}$$

This implies the following estimate

$$\|g\| = \max_{\bar{u}} \frac{\|gu\|}{\|u\|} = \left(\min_{\bar{u}} \sqrt{\frac{dg^{-1}\nu_0}{d\nu_0}(\bar{u})}\right)^{-2/k} \geq \left(\int_{\mathbb{P}^{k-1}} \sqrt{\frac{dg^{-1}\nu_0}{d\nu_0}(\bar{u})}\, d\nu_0(\bar{u})\right)^{-2/k} \tag{1.13}$$

We shall also make use of the following general facts.

Proposition (*) 1.17. *Let H be a locally compact group, μ a generating probability measure on H and let X be a compact metric space with a continuous H-action on it, which has no invariant probability measures. Then there exists a positive constant $\epsilon = \epsilon(X, H, \mu) > 0$, so that for any quasi-invariant probability measure ν on X, the average operator $\pi_\nu(\mu) = \int \pi_\nu(h)\, d\mu(h)$ on $L^2(X, \nu)$ corresponding to the quasi-regular unitary H-representation π_ν*

$$\pi_\nu(h) f(x) = \sqrt{\frac{dh\nu}{d\nu}(x)} \cdot f(h^{-1} \cdot x) \qquad (f \in L^2(X, \nu))$$

has a spectral gap $\|\pi_\nu(\mu)\|_{\mathrm{sp}} \leq 1 - \epsilon$, where $\|T\|_{\mathrm{sp}} = \lim \|T^n\|^{1/n}$ denotes the spectral radius of a bounded operator T.

Lemma (*) 1.18 ([11]). *Let π be a unitary representation of some group H on a Hilbert space $\mathcal{H}$, μ be a probability measure on H, and $\{X_n, n \geq 1\}$ be a sequence of independent μ-distributed H-valued random variables. Assume that $f_1, f_2 \in \mathcal{H}$ are two vectors with the property that $\langle \pi(h) f_1,\, f_2 \rangle \geq 0$ for every $h \in H$. Then with* P*-probability one*

$$\liminf_{n\to\infty} -\frac{1}{n} \log \langle \pi(S_n(\omega)) f_1,\, f_2 \rangle \geq \log \frac{1}{\|\pi(\mu)\|_{\mathrm{sp}}} \geq 0$$

Now consider the projective action of $G_\mu = \mathsf{grp}(\mu)$ on $\mathbb{P}^{k-1}$. Since G_μ has no invariant measures, by Proposition 1.17, the quasi-regular G_μ-representation π_{ν_0} on $L^2(\mathbb{P}^{k-1}, \nu_0)$ has a spectral gap $\|\pi_{\nu_0}(\mu)\|_{\mathrm{sp}} < 1$. Using the estimate (1.13) and Lemma 1.18, applied to the constant functions $f_1 = f_2 = \mathbf{1} \in L^2(\mathbb{P}^{k-1}, \nu_0)$, one has for P-a.e. $\omega \in \Omega$ the sequence $S_n = S_n(\omega)$ satisfies

$$\begin{aligned}
\liminf\nolimits_{n\to\infty} \quad \frac{1}{n} \log \|S_n\| &\geq \liminf_{n\to\infty} -\frac{2}{nk} \log \int_{\mathbb{P}^{k-1}} \sqrt{\frac{dS_n^{-1}\nu_0}{d\nu_0}(\bar{u})}\, d\nu_0(\bar{u}) \\
= \qquad & \liminf_{n\to\infty} -\frac{2}{nk} \log \langle \mathbf{1},\, \pi_{\nu_0}(S_n^{-1}) \mathbf{1} \rangle = \liminf_{n\to\infty} -\frac{2}{nk} \log \langle \pi_{\nu_0}(S_n) \mathbf{1},\, \mathbf{1} \rangle \\
\geq \qquad & \frac{2}{k} \log \frac{1}{\|\pi_{\nu_0}(\mu)\|_{\mathrm{sp}}} > 0
\end{aligned}$$

□

Observe that Theorem 1.14 is a corollary of Theorem 1.16: if $G_\mu = \mathsf{grp}(\mu)$ is strongly irreducible and there exists a G_μ-invariant probability measure ν on $\mathbb{P}^{k-1}$, then ν is proper (Lemma 1.7), while the stabilizers of proper measures in $\mathrm{PSL}_k(\mathbb{R})$-action on $\mathbb{P}^{k-1}$ are *compact* (see 1.27 below).

1.7. **Unitary representation approach II.** In Theorem 1.16 the positivity of the top Lyapunov exponent $\lambda_1(\mu) > 0$ was deduced from a spectral gap in the *quasi-regular* representation of G_μ, which followed from the absence of G_μ-invariant measures on $\mathbb{P}^{k-1}$. The latter is a manifestation of non-amenability of G_μ. It is natural therefore to try to deduce $\lambda_1(\mu) > 0$ directly from non-amenability of G_μ which can be characterized by a spectral gap in the *regular* representation π_{reg} of G_μ. This approach, taken up in a joint work with Y. Shalom [11], leads to a lower estimate of $\lambda_1(\mu)$ in terms of an intrinsic spectral gap of G_μ, which might be easier to compute.

Consider the following general setup: let G be locally compact group with a left invariant Haar measure m_G, and assume that G has a left invariant metric d with finite growth $\delta(G, d) < \infty$, where

$$\delta(G, d) = \limsup_{R\to\infty} \frac{1}{R} \log m_G \{g \in G \mid d(g, e) < R\} \tag{1.14}$$

A probability measure μ on G is said to have a *finite first moment* (with respect to a metric d) if

$$\int_G d(g,e)\,d\mu(g) < \infty$$

It follows from Kingman's subadditive ergodic theorem that for μ with finite first moment there exists a P-a.e. constant finite limit

$$\lambda^{(d)}(\mu) = \lim_{n\to\infty} \frac{1}{n} d(S_n, e) \tag{1.15}$$

often called the *escape rate* of the μ-random walk.

Theorem 1.19 (Furman-Shalom, [11], [63])**.** *Let d be a left invariant metric on a locally compact group G with finite growth $\delta(G,d) < \infty$, and let $\mu \in \mathcal{P}(G)$ be a probability measure on G. Assume that the closed group $H = \mathsf{grp}(\mu) \subseteq G$ is non-amenable. Then for P-a.e. ω one has*

$$\liminf_{n\to\infty} \frac{d(S_n(\omega), e)}{n} \geq \frac{2}{\delta(G,d)} \log \frac{1}{\|\pi_{\mathrm{reg}}(\mu)\|_{\mathrm{sp}}} > 0$$

where $\pi_{\mathrm{reg}}(\mu) = \int \pi_{\mathrm{reg}}(h)\,d\mu(h)$ denotes the average operator associated to the left regular H-representation π_{reg} on $L^2(H, m_H)$. If μ has finite first moment then the escape rate $\lambda^{(d)}(\mu)$ is bounded below by the positive constant on the RHS above.

Example 1.20. Let $F_r = \langle a_1, \ldots, a_r\rangle$ be a free group of rank $r > 1$, and G be a locally compact group with a left invariant metric d and growth $\delta(G,d) < \infty$. Then for any *discrete* embedding $j : F_r \to G$ and the measure μ on G given by $\mu(j(a_i)^{\pm 1}) = 1/2r$ one has

$$\lambda^{(d)}(\mu) \geq \frac{2}{\delta(G,d)} \log \frac{r}{\sqrt{2r-1}}$$

Proof of Theorem 1.19. Consider two unitary H-representations π_{reg} and π' on $L^2(H, m_H)$ and $L^2(G, m_G)$ respectively, defined by

$$\begin{aligned} (\pi^H_{\mathrm{reg}}(h)f)(h') &= f(h^{-1}h') \qquad (f \in L^2(H, m_H)), \\ (\pi'(h)f)(g) &= f(h^{-1}g) \qquad (f \in L^2(G, m_G)) \end{aligned}$$

and the associated operators $\pi_{\mathrm{reg}}(\mu)$, $\pi'(\mu)$, respectively. The left H-action on G is free and admits a measurable cross section $G = H \cdot X$ with some σ-finite measure η on X, so that $m_G = m_H \times \eta$. Writing $f \in L^2(G, m_G)$ as $f(h', x)$ with $h' \in H$, $x \in X$, one has $\pi'(h)f(h',x) = f(h^{-1}h', x)$, so that the representation π' of H is just an integral of π_{reg}-representations $\pi' = \int_X \pi_{\mathrm{reg}}\,d\eta(x)$, which implies $\|\pi'(\mu)\| = \|\pi_{\mathrm{reg}}(\mu)\|$ and $\|\pi'(\mu)\|_{\mathrm{sp}} = \|\pi_{\mathrm{reg}}(\mu)\|_{\mathrm{sp}}$.

Non-amenability of H implies (basically is equivalent to) the spectral gap condition $\|\pi_{\mathrm{reg}}(\mu)\|_{\mathrm{sp}} < 1$ (see Theorem 2.3 of Derriennic and Guivarc'h). Hence we have

$$\|\pi'(\mu)\|_{\mathrm{sp}} = \|\pi_{\mathrm{reg}}(\mu)\|_{\mathrm{sp}} < 1$$

Let U be some bounded subset of G of positive Haar measure and let f_U denote the function $m_G(U)^{-1}\cdot\mathbf{1}_U$ where $\mathbf{1}_U$ is the characteristic function of U. Fix some $\delta>\delta(G,d)$ and let

$$f(g)=e^{-\delta/2\,d(g,e)}$$

Then both f_U and f are in $L^2(G,m_G)$. Observe that for any $g\in G$ one has

$$\begin{aligned}-\log\langle\pi'(g)f,\,f_U\rangle &= -\log\left(\frac{1}{m_G(U)}\int_U e^{-\delta/2\,d(g',g)}\,dm_G(g')\right)\\ &\leq \frac{1}{m_G(U)}\int_U\frac{\delta}{2}\,d(g',g)\,dm_G(g')\leq\frac{\delta}{2}\,(d(g,e)-R)\end{aligned}$$

where $R=\sup\{(e,g')\mid g'\in U\}$. Using this inequality for $g=S_n(\omega)$ and applying Lemma 1.18, one has for P-a.e. ω

$$\liminf_{n\to\infty}\frac{1}{n}\,d(S_n(\omega),e)\geq\frac{2}{\delta}\,\log\frac{1}{\|\pi'(\mu)\|_{\mathrm{sp}}}=\frac{2}{\delta}\,\log\frac{1}{\|\pi_{\mathrm{reg}}(\mu)\|_{\mathrm{sp}}}$$

Since δ can be chosen arbitrarily close to $\delta(G,d)$, the theorem is proved. □

1.8. **Simplicity of the spectrum.** Let μ be a probability measure on $G=\mathrm{SL'}_k(\mathbb{R})$ with a finite first moment. Note that since $\lambda_1(\mu)+\cdots+\lambda_k(\mu)=0$ condition $\lambda_1(\mu)>0$ means that not all of $\lambda_p(\mu)$, $p=1,\ldots,k$, vanish (i.e. the Lyapunov spectrum is *not trivial*). An important problem is to obtain conditions on μ which guarantee the *simplicity* of the *top* Lyapunov exponent, i.e. $\lambda_1(\mu)>\lambda_2(\mu)$, and more generally conditions for the *simplicity* of the whole Lyapunov spectrum:

$$\lambda_1(\mu)>\lambda_2(\mu)>\cdots>\lambda_k(\mu)$$

To state the main results in this direction we shall need the following definitions.

Definitions 1.21. Consider the group $G=\mathrm{SL'}_k(\mathbb{R})$ and its projective action on $\mathbb{P}^{k-1}$ and on $\mathbb{P}(\wedge^p\mathbb{R}^k)$ for $p=1,2,\ldots,k-1$. A sequence $\{g_n\}$ in G is called

(a) *contracting* (on $\mathbb{P}^{k-1}$) *to* $\xi\in\mathbb{P}^{k-1}$ if the $K=\mathrm{SO}(k)$ invariant measure ν_0 on $\mathbb{P}^{k-1}$ is contracted by g_n to the Dirac measure δ_ξ, i.e. if $g_n\nu_0\to\delta_\xi$ weakly.

(b) *p-contracting* (to an $\xi\in\mathbb{P}(\wedge^p\mathbb{R}^k)$) for some $p\in\{1,\ldots,k-1\}$ if $\wedge^p g_n$ is contracting on $\mathbb{P}(\wedge^p\mathbb{R}^k)$, i.e. if $g_n\nu_0\to\delta_\xi$ weakly, where ν_0 is the K-invariant measure on $\mathbb{P}(\wedge^p\mathbb{R}^k)$.

A (semi)group $T\subseteq G$ is called

(c) *contracting* (resp. *p-contracting*) if T contains a contracting (resp. p-contracting) sequence.

(d) *totally contracting* if T is p-contracting for all $p=1,2,\ldots,k-1$.

The measure ν_0 used in the above definition can be replaced by an arbitrary proper measure (see Lemma 1.30).

Remark 1.22. Let T be a topological semigroup acting continuously on a compact metric space Q.

(a) The T-action on compact metric Q is said to be *proximal* if for any $x, y \in Q$ there exists a sequence g_n in T and a point $z \in Q$ so that $\lim_{n\to\infty} g_n \cdot x = \lim_{n\to\infty} g_n \cdot y = z$. In view of the compactness, the following condition is equivalent to proximality: for any two point set F in Q there exists a sequence g_n in T so that $\lim_{n\to\infty} \mathrm{diam}(g_n F) = 0$. In fact, it can be shown that in this definition "two point sets F" can be replaced by "finite sets F".

(b) The T-action on a compact metric Q is said to be *strongly proximal* if for any probability measure ν on Q there exists a sequence $g_n \in T$ and a point $z \in Q$ so that the measures $g_n \cdot \nu$ converge weakly (with respect to $\mathcal{C}(Q)^*$-topology) to the Dirac measure δ_z. Strong proximality implies proximality (consider $\nu = (\delta_x + \delta_y)/2$), but in general this is a stronger notion.

(c) Let T be a semigroup in $\mathrm{SL'}_k(\mathbb{R})$ which acts (strongly) irreducibly on $\mathbb{R}^k$. Then it can be shown (for example using Lemma 1.30 below) that for the T-action on $Q = \mathbb{P}^{k-1}$ the three conditions *proximality*, *strong proximality* and *contraction* are equivalent. In fact, under strong irreducibility condition, these properties are also equivalent to an existence of $g \in T$ with a dominant eigenvalue.

Let us point out an important instance of the contraction phenomenon. Let μ be a measure on $\mathrm{SL'}_k(\mathbb{R})$ with finite first moment and $\lambda_p(\mu) > \lambda_{p+1}(\mu)$. Then for P-a.e. μ-random walk $S_n = X_n \cdots X_1$, the *transposed* sequence $S_n^t = X_1^t \cdots X_n^t$ is contracting on $\mathbb{P}(\wedge^p \mathbb{R}^k)$; while the sequence S_n itself is not necessarily contracting, yet always contains contracting subsequences. Indeed, write

$$S_n = U_n\, D_n\, V_n \qquad S_n^t = V_n^t D_n\, U_n^t$$

with $D_n = \mathrm{diag}\left[e^{a_1(n)}, \ldots, e^{a_k(n)}\right]$ and $U_n, V_n \in K = \mathrm{SO}(k)$. For P-a.e. ω one has $a_p(n)/a_{p+1}(n) \to \infty$ which forces the measure $D_n \nu_0 = D_n U_n^t \nu_0 = D_n V_n \nu_0$ to converge to the Dirac measure δ_{ξ_0} at $\xi_0 = e_1 \wedge \cdots \wedge e_p$. Oseledec Theorem also implies that $V_n^t \cdot \xi_0 = V_n^{-1} \cdot \xi_0$ converges to some subspace $\xi(\omega)$ (which is actually $E_{p+1}(\omega)^\perp$ where $\mathbb{R}^k = E_1 \supset E_2(\omega) \supset \ldots$ is the Oseledec filtration). Thus S_n^t is contracting: $S_n^t \cdot \nu_0 \to \delta_{\xi(\omega)}$. On the other hand, $U_n \xi_0$ converges only after passing to a subsequence, which explains why S_n itself is not necessarily contracting.

The transposed sequence S_n^t will indeed play a role in the proof of the following remarkable result of Guivarc'h and Raugi.

Theorem 1.23 (Guivarc'h - Raugi [27])**.** *Let μ be a probability measure on $G = \mathrm{SL'}_k(\mathbb{R})$ with a finite first moment, Assume that the semigroup $T_\mu = \mathsf{sgr}(\mu)$ is strongly p-irreducible. Then T_μ is p-contracting iff $\lambda_p(\mu) > \lambda_{p+1}(\mu)$. In particular, if T_μ is totally irreducible then T_μ is totally contracting iff the Lyapunov spectrum of μ is simple:*

$$\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_k(\mu)$$

Remark 1.24. Note the following phenomena emphasized by this result: under the (strong) irreducibility condition a typical random walk $S_n(\omega)$ (i) necessarily "detects" the contraction property of the ambient semi-group T_μ, and (ii) translates the "qualitative" contraction into the quantitative *exponential* contraction.

Unfortunately it is not easy to verify the condition of Theorem 1.23 because it is given in terms of the *semi-group* T_μ, which is often hard to identify. The next conceptual step was made by Goldsheid and Margulis who proved that in the conditions of Theorem 1.23 one can replace the semi-group T_μ by a larger and much more convenient object:

Theorem 1.25 (Goldsheid - Margulis, [23]). *Let μ be a probability measure on $G = \mathrm{SL}'_k(\mathbb{R})$ with finite first moment, and let H_μ denote the smallest real algebraic subgroup of G containing* $\mathrm{supp}(\mu)$ *(i.e. $H_\mu = \overline{\mathsf{grp}(\mu)}^Z$). If H_μ is strongly p-irreducible and p-contracting so is $T_\mu = \mathsf{sgr}(\mu)$, and therefore $\lambda_p(\mu) > \lambda_{p+1}(\mu)$. Hence, if $\mathsf{grp}(\mu)$ is Zariski dense in G then the Lyapunov spectrum of μ is simple:*

$$\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_k(\mu)$$

Remark 1.26. In a subsequent work [28] Guivarc'h and Raugi gave a general description of the multiplicities of the Lyapunov exponents of μ in terms of the algebraic closure H_μ of $\mathsf{grp}(\mu)$.

Observe that Zariski density condition $H_\mu = G$ is a relatively weak one. For a typical choice (in the sense of Haar measure, Baire category etc.) of a pair $(A, B) \in \mathrm{SL}_k(\mathbb{R}) \times \mathrm{SL}_k(\mathbb{R})$ the group generated by A and B is Zariski dense in $\mathrm{SL}_k(\mathbb{R})$, and therefore every measure μ with finite first moment and $A, B \in \mathrm{supp}(\mu)$ has a simple Lyapunov spectrum.

We shall outline the proofs of Theorems 1.23 and 1.25 in sections 1.11 and 1.12. The following sections contain some important notions and auxiliary facts needed for these proofs.

1.9. **Quasi-Projective transformations and Flag Varieties.** The contraction property, which is central to Theorems 1.23 and 1.25, describes limit behavior of a sequence of projective transformations of $\mathbb{P}^{k-1}$. In [13] Furstenberg has introduced a very useful notion of *quasi-projective* transformations (QP-transformations), which provides a convenient framework for the analysis of the contraction properties.

A transformation b of the projective space $\mathbb{P}^{k-1}$ is called *quasi-projective* (QP) if there exists a sequence of projective transformations $\bar{A}_n$ of $\mathbb{P}^{k-1}$, given by some matrices $A_n \in \mathrm{SL}'_k(\mathbb{R})$, so that for each $\xi \in \mathbb{P}^{k-1}$ one has

$$\bar{A}_n \cdot \xi \to b \cdot \xi$$

Let $A_n \in \mathrm{SL}'_k(\mathbb{R})$ be an arbitrary sequence. Consider the matrices $A'_n = \|A_n\|^{-1} \cdot A_n$ which belong to the compact set $M^1_{k\times k} = \{A' \in M_{k\times k}(\mathbb{R}) \mid \|A'\| = 1\}$. Being scalar multiples of A_n, the matrices A'_n define the same projective transformations $\overline{A}_n = \overline{A'}_n$.

Passing to a convergent subsequence, one can assume that $A'_n \to B_0$, where B_0 is a $k \times k$ matrix with $\|B_0\| = 1$ and $\det B_0 = \lim_{n\to\infty} \|A_n\|^{-k}$. If A_n are bounded, then $\bar{A}_n \to \bar{B}_0 \in \mathrm{PSL}_k(\mathbb{R})$ and the process terminates with $b = \bar{B}_0$. Otherwise B_0 is a singular matrix with $\|B_0\| = 1$. Denote $L_0 = \mathbb{R}^k$, $L_1 = \operatorname{Ker} B_0$, $k_1 = \dim L_1$. Note that for $\xi \in L_0 \setminus L_1$ one has

$$\lim_{n\to\infty} \overline{A}_n \cdot \xi = \lim_{n\to\infty} \overline{A'}_n \cdot \xi = \overline{B}_0 \cdot \xi$$

Next consider the restricted transformations $A^1_n = A_n|_{L_1} : L_1 \to \mathbb{R}^k$. Let $B_1 : L_1 \to \mathbb{R}^k$ be a limit of some convergent subsequence of the transformations $\|A^1_n\|^{-1} \cdot A^1_n : L_1 \to \mathbb{R}^k$ which, as before, belong to a compact set $M^1_{k_1\times k}$ of $k_1 \times k$-matrices of norm one. Set $L_2 = \operatorname{Ker} B_1 \subset L_1$. Continuing this procedure, one obtains a sequence of subspaces

$$\mathbb{R}^k = L_0 \supset L_1 \supset \cdots \supset L_{s-1} \supset L_r = \{0\}$$

and a sequence of linear transformations $\{B_i : L_i \to \mathbb{R}^k\}_{i=0}^r$ with $L_{i+1} = \operatorname{Ker} B_i$ such that for a suitable subsequence A_{n_j} of A_n one has

$$\lim_{j\to\infty} \bar{A}_{n_j} \cdot \xi = \overline{B_i} \cdot \xi \qquad \text{for} \qquad \xi \in \overline{L}_i \setminus \overline{L}_{i+1}$$

Hence, the above limit defines a QP-transformation $b = \lim_{j\to\infty} \bar{A}_{n_j}$ of the projective space $\mathbb{P}^{k-1}$.

Let us summarize some important facts which follow from these arguments:

(i) Any sequence of PT-s $\{\bar{A}_n\}$ contains a subsequence converging to a QP-transformation b. Such limit transformations b are called *quasi-projective limits* (QP-limits) of the sequence $\{A_n\}$.

(ii) A sequence $\{A_n\}$ in $G = \mathrm{SL}'_k(\mathbb{R})$ is bounded iff all QP-limits (of its subsequences) are actually projective transformations.

(iii) Any QP-transformation b of $\mathbb{P}^{k-1}$ admits a description by $\{B_i : L_i \to \mathbb{R}^k\}_{i=0}^r$, where $\mathbb{R}^k = L_0 \supset L_1 \supset \cdots \supset L_r = \{0\}$ are nested subspaces, $B_i : L_i \to \mathbb{R}^k$ are linear transformations with $L_{i+1} = \operatorname{Ker} B_i$ and $\|B_i\| = 1$. In this description $b : \mathbb{P}^{k-1} \to \mathbb{P}^{k-1}$ is given by

$$b \cdot \xi = \bar{B}_i \cdot \xi \qquad \text{for} \qquad \xi \in L_i \setminus L_{i+1} \tag{1.16}$$

(It should be pointed out that such a description need not be unique). Conversely, one can show that any system $\{B_i : L_i \to \mathbb{R}^k\}_{i=0}^r$ with the above properties defines a QP-transformation by (1.16).

Observe that if a QP-limit of b of a sequence $A_n \in \mathrm{SL}'_k(\mathbb{R})$ then for any $\nu \in \mathcal{P}(\mathbb{P}^{k-1})$ one has

$$\lim_{n\to\infty} \bar{A}_n \cdot \nu = b \cdot \nu \tag{1.17}$$

Already the first layer B_0 of b, described by $\{B_i : L_i \to \mathbb{R}^k\}_{i=0}^r$, contains important information on $\{A_n\}$. If A_n are unbounded then b is a genuine QP-transformation with $L_1 = \operatorname{Ker} B_0$ and $R_1 = \operatorname{Im} B_0$ being proper subspaces of $\mathbb{R}^k$. Writing $\nu = \nu_0 + \nu_1$

where $\nu_0 = \nu|_{\bar{L_0}}$ and $\nu_1 = \nu - \nu_0$, the measure $b \cdot \nu_1$ is supported on $\bar{R}_1$, while $b \cdot \nu_0 = \lim_{n\to\infty} \bar{A}_n \cdot \nu_0$ is supported on the limit of projective spaces $\overline{A_n(L_0)}$, which is also a proper projective subspace. This consideration leads to the following:

Lemma 1.27 (Furstenberg [12]). *Let $A_n \in G = \mathrm{SL'}_k(\mathbb{R})$, and $\nu, \nu' \in \mathcal{P}(\mathbb{P}^{k-1})$ be probability measures so that $\bar{A}_n \cdot \nu \to \nu'$. Then either A_n are bounded, or there exist proper subspaces $V, W \subset \mathbb{R}^k$ such that ν' is supported on $\overline{V} \cup \overline{W}$. In particular, the stabilizer* $\mathrm{Stab}_G(\nu) = \{g \in G \mid g\nu = \nu\}$ *of any proper measure ν is a compact subgroup of G.*

Remark 1.28. Further analysis of QP-limits along similar lines was used by Zimmer (see [72] 3.4.2) to prove that the stabilizer $\mathrm{Stab}_{\mathrm{PSL}_k(\mathbb{R})}(\nu)$ of any measure $\nu \in \mathcal{P}(\mathbb{P}^{k-1})$ has a normal subgroup of finite index which is *algebraic* (recall that compact subgroups of real algebraic groups are algebraic).

Typically, QP-transformations are not continuous. Following Goldsheid-Margulis [23], we shall denote by $M_1(b) \subseteq \mathbb{P}^{k-1}$ the closure of the set of the discontinuity points of a QP-transformation b, and by $M_0(b)$ the b-image of the set of its continuity points. With these definitions, $M_0(b)$ is always a projective subspace of $\mathbb{P}^{k-1}$, more precisely

Lemma 1.29 (see [23] 2.8). *If $M_0(b)$ contains more than one point, then $M_0(b) = \overline{\mathrm{Im}\, B_0}$ and $M_1(b) = \overline{L_1}$. If $M_0(b)$ is a single point, then either $M_1(b) = \overline{L_i}$ for some $1 \le i \le r$, or $M_1(b)$ is empty.*

The contraction property (Definition 1.21) has the following characterization:

Lemma 1.30 ([27]). *Let $\{A_n\}$ be a sequence in $G = \mathrm{SL'}_k(\mathbb{R})$, and ν be an arbitrary proper probability measure on $\mathbb{P}^{k-1}$. Write $A_n = U_n \,\mathrm{diag}\left[e^{a_1(n)}, \dots, e^{a_k(n)}\right] V_n$ with $U_n, V_n \in \mathrm{O}(k)$ and $a_1(n) \ge a_2(n) \ge \dots \ge a_k(n)$. The following conditions are equivalent*

(a) *$\{A_n\}$ is a contracting sequence with $\bar{A}_n \cdot \nu_0 \to \delta_\xi$ where $\xi \in \mathbb{P}^{k-1}$.*
(b) *$\bar{A}_n \cdot \nu \to \delta_\xi$.*
(c) $\lim_{n\to\infty} a_2(n)/a_1(n) = 0$ *and* $\lim_{n\to\infty} \bar{U}_n \cdot \bar{e}_1 = \xi$
(d) *All QP-limits b of $\{A_n\}$ have $M_0(b) = \{\xi\}$.*

Remark 1.31. Note that a semigroup $T \subseteq \mathrm{SL'}_k(\mathbb{R})$ is p-strongly irreducible and p-contracting iff the transposed semigroup $T^t = \{g^t \mid g \in T\}$ has these properties. Indeed strong irreducibility is preserved by the transpose operation, as well as the existence of contracting sequences (use part (c) of the Lemma).

The linear action of $G = \mathrm{SL'}_k(\mathbb{R})$ on $\mathbb{R}^k$ defines the corresponding projective actions not only on the projective space, but also on more general flag varieties. Let us recall the basic definitions of the latter. A *flag* of *type* $\tau = (\tau_1, \dots, \tau_r)$, where $1 \le \tau_1 < \tau_2 < \dots < \tau_r \le k$ are integers, is an r-tuple of nested linear subspaces $V_1 \subset V_2 \subset \dots \subset V_r$ of $\mathbb{R}^k$ with $\dim V_i = \tau_i$ for $i = 1, \dots, r$. The *flag variety* $\mathcal{F}_\tau$ of type $\tau = (\tau_1, \dots, \tau_r)$ is the collection of all flags of type τ. Typical examples of flag varieties are: $\mathcal{F}_{(1)}$ - the projective space $\mathbb{P}^{k-1}$, $\mathcal{F}_{(p)}$ - the Grassmannian Gr_p, and the *full flags* $\mathcal{F}_{(1,2,\ldots,k)}$. We shall also use $\mathcal{F}_{(1,2)}$ consisting of *projective line elements* (i.e. a point and a direction through the point) on $\mathbb{P}^{k-1}$. A flag variety $\mathcal{F}_\tau$ forms a closed subset of the compact projective space of the linear space $\oplus_{i=1}^{r} \wedge^{\tau_i} \mathbb{R}^k$. This provides it with the natural topology, projective $G = \mathrm{SL'}_k(\mathbb{R})$-action and with a notion of QP-transformations. Any fixed Euclidean norm $\|\cdot\|$ on $\mathbb{R}^k$ defines the corresponding norms on the exterior products $\wedge^p\mathbb{R}^k$, which allow to define cocycles $\sigma_1, \ldots, \sigma_k$ for the $G = \mathrm{SL'}_k(\mathbb{R})$-action on the full flag variety $\mathcal{F} = \mathcal{F}_{(1,\ldots,k)}$ as follows: for $g \in G$ and $\xi \in \mathcal{F}$ set

$$\sigma_p(g, \xi) = \log \frac{\|gu_1 \wedge \cdots \wedge gu_p\|}{\|u_1 \wedge \cdots \wedge u_p\|} \tag{1.18}$$

where ξ is the flag $(\overline{u_1}, \overline{u_1 \wedge u_2}, \ldots, \overline{u_1 \wedge \cdots \wedge u_p})$. One easily checks that:

(i) the cocycles σ_p are well defined;
(ii) each σ_p can be defined on the corresponding Grassmannian $\sigma_p : G \times \mathcal{F}_{(p)}$ in a way which is consistent with the natural G-equivariant quotient map $\mathcal{F} \to \mathcal{F}_{(p)}$;
(iii) the definition of σ_1 coincides with the one given in (1.6).

Next consider the cocycle $\sigma = \sigma_2 - 2\sigma_1 : G \times \mathcal{F}_{(1,2)} \to \mathbb{R}$, which can be explicitly defined as

$$\sigma(g, \xi) = \log \frac{\|gu \wedge gv\|}{\|gu\|^2} - \log \frac{\|u \wedge v\|}{\|u\|^2} \tag{1.19}$$

where $\xi = (\bar{u}, \overline{u \wedge v}) \in \mathcal{F}_{(1,2)}$. One can verify that $\exp \sigma(g, \xi)$ is the dilatation coefficient of the projective action of g on $\mathbb{P}^{k-1}$ in the direction of the projective line element ξ.

In the sequel we shall need the following property of contracting sequences:

Lemma 1.32 (see [27]). *Let $\{g_n\}$ be a sequence in $G = \mathrm{SL'}_k(\mathbb{R})$ contracting $\mathbb{P}^{k-1}$ towards $\bar{z} \in \mathbb{P}^{k-1}$. Then for every line element $\xi = (\bar{u}, \overline{u \wedge v}) \in \mathcal{F}_{(1,2)}$ with $u \notin z^\perp$, the transposed sequence $\{g_n^t\}$ satisfies $\sigma(g_n^t, \xi) \to -\infty$.*

1.10. **Some Auxiliary Results.** This section contain some basic ingredients of the proof of Theorem 1.23 (providing the grounds for the phenomena pointed out in Remark 1.24).

Th following fact plays a crucial role in Theorem 1.23 and in the notion of (G, μ)-boundaries discussed in the next chapter.

Lemma (*) 1.33 (Furstenberg [13], Guivarc'h-Raugi [27]). *Let G be a locally compact group, Q be a compact metric G-space, μ a probability measure on G, and let ν be a μ-stationary probability measure on Q. Denote by $Y_1, Y_2, \ldots$ a sequence of G-valued independent random variables with distribution μ. Then for P-a.e. sequence $\omega = (Y_1, Y_2, \ldots)$ the measures $Y_1 Y_2 \cdots Y_n \cdot \nu$ converge in the weak topology to a limit probability measure $\nu_\omega \in \mathcal{P}(Q)$, while ν is the average of these random measures*

$$\nu = \int \nu_\omega \, d\mathsf{P}(\omega) \qquad \text{i.e.} \qquad \int_Q f \, d\nu = \int_\Omega \int_Q f \, d\nu_\omega \, d\mathsf{P}(\omega) \quad (f \in \mathcal{C}(Q))$$

Moreover, with P*-probability one the same limit* ν_ω *is obtained by*

$$Y_1 Y_2 \cdots Y_n \, g \cdot \nu \to \nu_\omega$$

for μ_**-a.e.* $g \in G$*, where* $\mu_* = \sum_1^\infty 2^{-p} \mu^p$*.*

Note the order of applied transformations $Y_1 Y_2 \cdots Y_n \, g \cdot \nu$, and the fact that the limit $Y_1 \cdots Y_n \, g \cdot \nu = \nu_\omega$ depends only on $\omega = (Y_1, Y_2, \dots)$, but not on g. Applying this general fact to the projective action of $G = \mathrm{SL}'_k(\mathbb{R})$ on $Q = \mathbb{P}^{k-1}$ one obtains:

Theorem 1.34 ([27])**.** *Let* μ *be a probability measure on* $G = \mathrm{SL}'_k(\mathbb{R})$ *so that the semi-group* $T_\mu = \mathsf{sgr}(\mu)$ *(equivalently the algebraic group* $H_\mu = \overline{\mathsf{grp}(\mu)}^Z$ *) are contracting and strongly irreducible. Then there is a unique* μ*-stationary measure* ν *on* $\mathbb{P}^{k-1}$*; for* P*-a.e.* $\omega \in \Omega$ *the limit measure*

$$\nu_\omega = \lim_{n \to \infty} Y_1(\omega) \cdots Y_n(\omega) \cdot \nu$$

is a Dirac measure $\nu = \delta_{z(\omega)}$*, where the random point* $z(\omega) \in \mathbb{P}^{k-1}$ *has distribution* ν*. The set* $L = \mathrm{supp}(\nu) \subseteq \mathbb{P}^{k-1}$ *is the unique minimal set for the* T_μ*-action on* $\mathbb{P}^{k-1}$*, and* T_μ *acts minimally and strongly proximally on* L*.*

Proof. Let ν be some μ-stationary measure on $\mathbb{P}^{k-1}$. Lemma 1.33 and Fubini theorem ensure that for all ω from a subset $\Omega_0 \subseteq \Omega$ with $\mathsf{P}(\Omega_0) = 1$, one has

$$Y_1(\omega) \cdots Y_n(\omega) g \cdot \nu \to \nu_\omega \qquad \text{as} \qquad n \to \infty$$

for μ^*-a.e. $g \in G$. Fix an $\omega \in \Omega_0$, and let b be a QP-limit of some subsequence Z_{n_i} of $Z_n = Y_1(\omega) \dots Y_n(\omega)$. Since μ is strongly irreducible, ν is a proper measure (Lemma 1.7) and so are $g \cdot \nu$, for every $g \in G$. Hence for μ^*-a.e. $g \in G$ one has

$$\nu_\omega = \lim_{i \to \infty} Z_{n_i} g \cdot \nu = bg \cdot \nu$$

The relation $bg \cdot \nu = \nu_\omega$ clearly holds for all $g \in \mathrm{supp}(\mu^*) = T_\mu$. Since T_μ contains contracting sequences g_n with $g_n \cdot \nu \to \delta_{\bar{w}}$ one concludes that $\nu_\omega = \delta_{b \cdot \bar{w}}$ is a Dirac measure $\nu_\omega = \delta_{z(\omega)}$ at some $z(\omega) \in \mathbb{P}^{k-1}$. In particular $b \cdot \nu = \nu_\omega = \delta_{z(\omega)}$, and since ν is proper, one has $M_0(b) = \{z(\omega)\}$. This argument applies to all QP-limits b of the sequence $Z_n = Y_1(\omega) \cdots Y_n(\omega)$, which means that $\{Z_n\}$ is a contracting sequence with $Z_n \cdot \nu_0 \to \delta_{z(\omega)}$. In particular $z(\omega)$ is determined by $\omega \in \Omega_0$ only (and not by the choice of the μ-stationary measure ν); and since ν satisfies

$$\nu = \int \nu_\omega \, d\mathsf{P}(\omega) = \int \delta_{z(\omega)} \, d\mathsf{P}(\omega)$$

the measure ν is just the distribution of $z(\omega)$, and hence is unique.

Let $L' \subseteq \mathbb{P}^{k-1}$ be a non-empty closed T_μ-invariant set. Then L' supports a μ-stationary probability measure ν', which by uniqueness coincides with ν. Hence $L \subseteq L'$. This proves that L is the unique T_μ-minimal set. The strong proximality assertion for the projective action of T_μ on L follows from the contraction property. □

We shall also use the following general fact from ergodic theory:

Lemma 1.35 (Kesten [40], also Guivarc'h-Raugi [26]). *Let T be a measure preserving transformation of a probability space (X, m) and assume that a function $f \in L^1(X, m)$ satisfies $\sum_{k=0}^{n-1} f(T^k x) \to -\infty$ for m-a.e. $x \in X$. Then $\int_X f\, dm < 0$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} f(T^k x) < 0$, for m-a.e. $x \in X$.*

This Lemma allows to translate a qualitative divergence to a divergence with a *speed*. In the context of Markov processes this gives the following

Proposition (*) 1.36 (Guivarc'h - Raugi [27]). *Let M be a compact metric space, $\sigma : M \to \mathbb{R}$ a continuous function, and P a Markov operator on M. Let $\theta \in \mathcal{P}(M)$ be a P-stationary measure, and let $\{Z_n^\theta, n \geq 0\}$ denote the corresponding stationary Markov process on M. Suppose that with probability one*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \sigma(Z_k^\theta) = -\infty \tag{1.20}$$

Then with probability one $\lim_{n\to\infty} n^{-1}\sum_{k=0}^{n} \sigma(Z_k^\theta) < 0$. Furthermore, if (1.20) holds for all P-stationary measures θ, then there exists a $\gamma < 0$ so that

$$\lim_{n\to\infty} \sup_{x\in M} \frac{1}{n}\sum_{k=0}^{n} P^k\sigma(x) = \gamma < 0$$

1.11. **From contractions to the simplicity of the spectrum.** With these preliminaries we can present the proof of Theorem 1.23, following Guivarc'h and Raugi [27].

If $\lambda_{p+1}(\mu) > \lambda_p(\mu)$ then T_μ is p-contracting as follows from the discussion after Remark 1.22. The main content of the theorem is the sufficiency of p-contracting for $\lambda_p(\mu) > \lambda_{p+1}(\mu)$. Clearly, the case of $p > 1$ can be reduced to the case $p = 1$ by passing to the G-action on the p-th exterior power $\wedge^p\mathbb{R}^k$. Hence, it is enough to show that if μ has finite first moment and $T_\mu = \mathsf{sgr}(\mu)$ is contracting and strongly irreducible, then $\lambda_1(\mu) > \lambda_2(\mu)$. To avoid estimates we shall assume that μ is compactly supported.

Denote $Y_n = X_n^t$. Then $\{Y_n\}$ is a sequence of independent random variables with common distribution μ^t, where $d\mu^t(g) = d\mu(g^t)$. Let ν and ν' be stationary measures on $\mathcal{F}_{(1)} = \mathbb{P}^{k-1}$ for μ and μ^t respectively. Since T_μ is strongly irreducible and contracting the same applies to $T_{\mu^t} = T_\mu^t$ (Remark 1.31) and Theorem 1.34 shows that ν and ν' are uniquely determined and proper (Lemma 1.7).

For P-a.e. $\omega \in \Omega$ random walk $\{S_n = X_n \cdots X_2 X_1\}$ the transposed sequence is $\{S_n^t = (X_n \cdots X_1)^t = Y_1 \cdots Y_n\}$, and Theorem 1.34 implies that

$$S_n^t \cdot \nu' = Y_1 \cdots Y_n \cdot \nu' \to \delta_{\bar{z}(\omega)}$$

is a Dirac measure at $\bar{z}(\omega) \in \mathbb{P}^{k-1}$. By Lemma 1.30 the sequence $\{S_n^t\}$ is contracting. Since the distribution ν' of $\bar{z}(\omega)$ is proper, for any fixed non-zero vector $u \in \mathbb{R}^k$ with P-probability one $u \notin z(\omega)^\perp$, and therefore by Lemma 1.32 for any projective line element $\xi \in \mathcal{F}_{(1,2)}$

$$\sigma(S_n, \xi) = \sigma((Y_1 \cdots Y_n)^t, \xi) \ \to \ -\infty \tag{1.21}$$

with P-probability one, where $\sigma = \sigma_2 - 2\sigma_1$ was defined by (1.19).

Now consider the cocycle σ as a function on the compact (!) space $M = \mathrm{supp}(\mu) \times \mathcal{F}_{(1,2)}$, equipped with the Markov operator $\tilde{P}$

$$\tilde{P} f(g, \xi) = \int f(g', g' \cdot \xi)\, d\mu(g')$$

All $\tilde{P}$-stationary measures on M are of the form $\mu \times \eta$ where η is a μ-stationary measure on $\mathcal{F}_{(1,2)}$ (Lemma 1.4). Fix such η and denote by $\{Z_n^\eta = (X_n, \xi_n), n \geq 0\}$ the corresponding stationary Markov process on M. Then (1.21) implies that with probability one

$$\sum_{j=0}^{n-1} \sigma(Z_j^\eta)) = \sum_{j=0}^{n-1} \sigma(X_{j+1},\, S_j \cdot \xi) = \sigma(S_n, \xi) \to -\infty$$

and this property holds for all μ-stationary η. Proposition 1.36 gives

$$\lim_{n\to\infty} \sup_{(g,\xi)\in M} \frac{1}{n} \sum_{j=0}^{n} \tilde{P}^j \sigma(g, \xi) = \gamma < 0$$

which, in view of the cocycle property of σ, yields

$$\lim_{n\to\infty} \sup_{\xi\in\mathcal{F}_{(1,2)}} \frac{1}{n} \int_G \sigma(g, \xi)\, d\mu^n(g) = \gamma < 0 \tag{1.22}$$

Denoting by ν_0 the $K = \mathrm{SO}(k)$-invariant probability measure on $\mathcal{F}_{(1,2)}$ one can check that there exists a constant C, so that for any $g \in G$

$$\log \|g \wedge g\| \leq C + \int \sigma_2(g, \xi)\, d\nu_0(\xi)$$

Since obviously $\log \|g\| = \sup_\xi \sigma_1(g, \xi) \geq \int \sigma_1(g, \xi)\, d\nu_0$, one has

$$\begin{aligned} \int_G (\log \|g \wedge g\| - 2 \log \|g\|) d\mu^n(g) \ &\leq \ C + \int\int \sigma(g, \xi)\, d\nu_0(\xi)\, d\mu^n(g) \\ &\leq \ C + \sup_{\xi\in\mathcal{F}_{(1,2)}} \int_G \sigma(g, \xi)\, d\mu^n(g) \end{aligned}$$

Dividing by n and taking $n \to \infty$, one finally obtains from (1.22) that

$$\lambda_2(\mu) - \lambda_1(\mu) = (\lambda_1(\mu) + \lambda_2(\mu)) - 2\lambda_1(\mu) \leq \gamma < 0$$

Thereby proving the theorem. □

The contraction property $\lambda_1 > \lambda_2$, established by the theorem, has the following quantitative form

Proposition 1.37 (LePage [46]). *Under the assumption of Theorem 1.23 one has*

$$\lim_{n\to\infty} \sup_{\bar{u}_1 \neq \bar{u}_2 \in \mathbb{P}^{k-1}} \frac{1}{n} \int_G \log \frac{\delta(g\cdot\bar{u}_1, g\cdot\bar{u}_2)}{\delta(\bar{u}_1,\bar{u}_2)}\, d\mu^n(g) < 0$$

where $\delta(\cdot,\cdot)$ *is the natural metric on* $\mathbb{P}^{k-1}$*, given by* $\delta(\bar{u}_1,\bar{u}_2) = \|u_1 \wedge u_2\|/\|u_1\|\cdot\|u_2\|$.

Proof. The Proposition follows from (1.22) and the following identity, satisfied by all $\bar{u}_1 \neq \bar{u}_2 \in \mathbb{P}^{k-1}$ and all $g \in G = \mathrm{SL'}_k(\mathbb{R})$

$$\log \frac{\delta(g\cdot\bar{u}_1, g\cdot\bar{u}_2)}{\delta(\bar{u}_1,\bar{u}_2)} = \log \frac{\|gu_1 \wedge gu_2\|\cdot\|u_1\|\cdot\|u_2\|}{\|gu_1\|\cdot\|gu_2\|\cdot\|u_1\wedge u_2\|} = \frac{\sigma(g,\xi_1)+\sigma(g,\xi_2)}{2}$$

where $\xi_i = (\overline{u_i}, \overline{u_1 \wedge u_2})$, $i = 1, 2$. □

1.12. **Zariski closures and the contraction properties.** This section contains an outline of the proof of Theorem 1.25. For the details the reader is referred to the original paper [23] (see also [21]).

Real algebraic groups have the property that algebraic closure of a sub-semigroup forms an algebraic subgroup. Thus Theorem 1.25 can be deduced from

Theorem 1.38 (Goldsheid-Margulis, [23]). *Let H be an algebraic closure of a semigroup $T \subseteq \mathrm{SL}_k(\mathbb{R})$. If H is p-strongly irreducible and p-contracting, then the same applies to the semigroup T.*

Passing to the p-th exterior power, the Theorem is reduced to the case $p = 1$. Since (strong) irreducibility is preserved by algebraic closures, it suffices to prove that if a semigroup $T \subseteq \mathrm{SL}_k(\mathbb{R})$ is strongly irreducible and its algebraic closure H contains contracting sequences then so does T itself. The proof of this fact is based on the analysis of QP-limits (on $\mathbb{P}^{k-1}$)) of sequences in T. In general, given a semigroup $T \subseteq G = \mathrm{SL}_k(\mathbb{R})$ we shall denote by $\overline{T}^{qp}$ the collection of all QP-limits (on $\mathbb{P}^{k-1}$) of sequences in T. The set $\overline{T}^{qp}$ will be called the *QP-closure* of T.

Lemma 1.39 ([23] 2.7, 2.10). *The QP-closure $\overline{T}^{qp}$ of a semigroup $T \subseteq \mathrm{SL}_k(\mathbb{R})$ forms a semigroup, which is closed with respect to pointwise convergence.*

In order to show that T contains a contracting sequence, it is enough (Lemma 1.30) to find a QP-limit $b \in \overline{T}^{qp}$ with $M_0(b) \subseteq \mathbb{P}^{k-1}$ being a point. Assuming that this is not the case, let $d > 0$ denote the minimal (projective) dimension of the spaces $M_0(b)$

as b varies over $\overline{T}^{qp}$; and let $Q \subseteq \overline{T}^{qp}$ consist of all $b \in \overline{T}^{qp}$ with $M_0(b)$ achieving this minimal dimension d.

Claim (*) 1.40. *For any $b \in Q$ either $M_0(b) \subseteq M_1(b)$ or $M_0(b) \cap M_1(b) = \emptyset$.*

Claim (*) 1.41. *There exist $b \in Q$ with $M_0(b) \cap M_1(b) = \emptyset$.*

QP-transformations b with $M_0(b) \cap M_1(b) = \emptyset$ enjoy the following useful property

Lemma 1.42 ([23] 2.9). *Let b be a QP-transformation with $M_0(b) \cap M_1(b) = \emptyset$. Denote by V the linear subspace generated by the lines in $M_0(b)$. Then the restriction of b to $\overline{V}$ is an invertible projective transformation $\beta(b) \in \mathrm{PGL}(V)$.*

Now fix a QP-transformation $b \in Q$ with disjoint $M_0(b)$ and $M_1(b)$, and consider the set

$$H(b) = \{h \in H \mid M_0(b\bar{h}) \cap M_1(b\bar{h}) = \emptyset\}$$

Let Φ and Φ_0 denote the semi-groups of all QP-transformations generated by $\{b\bar{h} \mid h \in H(b)\}$ and $\{b\bar{h} \mid h \in H(b) \cap T\}$ respectively, and denote by $V \subset \mathbb{R}^k$ the subspace spanned by the directions of $M_0(b)$, i.e. $M_0(b) = \overline{V}$. By Lemma 1.42, for each ϕ in the generating set of Φ, there is a uniquely defined element $\beta(\phi) \in \mathrm{PGL}(V)$ such that the actions of ϕ and $\beta(\phi)$ agree on $M_0(b) = \overline{V}$. Extending β to the whole semigroup Φ one obtains a homomorphism $\beta : \Phi \to \mathrm{PGL}(V)$. The next crucial claim is:

Claim (*) 1.43. *The (semigroup) homomorphism $\beta : \Phi \to \mathrm{PGL}(V)$ maps $\Phi_0 \subset \Phi$ to a relatively compact semigroup $\beta(\Phi_0)$ of $\mathrm{PGL}(V)$.*

Hence the closure K of $\beta(\Phi_0)$ in $\mathrm{PGL}(V)$ forms a compact semigroup of the real algebraic group $\mathrm{PGL}(V)$. It is therefore a real algebraic subgroup of $\mathrm{PGL}(V)$, due to the following general facts: (i) any compact sub-semigroup of a topological group forms a (compact) subgroup, and (ii) any compact subgroup of a real algebraic group is a real algebraic subgroup.

The correspondence $h \mapsto \beta(b\bar{h})$ of $H(b)$ to $\mathrm{PGL}(V)$ is a *rational* map. It maps $T \cap H(b)$ into the algebraic subgroup $K \subset \mathrm{PGL}(V)$. Since T is Zariski dense in H, one concludes that $\beta(bh) \in K$ for all $h \in H(b)$. This is used to reach a contradiction to the assumption that H is contracting, by showing that $H(b)$ contains a sequence h_n with a QP-limit $b' = \lim_{n\to\infty} \bar{h}_n$ such that $M_0(bb')$ has strictly smaller dimension than the dimension $d > 0$ of $M_0(b)$, contradicting the minimality of d. This completes the sketch of the proof of Theorem 1.25. For the proofs of the claims 1.40–1.43 see section 4.

1.13. **Regularity of the Lyapunov spectrum.** Consider the regularity properties of the maps

$$\mu \mapsto \lambda_1(\mu) \qquad \mu \mapsto (\lambda_1(\mu), \lambda_2(\mu), \dots, \lambda_k(\mu))$$

which are defined for all $\mu \in \mathcal{P}(\mathrm{SL}_k(\mathbb{R}))$ with finite first moments. It is not difficult to see that there is a *lower semi-continuity* in the following sense

Lemma 1.44. *Let $\mu_n \to \mu$ be a weakly convergent sequence of probability measures on $G = \mathrm{SL}_k(\mathbb{R})$, such that μ and μ_n have finite first moments. Then*

$$\limsup_{n\to\infty} \lambda_1(\mu_n) \leq \lambda_1(\mu)$$

Proof. By Kingman's subadditive ergodic theorem, one has

$$\lambda_1(\mu) = \inf_{p\in\mathbb{N}} \frac{1}{p} L(\mu^p) \qquad \text{where} \qquad L(\mu) = \int_G \log \|g\| \, d\mu(g)$$

For every fixed $p \in \mathbb{N}$, one has $L(\mu_n^p) \to L(\mu^p)$ as $n \to \infty$, hence the lemma follows from the general inequality $\limsup_{n\to\infty} \inf_p L(\mu_n^p) \leq \inf_p \lim_{n\to\infty} L(\mu_n^p)$. □

However, in general, one should not expect the map $\mu \mapsto \lambda_1(\mu)$ to be continuous:

Example 1.45. Let $G = \mathrm{SL}_2(\mathbb{R})$ and consider the matrices

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For $0 < t \leq 1$ let μ_t be defined by $\mu_t(\{A\}) = t$ and $\mu_t(\{F\}) = 1 - t$. Obviously $\lambda_1(\mu_1) = \log 2$, but one can check that $\lambda_1(\mu_t) = 0$ for any $t < 1$. To see the latter fact intuitively, observe that while A expands and contracts the vectors e_1, e_2 by the factor of 2, F flips their directions. Applying long random product $S_n = X_n \cdots X_1$, which contains many A-s and some roughly equally spaces flips F, to vectors e_1 and e_2, each of these vectors experiences long alternating periods of expansion and contraction resulting in a sub-exponential growth.

Note that in the above example the group G_{μ_1} is reducible.

Theorem 1.46 (Furstenberg - Kifer, [19])**.** *Let $\mu_n \to \mu$ be a weakly convergent sequence of probability measures on $G = \mathrm{SL}_k(\mathbb{R})$, such that μ_n have uniformly bounded first moments*

$$\sup_n \int_G \log \|g\| \, d\mu_n(g) < \infty$$

Assume that $G_\mu = \mathsf{grp}(\mu)$ has at most one invariant subspace in its linear action on $\mathbb{R}^k$. Then $\lambda_1(\mu_n) \to \lambda_1(\mu)$.

Proof. Here we shall give the proof for the case of an irreducible G_μ, and refer to [19] for the case of a single invariant subspace and further results.

Theorem 1.11 in particular implies that for each n there exists a μ_n-stationary measure $\nu_n \in \mathcal{P}(\mathbb{P}^{k-1})$ such that $\lambda_1(\mu_n) = \alpha_{\mu_n,\nu_n}$. Due to the compactness of $\mathcal{P}(\mathbb{P}^{k-1})$ one can assume that some subsequence ν_{n_i} converges weakly to a probability measure ν on $\mathbb{P}^{k-1}$. This limit measure ν is necessarily μ-stationary, because

$$\nu = \lim_{i\to\infty} \nu_{n_i} = \lim_{i\to\infty} \mu_{n_i} * \nu_{n_i} = \mu * \nu$$

The uniform bound on the first moments of μ_n enables to apply Lebesgue dominated convergence theorem to deduce that

$$\lim_{i\to\infty} \lambda_1(\mu_{n_i}) = \lim_{i\to\infty} \alpha_{\mu_{n_i},\nu_{n_i}} = \alpha_{\mu,\nu}$$

In general the right hand side gives just a lower bound for $\lambda_1(\mu)$ (cf. Lemma 1.44), but under the assumption of irreducibility of the G_μ-representation on $\mathbb{R}^k$, Theorem 1.11 states that $\lambda_1(\mu) = \alpha_{\mu,\nu}$, and therefore $\lambda_1(\mu_{n_i}) \to \lambda_1(\mu)$. This argument shows that any subsequence of $\{\mu_n\}$ contains a sub-subsequence with $\lambda_1(\mu_{n_{i_j}}) \to \lambda_1(\mu)$, which means that $\lim_{n\to\infty} \lambda_1(\mu_n) = \lambda_1(\mu)$. □

There exists a variety of results on regularity of the Lyapunov spectrum under various assumptions on μ (cf. LePage [45], Ruelle [61] etc.) Let us mention the following regularity result, due to Y. Peres, for measures with a *fixed finite support*:

Theorem 1.47 (Peres, [57])**.** *Let $S = \{A_1, \dots, A_a\}$ be some fixed finite set of matrices in $G = \mathrm{SL}_k(\mathbb{R})$. Consider all probability measures $\mu_{\bar{w}}$ with $\mathrm{supp}(\mu) = S$ and weights $\mu_{\bar{w}}(A_i) = w_i > 0$ where $\bar{w} = (w_1, \dots, w_a)$ is a probability vector. If $\lambda_p(\mu_{\bar{w}})$ is a simple Lyapunov exponent, i.e. $\lambda_{p-1}(\mu_{\bar{w}}) > \lambda_p(\mu_{\bar{w}}) > \lambda_{p+1}(\mu_{\bar{w}})$, then locally $\lambda_p(\mu_{\bar{w}})$ is a real analytic function of the weights $(w_1, \dots, w_a)$. In particular, if $S = \{A_1, \dots, A_a\}$ generates a Zariski dense subgroup of G, then all $\lambda_p(\mu_{\bar{w}})$, $p = 1, \dots, k-1$, are locally real analytic functions of the weights $\bar{w}$.*

The proof of this theorem relies on Theorem 1.49 below.

1.14. **Further Limit Theorems.** So far the discussion of μ-random products $S_n = X_n \cdots X_1$ focused on forms of Laws of Large Numbers for matrix products. Other classical limit theorems such as (Functional) Central Limit Theorem, Iterated Logarithm Law, Large Deviation results etc. were also proved in the setting of matrix products. Main results in this direction were obtained by the works of LePage, Guivarc'h, Raugi, Bougerol, Goldsheid ([46], [27], [22]) after earlier results of Furstenberg-Kesten [18], Tutubalin [64] and Virtzer [68]. Here we shall briefly state some of these results and refer the reader to [27], [22], [4], [46] for details and further results.

Finer Limit theorems typically require stronger integrability assumption than Laws of Large Numbers. In the context of random matrices we shall impose the following condition: a probability measure μ on $G = \mathrm{SL'}_k(\mathbb{R})$ is said to have *finite exponential moment* if for some $\epsilon > 0$

$$\int_G \|g\|^\epsilon \, d\mu(g) < \infty \tag{1.23}$$

Theorem 1.48 (LePage, [46])**.** *Let μ be a probability measure on $G = \mathrm{SL}_k(\mathbb{R})$. Assume that μ has finite exponential moment, is strongly irreducible and contracting on $\mathbb{P}^{k-1}$. Then there exists a constant $\sigma > 0$ so that for any fixed $u \in \mathbb{R}^k \setminus \{0\}$ one has*

Central Limit Theorem: *The random variables*

$$\frac{\log\|S_n(\omega)\| - n\lambda_1}{\sigma\sqrt{n}} \quad \text{and} \quad \frac{\log\|S_n(\omega)u\| - n\lambda_1}{\sigma\sqrt{n}}$$

converge in distribution to the standard Gaussian distribution $N(0,1)$ *on* $\mathbb{R}$.

Functional CLT: *The linear interpolation functions* $\Sigma_n(\omega, t)$, $t \in [0,1]$, *determined by the discrete values*

$$\Sigma_n(\omega, m/n) = \frac{\log\|S_m(\omega)u\| - m\lambda_1}{\sigma\sqrt{n}} \qquad (m = 0, \dots, n)$$

converge in distribution to the canonical Wiener measure on $\mathcal{C}([0,1])$.

Joint Distribution: *The* $\mathbb{R} \times \mathbb{P}^{k-1}$*-random variables*

$$\left(\frac{\log\|S_n u\| - n\lambda_1}{\sigma\sqrt{n}},\, S_n \cdot \bar{u}\right)$$

converge in distribution to $N(0,1) \times \nu$ *with an error estimate of* $C/\sqrt{n}$.

Iterated Logarithm Law: *With* P*-probability one the sequences*

$$\frac{\log\|S_n(\omega)\| - n\lambda_1}{\sigma\sqrt{n\log\log n}} \quad \text{and} \quad \frac{\log\|S_n(\omega)u\| - n\lambda_1}{\sigma\sqrt{n\log\log n}}$$

have the interval $[-1,1]$ *as the set of cluster points.*

Proofs of these results go far beyond the scope of this paper. Here let us just point out the main ideas linking these results on products of random matrices to, by now well established, techniques in the classical probability theory, and refer the reader to the above mentioned papers for the full exposition of these ideas. The main device used in the proof of Theorem 1.48 is the family

$$P_s\phi(\bar{u}) = \int_G e^{i\,s\cdot\sigma_1(g,\bar{u})}\,\phi(g\cdot\bar{u})\,d\mu(g)$$

of operators acting on a certain space L_α of Hölder functions ϕ on $\mathbb{P}^{k-1}$. (The relevance of these operators become clear from the fact that $P_s^n\mathbf{1}(\bar{u})$ is the Fourier transform of the distribution of $\log\|S_n(\omega)u\|/\|u\|$). The finite exponential moment assumption (1.23) is needed for the following key result

Theorem 1.49 (LePage, [46])**.** *Let* μ *be a strongly irreducible and contracting probability measure on* $\mathrm{SL}'_k(\mathbb{R})$, *satisfying (1.23). Then there exists an* $\alpha_0 > 0$ *such that for any* $\alpha \in (0, \alpha_0)$ *one has*

$$\lim_{n\to\infty}\left(\sup_{\bar{u}_1\neq\bar{u}_2\in\mathbb{P}^{k-1}}\int_G\left(\frac{\delta(g\cdot\bar{u}_1, g\cdot\bar{u}_2)}{\delta(\bar{u}_1,\bar{u}_2)}\right)^\alpha d\mu^n(g)\right)^{1/n} = \rho < 1$$

This Theorem is a strengthening of Proposition 1.37, and it follows from the corresponding strengthening of (1.22) which can be proved under condition (1.23).

Second major ingredient is the following

Theorem 1.50 (see Guivarc'h-Raugi [27] pp. 44–45, Goldsheid-Guivarc'h [22] section 4)**.** *Let μ satisfy the assumptions of Theorem 1.49. If a function $\psi \in L_\alpha$ satisfies the equation*

$$e^{i\, s\cdot\sigma_1(g,\bar{u})} = e^{i\theta}\psi(g\cdot\bar{u})/\psi(\bar{u})$$

for some $s,\theta \in \mathbb{R}$ and all $g \in \operatorname{supp}(\mu)$, then $s = 0$, $e^{i\theta} = 1$ and ψ is a constant.

Theorems 1.49 and 1.50 together allow to prove certain *spectral gap* properties of the operators P_s acting on the Hölder space L_α (see [27] and [4] Chapter V), and this spectral gap property is used to derive the limit theorems.

Similar techniques allow to prove multidimensional Limit Theorems (replacing projective spaces by more general flag varieties). More precisely, if $S_n = X_n \cdots X_1$ is written in a polar form as $S_n = U_n A_n V_n$ where $U_n, V_n \in \emptyset(k)$ and $A_n = \operatorname{diag}[e^{a_1(n)}, \dots e^{a_k(n)}]$ with $a_1 \geq \cdots \geq a_k$, consider the full flags $\mathcal{U}_n, \mathcal{V}_n \in \mathcal{F} = \mathcal{F}_{(1,2,\dots,k)}$ defined by

$$\begin{aligned} \mathcal{U}_n &= U_n \cdot (\overline{e_k}, \overline{e_k \wedge e_{k-1}}, \dots, \overline{e_k \wedge \cdots \wedge e_1}) \\ \mathcal{V}_n &= V_n^{-1} \cdot (\overline{e_1}, \overline{e_1 \wedge e_2}, \dots, \overline{e_1 \wedge \cdots \wedge e_k}) \end{aligned}$$

These flags are well defined if $a_1(n) > a_2(n) > \cdots > a_k(n)$ as it will be the case in the theorem below, due to Theorem 1.25 which guarantees the simplicity of the Lyapunov spectrum

$$\Lambda = (\lambda_1(\mu) > \lambda_2(\mu) > \cdots > \lambda_k(\mu))$$

Theorem 1.51 (Goldsheid - Guivarc'h [22], see also Guivarc'h -Raugi [27])**.** *Let μ be a probability measure on $\mathrm{SL}_k(\mathbb{R})$. Assume that μ has finite exponential moment (1.23) and $\operatorname{supp}(\mu)$ generates a Zariski dense subgroup in $\mathrm{SL}_k(\mathbb{R})$. Then*

1. *With probability one:* $\lim_{n\to\infty} n^{-1} \log A_n = \Lambda$.
2. *The random vectors* $\epsilon_n = n^{-1/2}(\log A_n - n\Lambda)$ *converge in distribution to a Gaussian distribution N with the full support on $\mathbb{R}^k_0 = \{x \in \mathbb{R}^k \mid x_1 + \cdots + x_k = 0\}$.*
3. *The sequence of flags $\{\mathcal{V}_n\}$ converges with probability one to a flag $\mathcal{V}$ in $\mathcal{F}$. The distribution of the limit point $\mathcal{V}$ is the unique $\check{\mu}$-stationary measure ν' on $\mathcal{F}$.*
4. *The sequence of flags $\{\mathcal{U}_n\}$ converges in law to the unique μ-stationary measure ν on $\mathcal{F}$.*
5. *The three random variables $\mathcal{V}_n$, ϵ_n, $\mathcal{U}_n$ are asymptotically independent, i.e.*

$$\lim_{n\to\infty} \mathsf{P}\{\mathcal{U}_n \in E,\ \epsilon_n \in F,\ \mathcal{V}_n \in G\} = \nu(E)\cdot N(F)\cdot \nu'(G).$$

Part (a) in the theorem is the conclusion of Theorem 1.25 which is stated here for completeness. Statements (c) and (d) are deduced from the Oseledec Theorem and the multidimensional analogues of Theorem 1.34. The main assertions of the Theorem are (b), stating that the limit Gaussian distribution is *non degenerate*, i.e. is not supported on a proper subspace of $\mathbb{R}^k_0$, and (d) which describes the limit joint distribution. In [22] Goldsheid and Guivarc'h further prove that if μ has finite exponential moment and the algebraic closure H_μ of G_μ is semisimple (but is possibly smaller than $\mathrm{SL}_k(\mathbb{R})$) then the dimension of the limit Gaussian distribution of the diagonal part A_n is the real rank of H_μ. A cohomological equation as in Theorem 1.50 plays a crucial role in the analysis of the dimension of the Gaussian distribution (see [22] section 4).

1.15. **Additional Remarks.** The phenomena discussed in this section depend heavily on the assumption that $\{X_n\}$ is a sequence of *independent* identically distributed matrices, rather than a general *stationary* sequence of matrices. In a recent paper [44] Kifer studied a mixed model, where $\{\mu_n\}$ is some *stationary* process taking values in $\mathcal{P}(\mathrm{SL}_k(\mathbb{R}))$, and X_n are matrices chosen independently according to the "random" distribution μ_n. Kifer has generalized many of the results of this section to this more general setting.

Some of the techniques and ideas in the study of products of random matrices found applications in other areas unrelated to random transformations. Let us mention here some of these applications.

- The important phenomenon behind Lemma 1.27 led Furstenberg to a short and elegant proof [16] of Borel's Density Theorem.
- Similar phenomena were exploited by Zimmer to prove the very important property of *tameness* of group actions on spaces of probability measures, arising from algebraic actions on projective varieties ([72] section 3.2).
- Non-vanishing of Lyapunov exponents, using a spectral approach similar to to the one described in section 1.6, was used in the original proof of Margulis' superrigidity (see [54] V.4).
- There exist "random walk" approaches to the proofs of superrigidity theorems of Margulis and Zimmer (see Furstenberg [17] and Margulis [54] chapter VI).
- In [24] Guivarc'h used random walks on $\mathrm{SL}_k(\mathbb{R})$ to investigate properties of linear groups, in particular giving a shorter proof of the Tits alternative.
- Recently, Abels, Margulis and Soifer [1] applied quasi-projective transformations (in a quantitative form) to an investigation of affine linear groups in the context of Auslander conjecture.

2. Random Walks on General Groups

We turn now to a discussion of random walks $S_n = X_n \cdots X_1$ on general countable discrete and locally compact groups G.

2.1. **Recurrence of random walks.**

Let Γ be a countable discrete group with a generating symmetric probability measure μ. Let $S_n = X_n \dots X_1$ denote a path of μ-generated random walk. If $S_n = e$ for some $n \geq 1$ with probability one then S_n returns to e *infinitely often* with probability one. In this case the μ-random walk is called *recurrent*. Otherwise, the probability of infinitely many returns to e is zero and the μ-random walk is called *transient*. It is well known that the simple random walk on $\mathbb{Z}^k$ is recurrent iff $k = 1$ or $k = 2$. The following is a far reaching generalization of this classical fact:

Theorem 2.1 (Varopoulos, [65]). *Let μ be a symmetric finitely supported generating probability measure on a countable discrete group Γ. The μ-random walk is recurrent on Γ if and only if Γ contains a subgroup of finite index Γ_0 which is isomorphic to either $\mathbb{Z}$ or to $\mathbb{Z}^2$.*

Hence for most infinite discrete groups symmetric random walks are transient. One may consider the quantitative characteristics of the transience behavior. One of such characteristics is the (exponential) *rate of decay* of the return probabilities

$$p_n = \mathsf{P}\{\omega \mid S_n(\omega) = e\} = \mu^n(\{e\})$$

Clearly $p_{n+m} \geq p_n \cdot p_m$ for all $n, m \in \mathbb{N}$, which allows to consider the limit of $(p_n)^{1/n}$ as $n \to \infty$, provided p_n do not vanish. For a symmetric measure μ one always has $p_{2n} > 0$ and therefore the limit

$$R(\Gamma, \mu) = \lim_{n\to\infty} (p_{2n})^{1/2n}$$

is a well defined quantity $R(\Gamma, \mu) \in (0, 1]$.

Theorem 2.2 (Kesten, [38], [39]). *Let μ be a symmetric generating probability measure on a countable discrete group Γ. Then $R(\Gamma, \mu) = \|\pi_{reg}(\mu)\| = \|\pi_{reg}(\mu)\|_{\mathrm{sp}}$, where π_{reg} denotes the regular Γ-representation on $l^2(\Gamma)$. In particular, $R(\Gamma, \mu) < 1$ if and only if the group Γ is non-amenable.*

Proof. Consider the semi-contraction $P = \pi_{reg}(\mu)$ on $l^2(\Gamma)$. Since μ is symmetric P is self-adjoint and $\|P\| = \|P\|_{\mathrm{sp}}$. Note that for any $\gamma \in \Gamma$, the action of P on the Dirac function $\delta_\gamma \in l^2(\Gamma)$ satisfies

$$\|P^n \delta_\gamma\|^2 = \langle P^n \delta_\gamma,\, P^n \delta_\gamma\rangle = \langle P^{2n} \delta_\gamma,\, \delta_\gamma\rangle = \langle P^{2n} \delta_e,\, \delta_e\rangle = p_{2n}$$

Hence

$$R(\Gamma, \mu) = \lim_{n\to\infty} (p_{2n})^{1/2n} = \lim_{n\to\infty} \|P^n \delta_e\|^{1/n} \leq \|P\|$$

On the other hand for any finitely supported $f = \sum_{i=1}^{N} a_i \delta_{\gamma_i}$ one has

$$\lim_{n\to\infty} \|P^n f\|^{1/n} \leq \lim_{n\to\infty} \left(\sum_{i=1}^{N} |a_i| \sqrt{p_{2n}} \right)^{1/n} = R(\Gamma, \mu)$$

which implies $\|P\| \leq R(\Gamma, \mu)$ using the spectral theorem. □

Kesten's theorem can also be stated for locally compact groups, in which case one should consider the probabilities $\mu^n(V)$ of the returns to bounded neighborhoods V of the identity. In the statement of Kesten's theorem we have used the fact that condition $\|\pi_{\text{reg}}(\mu)\|_{\text{sp}} = 1$ is equivalent to *amenability* of the group, generated by μ. The following is a more standard definition of amenability of a locally compact group G: there exists a (left) invariant *mean* (i.e. positive, normalized, finitely additive functional) on the space $BC(G)$ of bounded continuous functions on G.

Theorem 2.3 (Derriennic-Guivarc'h [7]). *Let G be a locally compact group and μ be a probability measure on G with $\mathsf{grp}(\mu) = G$. Then G is amenable if and only if $\|\pi_{\text{reg}}(\mu)\|_{\text{sp}} = 1$.*

Note that in this result the generating measure μ is not assumed to be symmetric. The proof is similar to that of Proposition 1.17 and Theorem 3.17 below (actually, in the proofs of the latter facts we used an idea from [7] of Derriennic-Guivarc'h).

Returning to the context of discrete countable groups, let Γ be a finitely generated group and S be a generating set. The associated left invariant *word metric* on Γ is defined by

$$d_S(g_1, g_2) = \min\{n \in \mathbb{N} \mid g_2^{-1} g_1 \in (\{e\} \cup S \cup S^{-1})^n\}$$

The corresponding *growth function* is $V_S(n) = |\{g \in \Gamma \mid d_S(g, e) \leq n\}|$. It is well known (exercise) that if S and T are two finite generating sets for Γ then the metrics d_T and d_S are bi-Lipschitz equivalent, i.e. there exists a constant $0 < c < \infty$ so that $c^{-1} \cdot d_S \leq d_T \leq c \cdot d_S$. Let us say that two functions $\phi, \phi' : \mathbb{N} \to \mathbb{R}_+$ are *roughly equivalent* (notation $\phi \simeq \phi'$) if there exists a constant $0 < c < \infty$ so that

$$c^{-1} \cdot \phi([n/c]) \leq \phi'(n) \leq c \cdot \phi([cn])$$

The above bi-Lipschitz equivalence of word metrics implies that the growth functions $V_S(n)$ and $V_T(n)$ are roughly equivalent, and their rough equivalence class can be denoted by $V_\Gamma(n)$.

In a recent paper Pittet and Saloff-Coste [58], in particular, proved that return probability functions $\phi_\mu(n) = \mu^{2n}(\{e\})$ are also independent of μ, up to rough equivalence. More precisely,

Theorem 2.4 (Pittet Saloff-Coste, [58]). *For any finitely generated group Γ and any two symmetric finitely supported generating probability measures μ, μ' on Γ the return probability functions $\phi_\mu(n)$ and $\phi_{\mu'}(n)$ are roughly equivalent: $\phi_\mu \simeq \phi_{\mu'}$.*

Remarks 2.5. (a) Non amenable groups Γ have exponential growth $V_\Gamma(n) \simeq e^n$ (but there exist many amenable, even polycyclic, groups of exponential growth). Kesten's theorem states that Γ is non-amenable iff $\phi_\Gamma(n) \simeq e^{-n}$.
(b) Polycyclic groups which are not virtually nilpotent have exponential growth $V_\Gamma(n) \simeq e^n$ (Milnor - Wolf), and have $\phi_\Gamma \simeq e^{-n^{1/3}}$ (Varopoulos).
(c) Virtually nilpotent groups Γ are characterized (Gromov) by polynomial growth $V_\Gamma(n) \simeq n^d$. For these groups Varopoulos showed that $\phi_\Gamma(n) \simeq n^{-d/2}$.
(d) Grigorchuck constructed groups with intermediate growth $V_\Gamma(n)$ (which is slower than any exponential, but faster than any polynomial). For these groups the above result is particularly interesting since the type $\phi_\Gamma(n)$ is still unknown.
(e) Pittet and Saloff-Coste also prove that the rough equivalence class of the return probability function is a quasi-isometric invariant of groups.

2.2. **Harmonic functions and (G,μ)-spaces.** Let G be a locally compact group, and μ be a probability measure on G. It will be convenient, although not necessary, to impose the following regularity assumption on μ

Definition 2.6. We shall say that a probability measure μ on a locally compact second countable group G is *admissible* if $\mathsf{sgr}(\mu) = G$ and μ is *spread out*, which means that some convolution power μ^p of μ is not singular with respect to the Haar measure on G.

Given a a locally compact group G with an admissible probability measure μ on it, one can study the Markov operator P

$$(Pf)(g) = \int_G f(gg')\, d\mu(g')$$

on various spaces of functions on G (subject to an appropriate integrability condition). Functions f fixed by P are called *μ-harmonic*, and are characterized by the *μ-mean value property*:

$$f(g) = \int_G f(gg')\, d\mu(g'), \qquad (g \in G) \tag{2.1}$$

We shall be especially interested in *bounded* μ-harmonic functions on G. It turns out that due to the assumption that μ is admissible, any bounded μ-harmonic function is necessarily *continuous* on G. The space of all bounded μ-harmonic functions on G will be denoted by $\mathcal{H}^\infty(G,\mu)$.

We shall also have an occasion to consider the space $B_{luc}(G)$ consisting of all bounded *left uniformly continuous* functions on G. Recall that a function f on G is l.u.c. if for any $\epsilon > 0$ there is a neighborhood of the identity $V = V(\epsilon)$ in G s.t.

$$|f(gg') - f(g')| < \epsilon \qquad \text{for} \qquad g \in V,\ g' \in G$$

The subspace of l.u.c. bounded μ-harmonic functions on G will be denoted by $\mathcal{H}^\infty_{luc}(G,\mu)$. Any bounded μ-harmonic function is a pointwise limit of l.u.c. bounded

μ-harmonic functions. Indeed one can verify that for any $f \in \mathcal{H}^\infty(G,\mu)$ and any non-negative compactly supported continuous $h \in \mathcal{C}_c(G)$, the convolution function $h * f$, defined by

$$(h * f)(g) = \int_G h(g')f(g'^{-1}g)\, dm_G(g')$$

is in $\mathcal{H}^\infty_{luc}(G,\mu)$. Taking a sequence $h_n \in \mathcal{C}_c(G)$ with $h_n \geq 0$, $\int h_n = 1$ and $\mathrm{supp}(h_n)$ shrinking to the identity, one obtains $h_n * f \to f$.

The following Lemma provides a connection between $\mathcal{H}^\infty(G,\mu)$ and the *right* μ-random walks on G:

Lemma 2.7. *For every $f \in \mathcal{H}^\infty(G,\mu)$ the limits*

$$\tilde{f}(g,\omega) = \lim_{n\to\infty} f(g\omega_1\omega_2\cdots\omega_n) \tag{2.2}$$

exist for all $g \in G$ and P-a.e. sequence ω. We shall denote $\tilde{f}(\omega) = \tilde{f}(e,\omega)$.

Proof. Fix $f \in \mathcal{H}^\infty(G,\mu)$ and $g \in G$ and consider the random variables $W_n = f(g\omega_1\cdots\omega_n)$ on (Ω,P). Observe that the sequence $\{W_n\}$ forms a bounded martingale with respect to the increasing sequence $\mathcal{A}_n$ of the σ-algebras generated by $\omega_1,\dots,\omega_n$. Indeed

$$\mathbb{E}(W_{n+1} \mid \mathcal{A}_n) = \int f(g\omega_1\cdots\omega_n h)\, d\mu(h) = f(g\omega_1\cdots\omega_n) = W_n$$

By the martingale theorem W_n converge pointwise on (Ω,P) to some measurable bounded function W_∞, which we have denote by $\tilde{f}(g,\omega)$. □

Remark 2.8. In view of the preceding lemma, we shall throughout this chapter focus on the *right* μ-random walks $\omega_1\cdots\omega_n$ rather than previously considered left random walks $S_n = \omega_n\cdots\omega_1$. Clearly these two settings are equivalent by $g \mapsto g^{-1}$. See also section 2.3 for a better perspective.

Proposition 2.9 below describes an important construction of bounded μ-harmonic functions on (G,μ), using (G,μ)-spaces. Let (M,ν) be a Lebesgue probability space with a measurable G-action such that ν is *μ-stationary* in the sense that $\mu * \nu = \nu$ where the convolution measure $\mu * \nu$ is the image of $\mu \times \nu$ under the action map $G \times M \to M$. Such a space (M,ν) with the G-action will be called a *(G,μ)-space.*

A natural source for (G,μ)-spaces are compact (metric) *G-spaces*, i.e. compact (metric) spaces M with a continuous action $G \times M \to M$. Any such G-space M has at least one μ-stationary probability measure ν, which gives rise to a (G,μ)-space (M,ν). Since one can restrict the continuous G-action to the compact G-invariant set $M_0 = \mathrm{supp}(\nu)$, we shall always assume that the considered μ-stationary measure ν has full support on the compact G-space M. Such a pair (M,ν) will be called a *compact (G,μ)-space.*

Proposition 2.9 (Furstenberg). *Given a measurable (G,μ)-space (M,ν) consider the transform $F_{(M,\nu)}:\phi\mapsto f_\phi$ defined for all $\phi\in L^\infty(M,\nu)$ by*

$$f_\phi(g)=\int_M \phi(g\cdot x)\,d\nu(x)=\int_M \phi\,dg\nu=\int_M \phi(x)\cdot\Pi(g,x)\,d\nu(x) \tag{2.3}$$

where $\Pi(g,x)=dg\nu/d\nu(x)$ is the Radon-Nikodym cocycle. Then

(a) *$f_\phi\in\mathcal{H}^\infty(G,\mu)$ for all $\phi\in L^\infty(M,\nu)$.*
(b) *f_ϕ are constant functions on G for all $\phi\in L^\infty(M,\nu)$ iff ν is G-invariant.*

If (M,ν) is a compact (G,μ)-space, then

(a') *$f_\phi\in\mathcal{H}^\infty_{luc}(G,\mu)$ for all $\phi\in\mathcal{C}(M)$.*
(b') *f_ϕ are constant functions on G for all $\phi\in\mathcal{C}(M)$ iff ν is G-invariant.*

Proof. Given $\phi\in L^\infty(M,\nu)$ the function f_ϕ is bounded by $\|\phi\|_\infty$ and satisfies

$$\int_G f_\phi(gg')\,d\mu(g')=\int_G\int_M \phi(g\cdot x)\,d\mu*\nu(x)=f_\phi(g)$$

Hence $f_\phi\in\mathcal{H}^\infty(G,\mu)$, proving statement (a). Statement (b) is obvious from the definition. If M is a compact G-space and $\phi\in\mathcal{C}(M)$, then for any $\epsilon>0$ there is an open neighborhood V of the identity in G so that for all $g\in V$ and $x\in M$ $|\phi(gx)-\phi(x)|<\epsilon$, and therefore for all $g'\in G$

$$|f_\phi(gg'x)-f_\phi(g')|\le\int_M|\phi(gy)-\phi(y)|\,dg'\nu(y)<\epsilon$$

Hence f_ϕ is a l.u.c. function, which proves (a'), while (b') is immediate. □

2.3. **(G,μ)-boundaries.** Consider a compact (G,μ)-space (M,ν). Lemma 1.33 (the proof of which basically consists of Lemma 2.7 and Proposition 2.9), states that for P-a.e. path $\omega=(\omega_1,\omega_2,\dots)$ of a (right) μ-random walk on G there exists a limit probability measure ν_ω on M

$$\nu_\omega=\lim_{n\to\infty}\omega_1\cdots\omega_n\cdot\nu$$

In the notation of Lemma 2.7 and 2.9 one has

$$\tilde f_\phi(g,\omega)=\int_M\phi\,dg\nu_\omega\qquad\text{and}\qquad \tilde f_\phi(\omega)=\int_M\phi\,d\nu_\omega \tag{2.4}$$

Definition 2.10. A compact (G,μ)-space (M,ν) is a *compact (G,μ)-boundary* if ν_ω is a Dirac measure $\nu_\omega=\delta_{z(\omega)}$ for P-a.e. $\omega\in\Omega$. An abstract measurable (G,μ)-space (M,ν) is a *(G,μ)-boundary*, if the transform $F_{(M,\nu)}:L^\infty(M,\nu)\to\mathcal{H}^\infty(G,\mu)$, $F(M,\nu):\phi\mapsto f_\phi$ satisfies

$$\tilde f_{\phi\cdot\psi}(\omega)=\tilde f_\phi(\omega)\cdot\tilde f_\psi(\omega) \tag{2.5}$$

for any $\phi,\psi\in L^\infty(M,\nu)$ and for P-a.e. ω.

It is straightforward to see that if (M,ν) is a (G,μ)-boundary then the transform $F_{(M,\nu)} : L^\infty(M,\nu) \to \mathcal{H}^\infty(G,\mu)$ is an isometric embedding, and if (M,ν) is a compact (G,μ)-boundary then also $F_{(M,\nu)} : \mathcal{C}(M) \to \mathcal{H}^\infty_{luc}(G,\mu)$ is an (isometric) embedding.

Note that any (G,μ)-boundary (M,ν) gives rise to a measurable map of Lebesgue spaces $\mathsf{bnd}_{(M,\nu)} : (\Omega,\mathsf{P}) \to (M,\nu)$, defined via its action on the commutative von Neumann algebra $L^\infty(M,\nu)$

$$\phi(\mathsf{bnd}_{(M,\nu)}(\omega)) = \tilde{f}_\phi(\omega) = \lim_{n\to\infty} f_\phi(\omega_1\cdots\omega_n) \tag{2.6}$$

In the setup of compact (G,μ)-boundaries one has $\mathsf{bnd}_{(M,\nu)}(\omega) = z(\omega) \in M$ where $\nu_\omega = \delta_{z(\omega)}$.

To further clarify the above concepts consider the skew-product construction $(\Omega\times M, \mathsf{P}\times\nu, T)$, $T : (\omega,x) \mapsto (\theta\omega, \omega_1\cdot x)$, discussed in Proposition 1.3. The transformation T of $(\Omega\times M, \mathsf{P}\times\nu)$ is measure-preserving but not invertible. Let $\bar\theta$ denote the two-sided shift $(\bar\theta\bar\omega)_i = \bar\omega_{i+1}$, $i\in\mathbb{Z}$, acting on the space $(\bar\Omega,\bar{\mathsf{P}}) = (G^{\mathbb{Z}},\mu^{\mathbb{Z}})$. This is the natural extension of the one-sided shift θ on (Ω,P). The natural extension of the skew-product $(\Omega\times M,\mathsf{P}\times\nu,T)$ can be realized in the form $(\bar\Omega\times M,\bar m,\bar T)$ with

$$\bar T : (\bar\omega, x) \mapsto (\bar\theta\bar\omega, \bar\omega_1\cdot x)$$

and $\bar m$ being the unique $\bar T$-invariant measure which extends $\mathsf{P}\times\nu$ from $\mathcal{B}(\Omega\times M)$ to $\mathcal{B}(\bar\Omega\times M)$, where $\mathcal{B}$ denotes the Borel σ-algebra of the corresponding spaces. Note that the projection $\bar\Omega\times M\to\bar\Omega$ maps $\bar m$ to $\bar{\mathsf{P}}$, because $\bar{\mathsf{P}}$ is the only $\bar\theta$-invariant probability measure which extends P from $\mathcal{B}(\Omega)$ to $\mathcal{B}(\bar\Omega)$. Let $\{m_{\bar\omega}\in\mathcal{P}(M)\}_{\bar\omega\in\bar\Omega}$ be the disintegration of $\bar m$ with respect to $\bar{\mathsf{P}}$. The $\bar T$ invariance of $\bar m$ translates into the identity $\omega_1 m_{\bar\omega} = m_{\bar\theta\bar\omega}$, and the fact that $\bar m$ extends $\mathsf{P}\times\nu$ to the σ-algebra of $\bar\Omega\times M$ means that

$$\int m_{\bar\omega}\, d\bar{\mathsf{P}}(\bar\omega) = \nu$$

and that $m_{\bar\omega}$ depends only on $(\ldots,\bar\omega_{-1},\bar\omega_0)$.

In fact, every $\bar T$-invariant measure $\bar m$ on $\bar\Omega\times M$ which projects onto $\bar{\mathsf{P}}$ and has the property that the fiber measures $m_{\bar\omega}$ depend only on $\bar\omega_i$, $i\le 0$, defines a measure on $\Omega\times M$ of the form $\mathsf{P}\times\nu$, where ν is a μ-stationary measure.

Hence the conditional measure of $\bar m$ with respect to $\{\bar\omega_k \mid k\ge -n\}$ is $\bar\omega_0\bar\omega_{-1}\cdots\bar\omega_{-n}\nu$. As $n\to\infty$ this conditional measure converges to $m_{\bar\omega}$, but the same limit measure was also described in Lemma 2.7 as

$$\nu_{(\omega_0,\omega_{-1},\ldots)} = \lim_{n\to\infty} \omega_0\omega_{-1}\cdots\omega_{-n}\nu$$

Using this description, one can say

Proposition 2.11. *A (G,μ)-space (M,ν) forms a (G,μ)-boundary if and only if in the natural extension $(\bar\Omega\times M,\bar m,\bar T)$ of the non-invertible measure-preserving skew-product $(\Omega\times M,\mathsf{P}\times\nu,T)$ the measure $\bar m$ disintegrates into Dirac measures with respect to $\bar{\mathsf{P}}$, or equivalently if $\bar m$ is a lift of $\bar{\mathsf{P}}$ to a graph of a measurable function $\bar\Omega\to M$.*

The following idea of Furstenberg allows to construct (G,μ)-boundaries from arbitrary compact metric *G-spaces M*.

Proposition 2.12 (Furstenberg [14]). *Let M be an arbitrary compact metric G-space and μ be an (admissible) probability measure on G. Then the G-action on the compact space $V=\mathcal{P}(M)$ contains a (G,μ)-boundary.*

Proof. Consider $V=\mathcal{P}(M)$ as a compact convex set in $\mathcal{C}(M)^*$. Consider also the space $\mathcal{P}(V)=\mathcal{P}(\mathcal{P}(M))$, elements of which will be denoted by $\tilde{\nu}$. Let $\delta: V\to\mathcal{P}(V)$ stand for the natural embedding $\delta: v\mapsto\delta_v$ which has the set of extremal points $\mathrm{Ext}(\mathcal{P}(V))$ of $\mathcal{P}(V)$ as its image $\delta(V)$. Let us denote by $\mathsf{bar}(\nu)$ the natural barycenter map, which is a continuous extension of the affine map

$$\mathsf{bar}(\sum_{i=1}^n p_i\delta_{v_i})=\sum_1^n p_i v_i$$

Since M is a G-space, there exists a μ-stationary measure ν on M, and according to Lemma 1.33 there is a disintegration $\nu=\int_\Omega \nu_\omega\,d\mathsf{P}$. Let $\tilde{\nu}\in\mathcal{P}(V)$ be defined as

$$\tilde{\nu}=\int_\Omega \delta_{\nu_\omega}\,d\mathsf{P} \tag{2.7}$$

which implies that $\mathsf{bar}(\tilde{\nu})=\nu$ and that $\tilde{\nu}$ is a μ-stationary measure on V. Observe that for a.e. ω one has

$$\mathsf{bar}(\omega_1\cdots\omega_n\cdot\tilde{\nu})=\omega_1\cdots\omega_n\cdot\nu\to\nu_\omega \tag{2.8}$$

Since δ_{ν_ω} are extremal points in $\mathcal{P}(V)$, the two facts (2.7) and (2.8) imply that $\omega_1\cdots\omega_n\cdot\tilde{\nu}\to\delta_{\nu_\omega}$, which means that $\mathrm{supp}(\tilde{\nu})\subseteq V=\mathcal{P}(M)$ together with $\tilde{\nu}$ form a (G,μ)-boundary. □

2.4. **The Poisson boundary.** It is easy to see that any measurable G-equivariant quotient (M',ν') of a (G,μ)-boundary (M,ν) is a (G,μ)-boundary. It turns out that given a pair (G,μ) there exists a uniquely defined *maximal* and *universal* (G,μ)-boundary, associated to (G,μ).

Theorem 2.13 (Furstenberg). *Given a locally compact group G with an admissible probability measure μ, there exists a uniquely defined maximal measurable (G,μ)-boundary (B,ν), called the Poisson boundary of (G,μ), which is uniquely characterized by each of the following properties:*

Poisson Representation: *The transform $F_{(B,\nu)}: L^\infty(B,\nu)\to\mathcal{H}^\infty(G,\mu)$ is an* isometric *bijection. In particular every bounded μ-harmonic function f can be presented as $f_\phi=F_{(B,\nu)}(\phi)$ for a unique $\phi\in L^\infty(B,\nu)$, which can be defined by*

$$\phi(\mathsf{bnd}_{(B,\nu)}(\omega))=\tilde{f}(\omega)=\lim_{n\to\infty} f(\omega_1\omega_2\cdots\omega_n)$$

Universality: *The Poisson boundary (B, ν) is the* maximal *measurable (G, μ)-boundary: any measurable (G, μ)-boundary (B', ν') is a G-equivariant measurable quotient $p : (B, \nu) \to (B', \nu')$ of the Poisson boundary (B, ν). The quotient map p is uniquely defined, up to sets of ν-measure zero.*

Combined with Proposition 2.12, the universality of the Poisson boundary gives the following important

Corollary 2.14 (Furstenberg). *Let (B, ν) be the Poisson boundary for a locally compact group G with respect to some admissible measure μ on G. Then for any compact metric G-space M, there exists a measurable, with respect to ν, G-equivariant map $f : B \to \mathcal{P}(M)$.*

Remark 2.15. This is one of the variants of the so called *boundary maps* which play an important role in rigidity results for group actions. In [70] Zimmer introduced the notion of *amenability* of an *action* of a group G on a measure space (X, ν) with a quasi-invariant measure ν. Amenability of a G-action on (X, ν) (which is basically characterized by the existence of a measurable G-equivariant maps $f : X \to \mathcal{P}(M)$ for any G-space M) can often be verified by other more convenient means. For example, an action of a lattice Γ in a locally compact group G on a homogeneous space G/H is amenable iff H is an amenable group. Poisson boundaries are amenable G-spaces. Boundary maps play a crucial role in the superrigidity results (see Zimmer [72] and Margulis [54]).

There are several ways to construct the Poisson boundary (B, ν) for (G, μ). In the following construction, due to Furstenberg (the form described below is borrowed from Glasner's [20]), one exhibits a compact G-space $\bar{B}$ with a μ-stationary measure ν for which the transform $F_{(\bar{B},\nu)}$ gives isometric isomorphisms

$$\mathcal{C}(\bar{B}) \xrightarrow{\cong} \mathcal{H}^{\infty}_{luc}(G, \mu) \qquad \text{and} \qquad L^{\infty}(\bar{B}, \nu) \xrightarrow{\cong} \mathcal{H}^{\infty}(G, \mu).$$

Denote by $\mathcal{A}$ the collection of all $f \in B_{luc}(G)$ such that for all $g \in G$ for P-a.e. $\omega \in \Omega$ the limit $\tilde{f}(g, \omega) = \lim_{n\to\infty} f(g\omega_1 \cdots \omega_n)$ exists. $\mathcal{A}$ forms a commutative C^*-algebra with respect to pointwise operations and the sup norm. Let $\mathcal{Z}$ denote the closed ideal in $\mathcal{A}$ consisting of those $f \in \mathcal{A}$ for which the $\tilde{f}(g, \omega) = 0$, and by $\bar{B}$ the Gelfund dual of the quotient commutative C^*-algebra $\mathcal{A}/\mathcal{Z}$, so that $\mathcal{C}(\bar{B})$ becomes identified with $\mathcal{A}/\mathcal{Z}$. The natural G-action by left translations on $\mathcal{A}$ preserves $\mathcal{Z}$ and descends to $\mathcal{A}/\mathcal{Z} \cong \mathcal{C}(\bar{B})$. This G-action preserves the C^*-algebra operations and norms, and the fact that all functions involved are l.u.c. corresponds to *continuity* of the action. This defines a *continuous* G-action on $\bar{B}$. Next note that for each $g \in G$ the functional $L_g : f \in \mathcal{A} \mapsto \int_\Omega \tilde{f}(g, \omega)\, d\mathsf{P}(\omega)$ is a positive normalized functional on $\mathcal{A}$, which vanishes on $\mathcal{Z}$, and hence descends to a positive normalized functional on $\mathcal{A}/\mathcal{Z} \cong \mathcal{C}(\bar{B})$. Thus L_g corresponds to a probability measure ν_g on $\bar{B}$. It follows from the construction that: (i) $\nu_g = g\nu_e$ for every $g \in G$; (ii) the measure $\nu = \nu_e$ is μ-stationary; (iii) $(\bar{B}, \nu)$ is a (topological) (G, μ)-boundary (due to multiplicativity).

By Lemma 2.7, $\mathcal{H}=\mathcal{H}^\infty(G,\mu)\subseteq B_{luc}(G)$ is a closed subspace of $\mathcal{A}$, and we claim that in fact $\mathcal{A}=\mathcal{Z}\oplus\mathcal{H}$. Indeed, the martingale theorem shows that for $f\in\mathcal{H}$ one has

$$f(g)=\int_\Omega \tilde{f}(g,\omega)\,d\mathsf{P}(\omega)$$

and therefore $\mathcal{Z}\cap\mathcal{H}=\{0\}$. To show that $\mathcal{H}+\mathcal{Z}=\mathcal{A}$, take an arbitrary $h\in\mathcal{A}$ and denote by $\phi\in\mathcal{C}(\bar{B})$ its image in $\mathcal{A}\to\mathcal{A}/\mathcal{Z}\cong\mathcal{C}(\bar{B})$. Then ϕ defines a function $f_\phi\in\mathcal{H}$, which has the same "boundary values" $\tilde{f}_\phi(\omega)=\lim_{n\to\infty}f_\phi(\omega_1\cdots\omega_n)=\lim_{n\to\infty}h(\omega_1\cdots\omega_n)$ as h, so that $h-f_\phi\in\mathcal{Z}$.

Hence we have a natural *bijection* $\mathcal{H}\cong(\mathcal{H}\oplus\mathcal{Z})/\mathcal{Z}\cong\mathcal{A}/\mathcal{Z}\cong\mathcal{C}(\bar{B})$, i.e. $F_{(\bar{B},\nu)}:\mathcal{C}(\bar{B})\to\mathcal{H}^\infty_{luc}(G,\mu)$ is an isomorphism.

Finally, observe that this fact implies the universality (and thereby the uniqueness) of $(\bar{B},\nu)$ in the topological sense. Indeed, if (B',ν') is a compact G-space and forms a (G,μ)-boundary, then the linear operator

$$\mathcal{C}(B')\overset{F_{(B',\nu')}}{\longrightarrow}\mathcal{H}^\infty_{luc}(G,\mu)\overset{F^{-1}_{(\bar{B},\nu)}}{\longrightarrow}\mathcal{C}(\bar{B})$$

preserves positivity, maps $\mathbf{1}_{B'}$ to $\mathbf{1}_{\bar{B}}$, and respects multiplication (basically due to the fact that both $(\bar{B},\nu)$ and (B',ν') are (G,μ)-boundaries !). Hence it is induced by a uniquely defined continuous map $p:\bar{B}\to B'$. It is not hard to deduce from the construction that p is G-equivariant and satisfies $p_*\nu=\nu'$ (in particular p is onto, assuming ν' has full support on B').

The fact that every bounded μ-harmonic function is a limit of l.u.c. bounded μ-harmonic ones, mentioned at the beginning of the section, can be used to deduce that as a measurable (G,μ)-boundary $(\bar{B},\nu)$ defines a bijection

$$F_{(\bar{B},\nu)}:L^\infty(\bar{B},\nu)\to\mathcal{H}^\infty(G,\mu) \tag{2.9}$$

This fact implies the universality property (and hence the uniqueness) in this measurable setup, due to the fundamental correspondence between subalgebras of $L^\infty(B,\nu)$ and measurable factors of (B,ν).

Remarks 2.16. (i) Observe, that for a discrete group G any bounded function is automatically l.u.c., so that $\mathcal{C}(\bar{B})$ coincides with $L^\infty(\bar{B},\nu)$. This means that for a discrete G with a non-trivial Poisson boundary the compact Hausdorff space $\bar{B}$ described above is non metrizable. (We have ignored some technical difficulties caused by this fact in the above arguments). However, measure-theoretically the Poisson boundary is always a Lebesgue space.

(ii) This is one of the reasons to consider the Poisson boundary as a measure-theoretical rather than topological object. A more significant, but related, reason is the fact that in many cases as a measurable G-space the Poisson boundary can be realized on some nice familiar compact (metric) G-spaces naturally associated to G. Typically these spaces are much smaller than somewhat monstrous universal topological G-space $\bar{B}$ (cf. sections 2.6, 2.10).

(iii) The pair (G, μ) gives rise yet to another *topological G-space* - the so called Martin boundary - which is the universal object responsible for the representation of all *positive* μ-harmonic functions f on G in terms of associated positive measures ν_f on Δ. The Poisson boundary for (G, μ) can be identified with $(\Delta, \nu_{\mathbf{1}})$, where $\mathbf{1}$ is the constant function on G. Interested reader is referred to the paper [33] of Kaimanovich for a discussion of the Martin boundary in this context, and to the monograph [25] by Guivarc'h, Ji and Taylor for a detailed analysis of Martin boundaries for semisimple Lie groups and symmetric spaces.

The following is one of several purely measure-theoretical construction of the Poisson boundary, described by Kaimanovich and Vershik [36] (see also a more recent paper Kaimanovich [33] for more details and discussion). Given a pair (G, μ) consider an auxiliary probability measure ρ on G and let $y = (y_0, y_1, \dots)$ be a path of a (right) random walk

$$y_0, \qquad y_n = y_{n-1}\omega_n = y_0\omega_1 \cdots \omega_n, \qquad (n \geq 1)$$

where y_0 has distribution ρ and $\omega_1, \dots$ are chosen independently with distribution μ. Let Q_ρ denote the resulting distribution of the paths $y = (y_n)_{n\geq 0}$ on the path space $Y = \{y = (y_n)\}$. If ρ is a Dirac measure at $g \in G$ denote the corresponding Q_ρ by Q_g. The group G acts on Y by left multiplication coordinate-wise, so that g takes Q_e to Q_g, and one has $\mathsf{Q}_\rho = \int \mathsf{Q}_g \, d\rho(g)$ for every ρ. Let T denote the shift $(Ty)_n = y_{n+1}$. It commutes with the G-action. If ρ is in the measure class of the Haar measure on G, then Q_ρ is quasi-invariant with respect to T and one can consider the space (X, ν_ρ) of *T-ergodic components* of (Y, Q_ρ), which comes equipped with the G-action induced by the G-action on Y. Denote by ν_g the image of Q_g under the factor map $Y \to X$ and set $\nu = \nu_e$ (one can use the assumption that μ is admissible to verify that ν_g are indeed well defined). Then $\nu_g = g\nu$ and since $T\mathsf{Q}_e = \int g\mathsf{Q}_e \, d\mu(g)$ one has $\nu = \int g\nu \, d\mu(g)$, i.e. ν is μ-stationary. It also follows from the construction that (X, ν) is a (G, μ)-boundary. In order to show that (X, ν) is the Poisson boundary, observe that for any $f \in \mathcal{H}^\infty(G, \mu)$ and Q_e-a.e. $y = (y_n)$ there exists a finite limit $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} f(\omega_1 \cdots \omega_n)$, which is clearly T-invariant and therefore defines a unique function $\tilde{f} \in L^\infty(X, \nu)$, and one has $f(e) = \int_X \tilde{f} \, d\nu$. The equivariant G-action gives

$$f(g) = \int_X \tilde{f} \, d\nu_g = \int_X \tilde{f} \, dg\nu$$

so that (X, ν) gives Poisson representation as described in Theorem 2.13, and is therefore the (unique up to measurable isomorphism) Poisson boundary for (G, μ).

In [70] Zimmer described the the Poisson boundary as (a version of) the Mackey range of of the measurable cocycle $\alpha : \mathbb{N} \times (\Omega, \mathsf{P}) \to G$ given by $\alpha(n, \omega) = \omega_n \cdots \omega_1$. This is basically the same construction as the above one (the space of the ergodic components of the shift T) considered from a different point of view. In this paper

Zimmer introduced the above mentioned important notion of *amenable actions*, in particular illustrated by the G-action on the Poisson boundary/ies.

2.5. **Semi-simple Lie groups and their lattices.** The following fundamental results of Furstenberg are naturally formulated for semisimple Lie groups G (which are always assumed to be connected, to have finite center and no non-trivial compact factors), rather than for the special case of $\mathrm{SL}_k(\mathbb{R})$ which has been our focus so far. Semisimple Lie groups G admit Iwasawa decomposition $G = KAN$, where K is a maximal compact subgroup, N is nilpotent, A is Cartan subgroup. Let $P = MAN$, where M is the centralizer of A in K, and denote by $B(G) = G/P$. This is a compact homogeneous G-space. In the case of $G = \mathrm{SL}_k(\mathbb{R})$, one can take $K = \mathrm{SO}(k)$, P - the subgroup of upper triangular matrices, and $B(G) = \mathcal{F} = \mathcal{F}_{(1,\dots,k)}$ - the full flag variety.

Theorem 2.17 (Furstenberg, [12]). *Let G be a semisimple Lie group as above, and let μ be an absolutely continuous probability measure on G, which contains the identity in the interior of its support. Then there is a unique μ-stationary probability measure ν on $B(G)$, which belongs to the Lebesgue class, and $(B(G),\nu)$ is the Poisson boundary for (G,μ). Moreover, $B(G)$ is the universal topological (G,μ)-boundary, i.e. the Poisson representation $F_{(B(G),\nu)}$ defines isometric isomorphisms*

$$L^\infty(B(G),\nu)\xrightarrow{\cong}\mathcal{H}^\infty(G,\mu) \qquad \mathcal{C}(B(G))\xrightarrow{\cong}\mathcal{H}^\infty_{luc}(G,\mu)$$

The G-equivariant factors G/Q of $B(G) = G/P$ with $P \subseteq Q$, provide the complete list of all (G,μ)-boundaries.

Remark 2.18. The theorem in particular states that semisimple groups G act transitively on their Poisson boundaries (under mild assumptions on μ). One should not expect this to be the case for general groups. However Raugi [59] and more recently Jaworski [30], [31] proved that under certain conditions other classes of groups G and measures μ have the property that G acts transitively on the associated Poisson boundary, so that the latter can be presented as a homogeneous space G/H.

Consider the particular case of absolutely continuous measures μ which satisfy $m_K * \mu = \mu$, where $K \subset G$ is a maximal compact. For such μ one has $k\mu = \mu$ for all $k \in K$, and this implies that the unique K-invariant probability measure $\nu = \nu_0$ on $B(G) = G/P$ is the only μ-stationary measure, because $\nu = \mu * \nu = k\mu * \nu = k\nu$ for all $k \in K$. Hence all K-invariant μ on G as in Theorem 2.17 give rise to the same space $\mathcal{H}^\infty(G,\mu)$ of bounded μ-harmonic functions. Such bounded μ-harmonic functions are also left K-invariant, and can be considered as lifts of bounded functions on the symmetric space $K\backslash G$. In [12] Furstenberg showed that in these cases $\mathcal{H}^\infty(G,\mu)$ (resp. $\mathcal{H}^\infty_{luc}(G,\mu)$) coincide with the spaces of bounded (resp. l.u.c. bounded) classical *harmonic functions* on the symmetric space $K\backslash G$, i.e. solutions of the Laplace-Beltrami equation. Hence the Poisson representation in Theorem 2.17 describes all classical bounded (l.u.c. bounded) harmonic functions on symmetric spaces in terms of their *boundary values* on $B(G)$. This generalizes the classical Poisson representation of bounded harmonic functions on the unit disc in terms of bounded functions on its boundary - the circle (this case corresponds to $G = \mathrm{SL}_2(\mathbb{R})$). The space $B(G)$ is often referred to as the *Furstenberg boundary*, or *Satake-Furstenberg* boundary, of a semi-simple group G.

Remark 2.19. The space $B(G)$ is one of the compactifications of the symmetric space $K\backslash G$ of G. The subject of the recent book [25] by Guivarc'h, Ji and Taylor is the connections between various compactifications of the symmetric spaces. Random walks in the form of Brownian motion on the symmetric space and Martin boundaries play a central role in their analysis.

An important theme in the theory of semisimple Lie groups is understanding the relationship between a semisimple group G and lattices Γ contained in it. Recall that a lattice Γ in G is a discrete subgroup of (generally locally compact group) G which has a measurable fundamental domain of finite Haar measure (standard example is $\Gamma = \mathrm{SL}_k(\mathbb{Z})$ in $G = \mathrm{SL}_k(\mathbb{R})$). Furstenberg introduced the use of random walks as a tool relating lattices to the ambient semisimple groups. One of such connections is given by the following:

Theorem 2.20 (Furstenberg, [15] 5.1)**.** *Let $\Gamma \subset G$ be a lattice in a semisimple Lie group G, and let ν_0 be the K-invariant probability measure on $B(G)$. Then there exists a probability measure μ on on Γ with* $\mathrm{supp}(\mu) = \Gamma$ *so that $(B(G), \nu_0)$ is the Poisson boundary for (Γ, μ).*

One of the constructions of such a measure μ on Γ, discussed in [15], is based on a certain discretization procedure applied to the continuous time Markov process - the Brownian motion - on the symmetric space $X = K\backslash G$. (Relevance of the Brownian motion is clear from the fact that $(B(G), \nu_0)$ defines all *harmonic* functions on X, which are closely related to the Brownian motion). This discretization, in particular, allows to associate a bounded μ-harmonic function $\bar{f}$ on Γ to every bounded harmonic function h on $X = K\backslash G$.

In [52] Lyons and Sullivan studied similar questions in a more general geometric situation. In particular they consider a regular cover X of a compact Riemannian manifold $M = X/\Gamma$ where Γ is a discrete group of isometries of X. In this, and more general context (see [52]), a discretization of Brownian motion on X allows to construct a probability measure μ on Γ with $\mathrm{supp}(\mu) = \Gamma$, so that every bounded harmonic function h on X defines a bounded μ-harmonic function $\bar{h}$ on Γ. On the other hand Lyons and Sullivan describe a natural projection p from $L^\infty(X)$ to the space $\mathcal{H}^\infty(X)$ of bounded harmonic functions on X. Roughly speaking, the projection of $f \in L^\infty(X)$ is obtained by applying a fixed *invariant mean* ϕ on the abelian semi-group $\mathbb{R}_+$ to the function $\mathbf{P}_t f(x) = \int_X P(t,x,y) f(y)\, dy$ where $P(t,x,y)$ denotes the transition density of the Brownian motion. In [52] it is shown that

$$\phi[\mathbf{P}_t f] = \phi[\mathbf{P}_{t+s} f] = \phi[\mathbf{P}_s \mathbf{P}_t f] \qquad \text{and} \qquad \mathbf{P}_s \phi[\mathbf{P}_t f]$$

are equal, which means that the bounded function $\hat{f}(x) = \phi[\mathbf{P}_t f(x)]$ is harmonic.

The combination of the discretization procedure and the projection gives a bijection between the spaces $\mathcal{H}^\infty(X)$ and $\mathcal{H}^\infty(\Gamma, \mu)$. This in particular allowed Lyons and Sullivan to conclude that (i) a *non-amenable* cover X of a compact manifold M always has non-trivial bounded harmonic functions, while (ii) *nilpotent* covers X of compact manifolds (and more generally ω-*nilpotent* covers of *recurrent* Riemannian manifolds) admit only constant bounded harmonic functions (see 2.35 and 2.37.(b) below).

2.6. **The Poisson boundary for discrete linear groups.** Let us return to random walks on (subgroups of) the simple Lie group $G = \mathrm{SL}_k(\mathbb{R})$. Let μ be a probability measure with finite first moment. Denote $G_\mu = \mathsf{grp}(\mu)$, and let $\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_k(\mu)$ be the Lyapunov exponents of μ. Let $1 \leq \tau_1 < \tau_2 < \cdots < \tau_r = k$ be the indices of strict inequalities: $\lambda_{\tau_i}(\mu) > \lambda_{\tau_i+1}(\mu)$, and consider the flag variety $\mathcal{F}_\tau$ of type $\tau = (\tau_1, \ldots, \tau_r)$ consisting of the flags

$$\{0\} \subset E_1 \subset E_2 \subset \cdots \subset E_r = \mathbb{R}^k \qquad \text{with} \qquad \dim E_i = \tau_i$$

Recall the cocycles $\sigma_{\tau_i} : G \times \mathcal{F}_\tau \to \mathbb{R}$ defined in (1.18) by

$$\sigma_{\tau_i}(g, \xi) = \log \frac{\| \wedge_{j=1}^{\tau_i} g u_j \|}{\| \wedge_{j=1}^{\tau_i} u_j \|}$$

where $\{u_j\}_1^k$ are any vectors such that the i-th subspace of the flag $\xi \in \mathcal{F}_\tau$ is spanned by $u_1, \ldots, u_{\tau_i}$ for $i = 1, \ldots, r$.

Theorem 2.21 (Ledrappier [47], Kaimanovich [32], [34])**.** *There exists a unique probability measure ν on $\mathcal{F}_\tau$ such that*

$$\int_G \int_{\mathcal{F}_\tau} \sigma_{\tau_i}(g, \xi)\, d\nu(\xi)\, d\mu(g) = \sum_{j=1}^{\tau_i} \lambda_j(\mu) \qquad \text{for} \qquad i = 1, \ldots, r. \tag{2.10}$$

Let $B \subseteq \mathcal{F}_\tau$ be the support of ν. Then (B, ν) is a (G_μ, μ)-boundary. Moreover, if $\mathsf{grp}(\mu)$ *is a discrete subgroup Γ in $\mathrm{SL}_k(\mathbb{R})$, then (B, ν) is the Poisson boundary for (Γ, μ).*

Remarks 2.22. Let μ be supported on a discrete subgroup $\Gamma = G_\mu$ of $\mathrm{SL}_k(\mathbb{R})$.

(a) Theorem 2.21 was proved independently by Kaimanovich [32] and Ledrappier [47]. Both proofs are based on the entropy criterion 2.31.(c), due to Kaimanovich and Vershik, for a (Γ, μ)-boundary to be the maximal, i.e. the Poisson boundary. Kaimanovich' version of the result (the precise statement of which slightly differs from the above) concerns random walks on discrete groups in semisimple Lie groups with the generating measure μ satisfying finite *logarithmic* moment condition: $\int \log\log \|g\|\, d\mu(g) < \infty$. On the other hand, Ledrappier's proof, which is based on Oseledec theorem requiring finite first moment, provides additional important information linking the Hausdorff-type dimension of the stationary measure ν on $\mathcal{F}_\tau$, the Lyapunov exponents and the random walk entropy $h(\Gamma,\mu)$ (see Remark 2.34).

(b) Assuming that μ is symmetric and has finite first moment, amenability of the group G_μ is equivalent to the triviality of the Lyapunov spectrum: $\lambda_1(\mu) = \cdots = \lambda_k(\mu) = 0$. By Theorem 2.21 this is also equivalent to the triviality of the Poisson boundary of (G_μ,μ). Compare this with Proposition 2.35, Theorem 2.36 and Remarks 2.37.

Here we shall prove only the first part of the Theorem, and will follow Ledrappier's [47]. Section 2.10 indicates Kaimanovich' argument showing that (B,ν) is the Poisson boundary for a discrete Γ.

Let $(\bar\Omega,\bar{\mathsf{P}})$ denote the bi-infinite product $(G^{\mathbb{Z}},\mu^{\mathbb{Z}})$ with the (right) shift transformation $\bar\theta$ and its inverse $\bar\theta^{-1}$ i.e. $(\bar\theta^{-1}\bar\omega)_i = \bar\omega_{i-1}$, $i\in\mathbb{Z}$. Applying Oseledec theorem to the matrix valued function $A(\bar\omega) = \bar\omega_{-1}^{-1}$ on $(\bar\Omega,\bar{\mathsf{P}},\bar\theta^{-1})$, which describes the asymptotic of the sequences of matrices

$$A_n(\bar\omega) = A(\bar\theta^{-n+1}\bar\omega)\cdots A(\bar\theta^{-1}\bar\omega)A(\bar\omega) = \bar\omega_{-n}^{-1}\cdots\bar\omega_{-1}^{-1} = (\bar\omega_{-1}\cdots\bar\omega_{-n})^{-1}$$

one obtains

(a) Lyapunov exponents $\check\lambda_1 \geq \cdots \geq \check\lambda_k$ defined by $\bar{\mathsf{P}}$-a.e. constant limits

$$\check\lambda_1 + \cdots + \check\lambda_j = \lim_{n\to\infty}\frac{1}{n}\cdot\log\|\wedge^j A_n(\bar\omega)\|$$

which are easily seen to satisfy $\check\lambda_j = -\lambda_{k-j}(\mu)$, $j=1,\ldots,k$.

(b) The measurable splitting $\mathbb{R}^k = W_1(\bar\omega)\oplus\cdots\oplus W_r(\bar\omega)$ into subspaces such that a non-zero vector v belongs to $W_i(\bar\omega)$ iff

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\log\|A_n(\bar\omega)\| &= \lim_{n\to\infty}\frac{1}{n}\log\|\bar\omega_{-n}^{-1}\cdots\bar\omega_{-1}^{-1}\| = \check\lambda_{k-\tau_i} = -\lambda_{\tau_i}(\mu)\\ \lim_{n\to\infty}\frac{1}{n}\log\|A_{-n}(\bar\omega)\| &= \lim_{n\to\infty}\frac{1}{n}\log\|\bar\omega_n\cdots\bar\omega_0\| = -\check\lambda_{k-\tau_i} = \lambda_{\tau_i}(\mu)\end{aligned}$$

These subspaces have dimensions $\dim W_i(\bar\omega) = \tau_i - \tau_{i-1}$ and satisfy $W_i(\bar\theta^{-1}\bar\omega) = \bar\omega_{-1}^{-1}W_i(\bar\omega)$, or equivalently $\bar\omega_0 W_i(\bar\omega) = W_i(\bar\theta\bar\omega)$, for $i=1,\ldots,r$.

(c) Denote by $\xi(\bar\omega)$ the flag in $\mathcal{F}_\tau$ corresponding to the filtration

$$\{0\}\subset E_1(\bar\omega)\subset E_2(\bar\omega)\subset\cdots\subset E_r(\bar\omega) = \mathbb{R}^k$$

where $E_j(\bar\omega) = \oplus_{i=1}^j W_i(\bar\omega)$. The correspondence $\bar\omega\mapsto\xi(\bar\omega)$ is in fact measurable with respect to negative coordinates $\{\ldots,\bar\omega_{-2},\bar\omega_{-1}\}$.

Consider the transformation $\bar T$ of the space $\bar\Omega\times\mathcal{F}_\tau$ defined by $\bar T(\bar\omega,\xi) = (\bar\theta\bar\omega,\bar\omega_0\xi)$ and denote by $s_i : \bar\Omega\times\mathcal{F}_\tau\to\mathbb{R}$ the functions $s_i(\bar\omega,\xi) = \sigma_{\tau_i}(\bar\omega_0,\xi)$. Let $\bar\nu$ denote the measure on $\bar\Omega\times\mathcal{F}_\tau$ defined by

$$d\bar\nu(\bar\omega,\xi) = \int_{\bar\Omega}\delta_{\bar\omega}\times\delta_{\xi(\bar\omega)}\,d\bar{\mathsf{P}}(\bar\omega)$$

that is $\bar{\nu}$ is the lifting of the measure $\bar{\mathsf{P}}$ to the graph of $\bar{\omega} \mapsto \xi(\bar{\omega})$. Using Oseledec theorem one can verify that $\bar{\nu}$ is the *unique* $\bar{T}$-invariant probability measure on $\bar{\Omega} \times \mathcal{F}_\tau$ which projects onto $\bar{\mathsf{P}}$ on $\bar{\Omega}$ and satisfies

$$\int_{\bar{\Omega}\times\mathcal{F}_\tau} s_i \, d\bar{\nu} = \lambda_1(\mu) + \cdots + \lambda_{\tau_i}(\mu) \qquad \text{for} \qquad i = 1, \ldots, r. \tag{2.11}$$

The invertible system $(\bar{\Omega}, \bar{\mathsf{P}}, \bar{\theta})$ with the shift $(\bar{\theta}\bar{\omega})_i = \bar{\omega}_{i+1}$, $i \in \mathbb{Z}$, is the *natural extension* of the one-sided shift $(\theta\omega)_i = \omega_{i+1}$, $i \in \mathbb{N}$, on (Ω, P). Consider the non-invertible transformation $T(\omega, \xi) = (\theta\omega, \omega_0\xi)$ of $\Omega \times \mathcal{F}_\tau$, and the projection

$$\pi \times \mathrm{Id} : (\bar{\Omega} \times \mathcal{F}_\tau) \to (\Omega, \mathcal{F}_\tau)$$

corresponding to the projection $\pi : \bar{\Omega} \to \Omega$ and the identity map in the $\mathcal{F}_\tau$ coordinate. One clearly has $(\pi \times \mathrm{Id}) \circ \bar{T} = T \circ (\pi \times \mathrm{Id})$, and it is easy to see that the projection $(\pi \times \mathrm{Id})_* \bar{\nu}$ of the above defined $\bar{\nu}$ is a product measure $\mathsf{P} \times \nu$ where ν is a μ-stationary measure on $\mathcal{F}_\tau$. The constructions are summarized by the following commutative diagram of ergodic transformations

$$\begin{array}{ccc} (\bar{\Omega} \times \mathcal{F}_\tau, \bar{\nu}, \bar{T}) & \xrightarrow{\pi\times\mathrm{Id}} & (\Omega \times \mathcal{F}_\tau, \mathsf{P} \times \nu, T) \\ \downarrow & & \downarrow \\ (\bar{\Omega}, \bar{\mathsf{P}}, \bar{\theta}) & \xrightarrow{\pi} & (\Omega, \mathsf{P}, \theta) \end{array}$$

where the systems on the left hand sides are the natural extensions of the non-invertible systems on the right hand sides.

It follows from the construction that ν satisfies the relation (2.10). On the other hand, any μ-stationary probability measure ν' on $\mathcal{F}_\tau$ gives rise to a T-invariant measure $\mathsf{P} \times \nu'$ on $(\Omega \times \mathcal{F}_\tau)$, the natural extension of which can be uniquely realized as $(\bar{\Omega}\times\mathcal{F}_\tau, \bar{T}, \bar{\nu}')$ where $\bar{\nu}'$ is a $\bar{T}$-invariant probability measure $\bar{\nu}'$ with $(\pi\times\mathrm{Id})_*\bar{\nu}' = \mathsf{P}\times\nu'$. Since the measure $\bar{\nu}$ is the unique $\bar{T}$-invariant probability measure satisfying (2.11), the uniqueness of ν as in the proposition is established.

Finally, the fact the disintegration of $\bar{\nu}$ with respect to $\bar{\mathsf{P}}$ consists of Dirac measures at $\xi(\bar{\omega})$ means precisely (Proposition 2.11) that (B, ν) is a (G_μ, μ)-boundary. □

2.7. **Boundary entropy.** Let us return to the general setup of an admissible probability measure μ on a locally compact group G. Consider a measurable (G, μ)-space (M, ν), which is not necessarily assumed to be a (G, μ)-boundary.

As we have already mentioned, the assumption that μ is admissible implies that ν is G-quasi-invariant, which enables to define the following notion of *boundary entropy*:

$$h_\mu(M, \nu) = \int_G \int_M - \log \frac{dg^{-1}\nu}{d\nu}(x) \, d\nu(x) \, d\mu(g) \tag{2.12}$$

provided the integral is defined. The boundary entropy (also known as *Furstenberg entropy*) was introduced by Furstenberg [13] as a quantity measuring the extent of non-invariance of a μ-stationary measure ν. Indeed, $h_\mu(M,\nu)$ takes non-negative values

$$0 \leq h_\mu(M,\nu) \leq \infty$$

with $h_\mu(M,\nu) = 0$ iff ν is G-invariant. This follows from the strict convexity of $-\log x$, because

$$h_\mu(M,\nu) \geq \int_G -\log\left(\int_M \frac{dg^{-1}\nu}{d\nu}(x)\, d\nu(x)\right) d\mu(g) = \int_G \log 1\, d\mu(g) = 0$$

with the equality corresponding to $\rho_\nu(g,x) = -\log dg^{-1}\nu/d\nu(x)$ being ν-a.e. constant for μ-a.e. $g \in G$, which is equivalent to ν being G-invariant.

Remark 2.23. The quantity (2.12) can be defined for quasi-invariant probability measure ν, which are not necessarily μ-stationary. The computed value remains constant over all measures (with a suitable integrability condition) from a given measure class precisely if this measure class contains a μ-stationary measure (see Nevo-Zimmer [55]).

Proposition 2.24 ([36]). *Let (M,ν) be a (G,μ)-space. Then for any convolution power μ^p of μ the measure ν is μ^p-stationary and $h_{(\mu^p)}(M,\nu) = p\cdot h_\mu(M,\nu)$.*

Proof. The function $\rho_\nu : G\times M \to \mathbb{R}$ is a cocycle, i.e. satisfies $\rho_\nu(gh,x) = \rho_\nu(g,h\cdot x) + \rho_\nu(h,x)$. Thus for every $p\in\mathbb{N}$ one has

$$\begin{aligned}
h_{(\mu^p)}(M,\nu) &= \int_G\int_M \rho_\nu(g,x)\, d\nu(x)\, d\mu^p(g) \\
&= \int_G\int_M \rho_\nu(g,h\cdot x)\, d\nu(x)\, d\mu(h)\, d\mu^{p-1}(g) + \int_G\int_M \rho_\nu(h,x)\, d\nu(x)\, d\mu(h) \\
&= \int_G\int_M \rho_\nu(g,y)\, d\nu(y)\, d\mu^{p-1}(g) + h_\mu(M,\nu) \\
&= h_{(\mu^{p-1})}(M,\nu) + h_\mu(M,\nu) = \cdots = p\cdot h_\mu(M,\nu)
\end{aligned}$$

□

Proposition 2.25 ([36]). *Let $p : (M,\nu) \to (M_0,\nu_0)$ be a G-equivariant measurable map of (G,μ)-spaces, then $h_\mu(M,\nu) \geq h_\mu(M_0,\nu_0)$, with the equality achieved iff the quotient is* relatively measure-preserving, *i.e. $g\nu_x = \nu_{g\cdot x}$ for ν_0-a.e. $x\in M_0$ where $\nu = \int_{M_0} \nu_x\, d\nu_0(x)$ is the disintegration of ν with respect to ν_0.*

This statement follows from the cocycle property of Radon-Nikodym derivatives, relations $\mu * \nu = \nu$ and $\mu * \nu_0 = \nu_0$ and strong convexity of $-\log(\cdot)$.

Given a (G,μ)-space (M,ν) one can consider the quasi-regular G-representation π_ν of G on $L^2(M,\nu)$. Then one has the estimate

$$\begin{aligned} h_\mu(M,\nu) &= \int_G\int_M -\log\frac{dg^{-1}\nu}{d\nu}(x)\,d\nu(x)\,d\mu(g) \\ &\geq -2\log\left(\int_G\int_M\sqrt{\frac{dg^{-1}\nu}{d\nu}(x)}\,d\nu(x)\,d\mu(g)\right) \\ &= -2\log\langle\mathbf{1},\,\pi_\nu(\check{\mu})\mathbf{1}\rangle = -2\log\langle\pi_\nu(\mu)\mathbf{1},\,\mathbf{1}\rangle \geq 2\log\frac{1}{\|\pi_\nu(\mu)\|} \end{aligned}$$

which can be further improved by replacing $\|\pi_\nu(\mu)\|$ by the spectral radius $\|\pi_\nu(\mu)\|_{\mathrm{sp}} = \lim_{n\to\infty}\|\pi_\nu(\mu^p)\|^{1/p}$. This estimate together with Proposition 1.17 gives

Proposition 2.26. *Let M be a compact metric G-space which has no G-invariant probability measures. Then for any admissible measure μ on G*

$$\inf_{\mu*\nu=\nu} h_\mu(M,\nu) > 0$$

Remark 2.27. Recall Proposition 1.3 describing the measure-preserving skew-product system $(\Omega\times M, \mathsf{P}\times M, T)$. Assume that the logarithmic Radon-Nikodym derivative $\rho_\nu(g,x) = -\log dg^{-1}\nu/d\nu(x)$ is in $L^1(\mathsf{P}\times\nu)$. Since T-invariant sets can be only of the form $\Omega\times E$, $E\subseteq M$, the Ergodic Theorem implies that for P-a.e. ω:

$$\begin{aligned} h_\mu(M,\nu) &= \int_{\Omega\times M}\rho_\nu\,d\mathsf{P}\times\nu = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\int_M\rho_\nu(\theta^k\omega,\omega_k\cdots\omega_1 x)\,d\nu(x) \\ &= \lim_{n\to\infty}\frac{1}{n}\int_M\rho_\nu(\omega_n\cdots\omega_1,x)\,d\nu(x) \\ &= \lim_{n\to\infty}\frac{1}{n}\int_M -\log\frac{d(\omega_n\cdots\omega_1)^{-1}\nu}{d\nu}(x)\,d\nu(x) \end{aligned}$$

If the skew-product is ergodic then one has pointwise convergence

$$\lim_{n\to\infty} -\frac{1}{n}\log\frac{d(\omega_n\cdots\omega_1)^{-1}\nu}{d\nu}(x) = h_\mu(M,\nu)$$

Observe the similarity between these expressions and the formulae for the Lyapunov exponents (Theorem 1.8 and (1.12)). In particular, if G is a strongly irreducible subgroup of $\mathrm{SL}_k(\mathbb{R})$, μ - a probability measure on G with finite first moment, such that the Lebesgue measure ν_0 on $M = \mathbb{P}^{k-1}$ is μ-stationary, then

$$\begin{aligned} \lambda_1(\mu) &= \int_G\int_{\mathbb{P}^{k-1}}\log\frac{\|gu\|}{\|u\|}\,d\nu_0(\bar{u})\,d\mu(g) \\ &= k^{-1}\cdot\int_G\int_{\mathbb{P}^{k-1}} -\log\frac{dg^{-1}\nu_0}{d\nu_0}(\bar{u})\,d\nu_0(\bar{u})\,d\mu(g) \\ &= k^{-1}\cdot h_\mu(\mathbb{P}^{k-1},\nu_0) \end{aligned}$$

One can check that these equalities remain true if there is a μ-stationary measure in the Lebesgue *class* $[\nu_0]$.

2.8. **Random Walk Entropy.** In this section it will be more convenient to restrict our attention to discrete groups which will be denoted by Γ. For μ to be admissible on Γ just means that $\Gamma = \mathsf{sgr}(\mu)$, however in the current context this assumption can be relaxed to $\Gamma = \mathsf{grp}(\mu)$. Furthermore, some of the statements and proofs can be reformulated for general locally compact groups.

Given an arbitrary probability measure η on Γ define the entropy of η to be

$$H(\eta) = \sum_{\gamma\in\Gamma} -\eta(\gamma)\cdot\log\eta(\gamma)$$

with the usual convention $0\cdot\log 0 = 0$. We shall say that η has finite entropy if $H(\eta) < \infty$. Given two probability measures η_1, η_2 on Γ the *convolution* probability measure $\eta_1 * \eta_2$ is the image of $\eta_1 \times \eta_2 \in \mathcal{P}(\Gamma\times\Gamma)$ under the map $(g_1, g_2) \mapsto g_1 g_2$. The fact that the function $-x\cdot\log x$ is strictly convex, implies that

$$H(\eta_1 * \eta_2) \le H(\eta_1) + H(\eta_2) \qquad \text{for} \qquad \eta_1, \eta_2 \in \mathcal{P}(\Gamma) \tag{2.13}$$

and if both η_1, η_2 have finite entropies then the equality in (2.13) is achieved iff every $g \in \mathrm{supp}(\eta_1 * \eta_2)$ has a unique factorization $g = g_1 g_2$ with $g_i \in \mathrm{supp}(\eta_i)$, $i = 1, 2$.

Now consider the sequence of convolutions μ^n of the given probability measure μ on Γ. The subadditivity property (2.13) allows to define (Avez [2], Kaimanovich-Vershik [36]) the following *random walk entropy*

$$h(\Gamma,\mu) = \lim_{n\to\infty}\frac{1}{n}H(\mu^n) = \inf_{n\ge 1}\frac{1}{n}H(\mu^n)$$

where μ^n is the n-th convolution power of μ. Note that if $H(\mu) < \infty$ then $h(\Gamma,\mu) \le H(\mu)$ and the the equality is achieved iff for each $n \ge 1$ every $g \in \mathrm{supp}(\mu^n)$ has a unique factorization $g = g_1 \cdots g_n$ with $g_1, \dots, g_n \in \mathrm{supp}(\mu)$. Thus $h(\Gamma,\mu)$ describes the average information about the steps $X_1, \dots, X_n$ of a typical random walk, given its position of the product $X_1 \cdots X_n$.

The following analogue of Shannon-McMillan-Breiman theorem states that the μ^n-weights assigned to μ-random walks at time n are asymptotically equal, on the logarithmic scale. More precisely

Theorem 2.28 (Kaimanovich-Vershik [36], Derriennic [6])**.** *Let Γ be a discrete countable group, and μ be a probability measure with $H(\mu) < \infty$. Denote by $y_n(\omega) = \omega_1 \cdots \omega_n$, $\omega \in \Omega$, the path of the (right) μ-random walk on Γ. Then*

$$\lim_{n\to\infty} -\frac{1}{n}\log\mu^n(y_n(\omega)) = h(g,\mu)$$

the convergence being both pointwise w.r.to P and in $L^1(\Omega,\mathsf{P})$.

Proof. Observe that for every $n, m \in \mathbb{N}$ and every sequence $x = (x_i)_{i=1}^{\infty}$ in Γ one has $\mu^{m+n}(x_1 \cdots x_{n+m}) \ge \mu^n(x_1 \cdots x_n)\cdot\mu^m(x_{n+1}\cdots x_{n+m})$. Therefore the sequence of functions $f_n(\omega) = -\log \mu^n(y_n(\omega))$ satisfies $f_{n+m}(\omega) \le f_n(\omega) + f_m(\theta^n\omega)$, i.e. $\{f_n\}$ is a subadditive sequence over the ergodic system $(\Omega, \mathsf{P}, \theta)$. The finite entropy condition gives $f_1 \in L^1(\Omega, \mathsf{P})$ and Kingman's subadditive ergodic theorem completes the proof. □

Remark 2.29. One might view this result as the statement that the μ-random walk on Γ has the number $h(G, \mu)$ as its "large scale Hausdorff dimension", by which we mean the following. Denoting $h = h(\Gamma, \mu)$

(a) There exists a sequence of subsets $A_n \subset \Gamma$ with $\log |A_n| \le (h + \epsilon)n$, such that with P-probability one $g_n(\omega) \in A_n$ for all, but finitely many, n-s.
(b) For any sequence of subsets $B_n \subset \Gamma$ with $\log |B_n| \le (h - \epsilon)n$, with P-probability one $g_n(\omega) \in B_n$ for at most finitely many n-s.

Remark 2.30. In [49] (see also [50]) Ledrappier introduced the notion of *entropy profile* for a symmetric probability measure μ on Γ with finite entropy. Assume that $\mu^n(e) > 0$ for all n (this is always true for $\mu' = \mu * \mu$ when μ is symmetric). Given such a μ consider the paths of μ-random walk $y_n(\omega) = \omega_1 \cdots \omega_n$. Ledrappier proves that there exists a *convex* function $\beta_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ such that for P-a.e. ω for every $t \ge 0$ one has

$$\lim_{n\to\infty} -\log \mu^n(y_{[tn]}(\omega)) = \beta_\mu(t) \tag{2.14}$$

Observe that the entropy profile connects the spectral radius to the random walk entropy in a very natural way:

$$\beta_\mu(0) = -\log \|\pi_{\mathrm{reg}}(\mu)\| \qquad \text{and} \qquad \beta_\mu(1) = h(\Gamma, \mu)$$

See Theorem 2.33 below for further properties.

The following fundamental facts relate the entropy of random walk to the boundary entropy:

Theorem 2.31 (Kaimanovich-Vershik, [36])**.** *Let μ be a probability measure on a discrete countable group Γ with finite entropy. Then*

(a) *For any (Γ, μ)-space (M, ν) one has the inequality $h_\mu(M, \nu) \le h(\Gamma, \mu)$.*
(b) *For the Poisson boundary (B, ν) the equality holds $h_\mu(B, \nu) = h(\Gamma, \mu)$.*
(c) *If a (Γ, μ)-boundary (B', ν') satisfies $h_\mu(B', \nu') = h(\Gamma, \mu)$, then (B', ν') is (measurably isomorphic to) the Poisson boundary (B, ν).*
(d) *The Poisson boundary of (Γ, μ) is trivial if and only if $h(\Gamma, \mu) = 0$.*

Proof. Since $\nu = \mu * \nu = \sum_{g\in\Gamma} \mu(g) \cdot g\nu$ one has $dg\nu/d\nu(x) \le \mu(g)^{-1}$ for ν-a.e. $x \in M$. Thus the cocycle $\rho_\nu(g, x) = -\log dg^{-1}\nu/d\nu(x)$ satisfies for every $g \in \Gamma$ and ν-a.e. $x \in M$

$$\rho_\nu(g, x) = -\rho_\nu(g^{-1}, g \cdot x) = \log \frac{dg\nu}{d\nu}(g \cdot x) \le -\log \mu(g)$$

so that $h_\mu(M,\nu) \le \sum_{g\in\Gamma} -\mu(g)\log\mu(g) = H(\mu)$ and

$$h_\mu(M,\nu) = \frac{1}{n} h_{(\mu^n)}(M,\nu) \le \frac{1}{n} H(\mu^n) \to h(\Gamma,\mu)$$

which proves (a).

For the proof of (b) recall the Kaimanovich-Vershik construction of the Poisson boundary (B,ν) as the space of ergodic components of the shift T on (Y,Q_e) (section 2.4). One can also view (B,ν) as the factor $(Y,\mathsf{Q}_e)/\tau$ of (Y,Q_e) defined by the limit σ-algebra $\tau \lim_{n\to\infty} \tau_n$ of the decreasing sequence of measurable partitions τ_n, where $y \sim y' \mod (\tau_n)$ iff $y_k = y'_k$ for all $k \ge n$, or equivalently if $T^n y = T^n y'$. Denote by α_1 the partition of Y defined by the value of y_1. Let $\alpha_1(y)$, $\tau_n(y)$ denote the equivalence classes containing y with respect to the corresponding partitions. The computation of the conditional probabilities

$$\mathsf{Q}_e\{\alpha_1(y) \mid \tau_n(y)\} = \mu(y_1)\cdot\mu^{n-1}(y_1^{-1}y_n)/\mu^n(y_n)$$

gives the following relation on conditional entropies

$$H(\alpha_1 \mid \tau_n) = \int_Y -\log \mathsf{Q}_e\{\alpha_1(y) \mid \tau_n(y)\}\, d\mathsf{Q}_e(y) = H(\mu) + H(\mu^{n-1}) - H(\mu^n)$$

Since $\tau_{n+1} \prec \tau_n$ one has $H(\alpha_1 \mid \tau_n) \le H(\alpha_1 \mid \tau_{n+1})$ which means that the sequence $H(\mu^{n+1}) - H(\mu^n)$ is decreasing. Therefore

$$\begin{aligned} h(\Gamma,\mu) &= \lim_{n\to\infty} \frac{1}{n} H(\mu^n) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(H(\mu^k) - H(\mu^{k-1})\right) \\ &= \lim_{n\to\infty} \left(H(\mu^n) - H(\mu^{n-1})\right) \end{aligned}$$

so that

$$H(\alpha_1 \mid \tau) = \lim_{n\to\infty} H(\alpha_1 \mid \tau_n) = H(\mu) - \lim_{n\to\infty} \left(H(\mu^{n-1}) - H(\mu^n)\right) = H(\mu) - h(\Gamma,\mu)$$

In the following computation we shall denote by $b \in B$ the image of $y \in Y$ under the projection $(Y, \mathsf{Q}_e) \to (Y, \mathsf{Q}_e)/\tau = (B, \nu)$.

$$\begin{aligned}
h(\Gamma,\mu) &= H(\mu) - H(\alpha_1 \mid \tau) \\
&= \sum_{g\in\Gamma} -\mu(g)\log\mu(g) + \sum_{g\in\Gamma}\mu(g)\cdot\int_Y \log \mathsf{Q}_e(y_1 = g \mid \tau(y))\, d\mathsf{Q}_e(y) \\
&= \sum_{g\in\Gamma} -\mu(g)\log\mu(g) + \sum_{g\in\Gamma}\mu(g)\cdot\int_B \log\left(\mu(g)\frac{dg\nu}{d\nu}(b)\right) d\nu(b) \\
&= \sum_{g\in\Gamma}\mu(g)\int_B \log\frac{dg\nu}{d\nu}(b)\, d\nu(b) \\
&= \sum_{g\in\Gamma}\mu(g)\cdot\int_B -\log\frac{dg^{-1}g\nu}{dg\nu}(b)\, d\nu(b) \\
&= \sum_{g\in\Gamma}\mu(g)\cdot\int_B -\log\frac{dg^{-1}\nu}{d\nu}(b')\, d\nu(b') = h_\mu(B,\nu)
\end{aligned}$$

proving (b).

Claim (c). By the universality of the Poisson boundary (B, ν) there is a measurable Γ-equivariant map $p : (B, \nu) \to (B', \nu')$. Hence (a) and Proposition 2.25 imply

$$h(\Gamma,\mu) \geq h_\mu(B,\nu) \geq h_\mu(B',\nu') \geq h(\Gamma,\mu)$$

i.e. all the quantities are equal. Thus $p : (B, \nu) \to (B', \nu')$ is relatively measure-preserving, and using the fact that (B, ν) is a boundary one concludes that the disintegration of ν with respect to ν' consists of Dirac measures, i.e. p is an isomorphism.

Claim (d). If $h(\Gamma, \mu) = 0$ then $h_\mu(B, \nu) = 0$ by (a) and hence (B, ν) is trivial. The opposite implication follows from (b). □

We conclude with some general inequalities.

Proposition (*) 2.32 ([36]). *Let Γ be a finitely generated with a left invariant metric d, and let μ be a probability measure on Γ with finite first moment. Then μ has finite entropy and*

$$h(\Gamma,\mu) \leq \delta(\Gamma,d)\cdot\lambda^{(d)}(\mu)$$

where $\delta(\Gamma, d) = \lim_{n\to\infty} n^{-1} \log |\{g \in \Gamma \mid d(g, e) \leq n\}|$ denotes the growth exponent of Γ, and $\lambda^{(d)}(\mu) = \lim_{n\to\infty} d(S_n, e)/n$ - the escape rate.

Theorem 2.33 (Ledrappier, [49] see also [50]). *Let μ be a symmetric probability measure μ with finite entropy on Γ. Denote $r_\mu = \|\pi_{\mathrm{reg}}(\mu)\|$. Then*

$$4(1 - r_\mu) \leq h(\Gamma,\mu)$$

If, furthermore, there is a semigroup μ_t, $t > 0$, of symmetric measures so that $\mu_1 = \mu$, then

$$4\log(1/r_\mu) \leq h(\Gamma, \mu)$$

The entropy profile $\beta_\mu(t)$, defined in 2.30, is asymptotic to $\log(1/r_\mu) \cdot t + cnst$ as $t \to \infty$.

Remark 2.34. Let $\Gamma = \mathsf{grp}(\mu)$ be a discrete subgroup of $\mathrm{SL}_k(\mathbb{R})$ where μ is a probability measure with finite first moment. In [47] Ledrappier established a relationship between Lyapunov exponents, the entropy $h(\Gamma, \mu)$ and Hausdorff-like dimension $\dim(\nu)$ of the μ-stationary measure ν on the natural boundary $\mathcal{F}_\tau$. In the simplest case of $k = 2$ where $\mathcal{F}_\tau = \mathbb{P}^1$ (unless $\lambda_1(\mu) = 0$) the "dimension" of the stationary measure is

$$\dim(\nu) = \lim_{\delta\to 0} \limsup_{\epsilon\to 0} \frac{\log N(\epsilon, \delta, \nu)}{\log(1/\epsilon)}$$

where $N(\epsilon, \delta, \nu)$ is the minimal number of ϵ-intervals needed to cover $(1 - \delta)$ mass of ν on $\mathbb{P}^1$. In this case Ledrappier proves the inequalities

$$h(\Gamma, \mu) \leq 2\lambda_1(\mu) \cdot \dim(\nu) \leq h_\mu(\mathbb{P}^1, \nu) \tag{2.15}$$

where the discreteness assumption on $\Gamma = \mathsf{grp}(\mu) \subseteq \mathrm{SL}_2(\mathbb{R})$ is used only in the proof of the first inequality. Since one always has $h(\Gamma, \mu) \geq h_\mu(\mathbb{P}^1, \nu)$, there is an equality throughout in (2.15) which in particular shows that (B, ν) is the Poisson boundary for (Γ, μ) where $B = \mathrm{supp}(\nu) \subseteq \mathbb{P}^1$.

2.9. **Triviality of the Poisson boundary.** Let G be a non-amenable locally compact group, and let μ be an admissible probability measure on G. Non-amenability of G is equivalent to the existence of a compact (metric) G-space M which has no G-invariant probability measures on it. Any such space supports a μ-stationary measure ν, and by Proposition 2.9 the transform $F_{(M,\nu)}$ gives rise to a family of l.u.c. bounded μ-harmonic functions on G, not all of which are constant. This proves:

Proposition 2.35 (Furstenberg)**.** *For any admissible (or just generating) probability measure μ on a locally compact non-amenable group G, there exist non-constant bounded l.u.c. μ-harmonic functions.*

There exist many examples of *amenable* groups (as small as two-step solvable) with probability measures μ which have non-constant μ-harmonic functions. Furstenberg conjectured that for any amenable group G there *exists* a probability measure μ with $\mathrm{supp}(\mu) = G$, so that G has no non-constant bounded μ-harmonic functions. This conjecture was proved independently by Kaimanovich-Vershik and Rosenblatt:

Theorem 2.36 (Kaimanovich-Vershik [36], Rosenblatt [60])**.** *Let G be a locally compact amenable group. Then there exists a symmetric admissible probability measure μ with* $\mathrm{supp}(\mu) = G$ *such that any bounded μ-harmonic function on G is constant.*

In [36] Kaimanovich and Vershik give very interesting examples of finitely generated solvable groups (certain wreath products) for which any generating *symmetric* μ with *finite support* has $h(\Gamma,\mu)>0$, i.e. admits non constant bounded μ-harmonic functions.

Remarks 2.37. Let us point out some situations where a group Γ has no non-constant bounded μ-harmonic functions, i.e. the Poisson boundary of (Γ,μ) is trivial:

(a) Groups Γ of *sub-exponential* growth (i.e. those with $\delta(\Gamma,d)=0$) and measures μ with a finite first moment (Proposition 2.32). Recall that besides finitely generated nilpotent groups, which have polynomial growth, there are many finitely generated groups, constructed by Grigorchuck, with so called *intermediate* which is faster than polynomial but slower than exponential.
(b) Groups of *polynomial* growth and *any* probability measure μ. By the famous theorem of Gromov such groups are virtually nilpotent. For nilpotent groups the Poisson boundary is known to be trivial (cf. Furstenberg [14] 11.2, see also Margulis [53] for the corresponding results on positive μ-harmonic functions).
(c) General finitely generated groups Γ and measures μ with finite entropy, finite fist moment and zero escape rate: $\lambda^{(d)}(\mu)=0$ (Proposition 2.32). In fact, Varopoulos proved in [66] that for *finitely supported symmetric* measures μ on a general discrete group Γ, the condition $\lambda^{(d)}(\mu)=0$ not only implies, but is actually equivalent to $h(\Gamma,\mu)=0$.

2.10. **Identification of the Poisson boundary.** As we have seen, the Poisson boundary (B,ν) associated to a pair (G,μ) allows to describe all bounded μ-harmonic functions on G. In order to make this description tangible one seeks to realize this abstract measurable G-space (B,ν) on some concrete (topological) G-spaces naturally associated to G. Theorems 2.17 and Theorem 2.21 are examples of such realizations. In these examples the Poisson boundary is realized on some boundary $\partial G=\bar{G}\setminus G$ associated to an appropriate compactification $\bar{G}$ of G, i.e. a compact G-space which has an open dense G-orbit homeomorphic to G.

In many examples a locally compact (or discrete) group G with, say an admissible, probability measure μ, has a natural compactification $\bar{G}$ such that $\partial G=\bar{G}\setminus G$ supports a unique stationary measure ν on ∂G, so that $(\partial G,\nu)$ forms a (G,μ)-boundary. In such a situation a.e. path of μ-random walk $\omega_1\omega_2\cdots\omega_n$ on G converges in $\bar{G}$ to a unique point $\xi=\xi(\omega)\in\partial G$, where $X_1\cdots X_n\nu\to\delta_{\xi(\omega)}\in\mathcal{P}(\partial G)$. (Observe also that, by Lemma 1.33, the same limit point $\xi=\xi(\omega)$ is obtained as the limit $\omega_1\cdots\omega_n g\to\xi_\omega\in\partial G$ for every $g\in G$). In such a situation one would like to know whether $(\partial G,\nu)$ is (measurably) the *maximal* (G,μ)-boundary, i.e. the Poisson boundary. In [34] and [33] Kaimanovich developed very powerful geometric criteria for maximality of such a (G,μ)-boundaries. Consider the discrete situation. Let Γ be a finitely generated group with a left invariant word metric d. For $r\geq 0$ let $\mathsf{B}_r=\{g\in\Gamma\mid d(g,e)\leq r\}$ denote the centered ball of radius r in Γ.

Theorem 2.38 (**Strip Approximation**, Kaimanovich [34] 14.4)**.** *Let μ be a finite entropy probability measure on Γ as above. Assume that (B_+, ν_+) and (B_-, ν_-) are boundaries for (Γ, μ) and $(\Gamma, \check{\mu})$ respectively, where $\check{\mu}(g) = \mu(g^{-1})$. Assume that there exists a Γ-equivariant map S, assigning to $\nu_- \times \nu_+$-a.e. pair $(\xi_-, \xi_+) \in B_- \times B_+$ a non-empty subset $\mathsf{S}(\xi_-, \xi_+)$ of Γ, such that for $\nu_- \times \nu_+$-a.e. (ξ_-, ξ_+)*

$$\frac{1}{n} \log \left| \mathsf{S}(\xi_-, \xi_+) \cap \mathsf{B}_{d(g_n(\omega), e)} \right| \to 0$$

in probability with respect to P, where $g_n(\omega) = \omega_1 \cdots \omega_n$. Then (B_-, ν_-) and (B_+, ν_+) are the Poisson boundaries for (Γ, μ) and $(\Gamma, \check{\mu})$ respectively.

The proof of this criterion is based on Theorem 2.31.(c). Applicability of the criterion depends on the existence of the "strips" $\mathsf{S}(\xi_-, \xi_+)$ with sub-exponential intersection with the centered balls $\mathsf{B}_{d(g_n(\omega, e))}$. If μ has a finite first moment then $d(g_n(\omega), e)$ has a linear growth. Hence if one can find "strips" $\mathsf{S}(\xi_-, \xi_+)$ with sub-exponential growth the condition of the theorem would be satisfied. However, if one can find "strips" $\mathsf{S}(\xi_-, \xi_+)$ with a polynomial growth, the condition applies to a larger class of measures μ, namely those with a finite *first logarithmic moment*:

$$\sum \log d(g, e)\, d\mu(g) < \infty$$

In [34] Kaimanovich used this strip approximation criterion (and other related techniques) to realize the Poisson boundaries (B, ν) for several classes of groups Γ on certain boundaries $\partial\Gamma$ associated to some natural compactifications $\bar{\Gamma}$ of Γ. Here is a brief list of such results (see [34] for details):

1. Word hyperbolic groups Γ and μ with finite entropy, finite first logarithmic moment and non-elementary $\Gamma_\mu = \mathsf{grp}(\mu)$. The relevant "geometric boundary" $\partial\Gamma$ is the hyperbolic (ideal) boundary.
2. Groups Γ with infinitely many ends, and measures μ with finite entropy, finite first logarithmic moment and non-elementary $\Gamma_\mu = \mathsf{grp}(\mu)$. The relevant "geometric boundary" $\partial\Gamma$ is the the space of ends $\mathcal{E}(\Gamma)$.
3. Cocompact discrete groups Γ of isometries of rank one Cartan-Hadamard manifolds, and measures μ with finite first logarithmic moment, finite entropy and $\mathsf{grp}(\mu) = \Gamma$. The relevant "geometric boundary" $\partial\Gamma$ is the the visual boundary, homeomorphic to a sphere (see also Ballmann-Ledrappier [4]).
4. Non elementary subgroups in the Mapping Class Groups (joint work of Kaimanovich and Masur [35]). The "geometric boundary" in this context is Thurston's compactification of the Teichmüller space. This identification of the Poisson boundary allowed Kaimanovich and Masur to prove that a non-elementary subgroup of a mapping class group cannot be isomorphic to a higher rank lattice.
5. Discrete subgroups Γ of $\mathrm{SL}_k(\mathbb{R})$ and measures μ with finite entropy, *finite first logarithmic moments* and $\mathsf{grp}(\mu) = \Gamma$. The relevant boundary is an appropriate flag variety (Stake-Furstenberg compactification) (see Theorem 2.21 and the following remarks).

In [34] one can also find descriptions of the Poisson boundary for several other classes of groups, including polycyclic groups, some solvable groups and wreath products.

2.11. **Towards a Structure Theory for (G, μ)-spaces.** Measures, especially invariant ones, have proved to be an important tool in the study of group actions on topological spaces, and manifolds in particular. Since actions of non-amenable groups G on compact spaces do not always have invariant probability measures, but always admit stationary measures (for any probability measure on the group), it is highly desirable to develop a structure theory for (G, μ)-spaces, at least in the measurable category. This goal is still beyond reach, but one can speculate that (G, μ)-boundaries and measure-preserving systems are the potential building blocks of such a structure theory.

Let (M, ν) be a (measurable) (G, μ)-space where μ is an admissible measure on a locally compact group G. Let $\mathcal{B}$ denote the Lebesgue (complete) σ-algebra of (M, ν) and let $\mathcal{B}_{RN}$ denote the smallest complete sub-σ-algebra $\mathcal{B}_{RN}$ of $\mathcal{B}$ with respect to which the Radon-Nikodym derivatives $r_\nu(g, \cdot) = dg^{-1}\nu/d\nu(\cdot)$, $g \in G$, are all measurable. This σ-algebra $\mathcal{B}_{RN}$ is G-invariant (this basically follows from the cocycle property of r_ν) and therefore gives rise to a measurable G-equivariant factor (G, μ)-space

$$p : (M, \nu) \to (M_{RN}, \nu_{RN})$$

which is called the *Radon-Nikodym* factor. Radon-Nikodym factor was introduced by Kaimanovich and Vershik in [36] in the discrete groups setting. For the details of the construction in general case see Nevo and Zimmer in [55]. The Radon-Nikodym factor has the following nice properties with respect to boundary entropy:

Proposition 2.39 (Kaimanovich-Vershik [36], Nevo-Zimmer [55])**.** *Let (M, ν) be a (G, μ)-space with finite boundary entropy, and let (M_{RN}, ν_{RN}) denote the Radon-Nikodym factor. Then*

- (i) *$h_\mu(M, \nu) = h_\mu(M_{RN}, \nu_{RN})$ so that $p : (M, \nu) \to (M_{RN}, \nu_{RN})$ is relatively measure-preserving extension of G-spaces.*
- (ii) *Every non-trivial G-equivariant factor (M_0, ν_0) of the Radon-Nikodym factor (M_{RN}, ν_{RN}) has strictly smaller boundary entropy: $h_\mu(M_0, \nu_0) < h_\mu(M, \nu)$.*

In a sense the Radon-Nikodym factor filters out (the outer layer of) the measure-preserving part of the action.

Recently Nevo and Zimmer studied the structure of (G, μ)-spaces (M, ν) where G is a semisimple Lie group, μ is an admissible measure μ on G, and (M, ν) has finite *positive* boundary entropy, i.e. ν is not G-invariant. Let us just mention the following (the reader is referred to Nevo-Zimmer [55], [56] for further results):

- (i) If G is a simple Lie group with $\mathrm{rk}_{\mathbb{R}}(G) = 1$, there exists a (G, μ)-space (M, ν) with $h_\mu(M, \nu) > 0$ which does not have any non-trivial (G, μ)-boundary as a factor. In particular, for such spaces the Radon-Nikodym factor is not a (G, μ)-boundary (compare with [36] 3.6).

(ii) For (semi)simple Lie group G with $\mathrm{rk}_{\mathbb{R}}(G) \geq 2$ any (G, μ)-space (M, ν) with $h_\mu(M, \nu) > 0$ has a non-trivial measurable boundary factor $(M, \nu) \to (G/Q, \nu_{G/Q})$; moreover under some necessary assumptions (M, ν) has a boundary factor G/Q with the full boundary entropy $h_\mu(M, \nu) = h_\mu(G/Q, \nu_{G/Q})$, in which case the G-action on (M, ν) is an induced G-action from G/Q by a *measure-preserving* Q-action on some space (Y, η).

These results turn out to be closely related to Margulis' Normal Subgroup Theorem !

3. Random Transformations

Let Φ be a family of transformations of a space X, μ a probability measure on Φ, and $\{\phi_n, n \geq 0\}$ a sequence of independent Φ-valued μ-distributed random variables defined on the product probability space $(\Omega, \mathsf{P}) = (\Phi^{\mathbb{N}}, \mu^{\mathbb{N}})$. Let

$$S_n(\omega) = \phi_n(\omega) \circ \cdots \circ \phi_2(\omega) \circ \phi_1(\omega)$$

be the corresponding product of transformations. The subject of our discussion will be the typical (with respect to P) behavior of the sequence $\{S_n(\omega), n \geq 0\}$ of transformations of the space X. (To make this setup precise, a measurable structure should be introduced on Φ and X so that the action $\Phi \times X \to X$ is measurable). If the transformations $\phi \in \Phi$ are invertible, as we shall assume for convenience, one can talk about the *group* G of transformations of X, generated by the family Φ. Thus $\{S_n(\omega)\}$ become paths of μ-random walks on G. In the previous chapters we focused on the connections between the structure of the group G and properties of the random walks S_n on it, while here we shall be interested in understanding the *action* of G on X using the properties of random *transformations* S_n.

3.1. **The Random Ergodic Theorem.**

Let (X, ν) be a probability space, $\Phi = \{\phi\}$ a family of measure class preserving transformations of (X, ν), μ - a probability measure on Φ such that $(\phi, x) \mapsto \phi(x)$ is measurable map mapping $\mu \times \nu$ onto ν (in other words (X, ν) is a (G, μ)-space where G is the group generated by Φ). For $1 \leq p \leq \infty$ consider the Markov operator P acting on $L^p(X, \nu)$ by

$$(Pf)(x) = \int_\Phi f(\phi(x))\, d\mu(\phi)$$

Consider the product space $(\Omega, \mathsf{P}) = (\Phi^{\mathbb{N}}, \mu^{\mathbb{N}})$ with the shift θ, $(\theta\omega)_i = \omega_{i+1}$, and the skew-product *measure-preserving* transformation T of $(\Omega \times X, \mathsf{P} \times \nu)$ defined by $T(\omega, x) = (\theta\omega, \omega_1(x))$.

Theorem 3.1 (Random Ergodic Theorem). *With the notations as above, the following conditions are equivalent:*

(a) *Every measurable set $E \subseteq X$ for which $m(\phi^{-1}E \triangle E) = 0$ for μ-a.e. $\phi \in \Phi$, is measurably trivial, i.e. $\nu(E) = 0$ or $\nu(X \setminus E) = 0$.*
(b) *The Markov operator P on $L^p(X, \nu)$, $1 \leq p \leq \infty$, is ergodic, i.e. $Pf = f$ for $f \in L^p(X, \nu)$ iff $f(x) = c$ constant ν-a.e.*
(c) *The skew-product $(\Omega \times X, \mathsf{P} \times \nu, T)$ is ergodic.*

If these conditions are satisfied then for every function $f \in L^1(X, \nu)$ with P-probability one the sequence of random products $\{S_n(\omega) = \omega_n \circ \cdots \circ \omega_1\}_{n=1}^\infty$ satisfies

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(S_n(\omega) \cdot x) = \int_X f\, d\nu$$

the convergence being pointwise w.r.to ν and in $L^1(X, \nu)$.

Remarks 3.2. (a) The case of ν being invariant, rather than μ-stationary measure, is known as Kakutani's Random Ergodic Theorem [37]. The proof of the theorem in the general case below is borrowed from Kifer [41] pp. 19-21.

(b) If the skew-product $(\Omega\times X, \mathsf{P}\times\nu, T)$ is not ergodic, the T-invariant measure $\mathsf{P}\times\nu$ can be disintegrated into a family of T-ergodic probability measures, which can be shown to have a form $\mathsf{P}\times\nu_t$ where ν_t is a *μ-stationary ergodic* measure (see Kifer [41] A1). If X is a compact G-space (G-is a group generated by Φ) then this decomposition is precisely the presentation of the μ-stationary measure ν as a convex (integral) combination of the set of *extremal* μ-stationary measures ν_t.

(c) Let G be a locally compact group, M a compact metric G-space, μ - an admissible probability measure on G, then the Random Ergodic Theorem implies that for a μ-stationary measure ν on M the G-action on (M,ν) is ergodic iff ν is an extremal point in the convex compact set of μ-stationary measures on M.

(d) In the measure preserving case, where ν is an invariant measure, under fairly general conditions on μ (μ is symmetric or μ^n and μ^{n+1} are not mutually singular for some $n\geq 1$) the skew-product $(\Omega\times X, \mathsf{P}\times\nu, T)$ is not merely ergodic but is *exact*, i.e. its natural extension is a K-automorphism ([10] Appendix B). However, in general the skew-product is not Bernoullian (Kalikow's T, T^{-1}-theorem).

Proof of Theorem 3.1.

The implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) are evident. To show that (a) $\Rightarrow$ (b) let $f\in L^p(X,\nu)$ be P-invariant. Then $|f| = |Pf| \leq P|f|$ so that $P|f| - |f|$ is a non-negative function in $L^p(X,\nu)$, and since $P^*\nu = \nu$ one has

$$\int_X (P|f| - |f|)\, d\nu = \int_X |f|\, dP^*\nu - \int_X |f|\, d\nu = 0$$

which means that ν-a.e. $P|f|(x) = |f(x)|$, i.e. $|f|$ is P-invariant. Hence $f^+(x) = \max\{f(x), 0\} = (|f(x)| + |f(x)|)/2$ is also a P-invariant function, so the set $E = \{x \mid f(x)\geq 0\}$ satisfies $\nu(\phi^{-1}E\triangle E) = 0$ for μ-a.e. $\phi\in\Phi$ and is therefore measurably trivial by (a). Repeating this argument for the P-invariant function $f(x) - c$, where c is a constant, one concludes that all sets $\{x \mid f(x)\geq c\}$ are measurably trivial, i.e. f is a constant function, proving (b).

Now assume (b), and let $h(\omega, x)\in L^2(\Omega\times X, \mathsf{P}\times\nu)$ be a T-invariant function, so that $h(\theta\omega, \omega_1\cdot x) = h(\omega, x)$. For $n\geq 0$ let $\mathcal{B}_n$ denote the σ-algebra generated by $x, \omega_1, \ldots, \omega_n$ and let $h_0(x)$, $h_n(\omega_1,\ldots,\omega_n, x)$ denote the projections of h to $\mathcal{H}_n = L^2(\mathcal{B}_n)\subseteq\mathcal{H} = L^2(\mathsf{P}\times\nu)$. In probabilistic terms

$$h_n(\omega_1,\ldots,\omega_n, x) = E(h(\omega, x)\mid \omega_1,\ldots,\omega_n)$$

are the conditional expectations of h with respect to $\omega_1, \ldots, \omega_n$. Since $h(\omega, x) = h(\theta^n\omega, \omega_n \cdots \omega_1 \cdot x)$ we have

$$h_n(\omega_1, \ldots, \omega_n, x) = h_{n-1}(\omega_2, \ldots, \omega_n, \omega_1 \cdot x) = \cdots = h_0(\omega_n \cdots \omega_1 \cdot x) \tag{3.1}$$

In particular $h_0(x) = EE(h(\omega, x) \mid \omega_1) = Eh_1(\omega_1, x) = Eh_0(\omega_1 \cdot x) = Ph_0(x)$, which by (b) means that $h_0(x) = c$ is a constant ν-a.e. In view of (3.1) all functions $h_n(\omega_1, \ldots, \omega_n, x) = c$ are constant. Since $\mathcal{H}_n$ increase to the whole space $\mathcal{H} = L^2(\mathsf{P} \times \nu)$ the limit function $h(\omega, x)$ has to be constant c, and assertion (c) follows. □

The following is a topological analogue of the Random Ergodic Theorem. Recall, that a continuous action $G \times X \to X$ of a locally compact group G on a metric compact X is *minimal* iff *every* orbit $G \cdot x$, $x \in X$, is dense in X.

Theorem (*) 3.3. *Let G be a locally compact group with a continuous minimal action $G \times X \to X$ on a compact metric space X. Let μ be a probability measure on G so that $G = \mathsf{sgr}(\mu)$, and let $S_n(\omega) = f_n(\omega) \cdots f_1(\omega)$ denote the random products of the μ-generated random walk on G. Then there exists a set $\Omega_0 \subseteq \Omega$ with $\mathsf{P}(\Omega_0) = 1$ so that for all $\omega \in \Omega_0$ and every $x \in X$ the sequence $\{S_n(\omega) \cdot x, n \geq 0\}$ is dense in X.*

3.2. **Strong ergodicity and Rate of convergence.** Consider a *measure-preserving* action (X, m, G) of a locally compact group G, i.e. a jointly measurable action $G \times X \to X$ where each $g \in G$ acts as a measure-preserving transformation of the standard probability space (X, m). As usual such an action is called *ergodic* if if there are no measurable sets $E \subseteq X$ with $0 < m(E) < 1$ such that $m(g^{-1}E \triangle E) = 0$ for every $g \in G$. Random Ergodic Theorem 3.1 (in this case Kakutani's Theorem) implies

Corollary 3.4 (Kakutani' Random Ergodic Theorem)**.** *Let (X, m, G) be an ergodic measure-preserving system and μ a measure on G with $\mathsf{grp}(\mu) = G$. Then for every function $f \in L^1(X, m)$ P-a.e. path $\{S_n = \omega_n \cdots \omega_1\}$ of the μ-random walk on G*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(S_n \cdot x) = \int_X f\, dm$$

in $L^1(X, m)$ and for m-a.e. $x \in X$.

Proof. It suffices to show that any measurable set $E \subseteq X$ with $m(g^{-1}E \triangle E) = 0$ for μ-a.e. $g \in G$ has $m(E) \in \{0, 1\}$. The unitary G-representation π on $L^2(X, m)$

$$(\pi(g)f)(x) = f(g^{-1} \cdot x) \qquad (f \in L^2(X, m),\ g \in G)$$

is continuous (in the weak topology), and the indicator function $\mathbf{1}_E \in L^2(X, m)$ is a fixed vector for μ-a.e. $g \in G$. By continuity, it is fixed by all of $\mathsf{grp}(\mu) = G$. As the system (X, m, G) is ergodic, the only $\pi(G)$-invariant vectors are constant functions, which means that E is measurably trivial. □

Note the following features of the corollary:

Representative behavior: The ergodicity of the G-action forces a.e. paths of the μ-random walk on G to become equidistributed on (X, m).

One dimensional behavior: The paths of a random walk are indexed by $\mathbb{N}$ and therefore allow to obtain "random walk" analogues of the classical ergodic theorems for single transformations.

The dependence on μ: The random walks are defined using an auxiliary distribution μ on G, but the equidistribution phenomenon obtained is independent of μ, provided μ is generating G.

Let (X, m, G) be a measure-preserving system and π denote the unitary G-representation on $L^2(X, m)$. As the constant functions form an invariant subspace, the orthogonal complement

$$L^2_0(X, m) = \left\{ f \in L^2(X, m) \,\middle|\, \int_X f\, dm = 0 \right\}$$

is $\pi(G)$-invariant as well, and we shall denote by π_0 the restriction of π to $L^2_0(X, m)$. Given a probability measure μ on G form the average operator $\pi_0(\mu)$ on $L^2_0(X, m)$. Then $\pi_0(\mu)$ is a semi-contractions, i.e. $\|\pi_0(\mu)\| \le 1$, and if μ is symmetric then $\pi_0(\mu)$ is self adjoint. In any case, $\pi_0(\check{\mu} * \mu) = \pi_0(\mu)^* \pi_0(\mu)$ is a positive self-adjoint semi-contraction.

Suppose that μ is generating, i.e. $\mathsf{grp}(\mu) = G$. The following conditions are well known to be equivalent to ergodicity of the system (X, m, G):

(a) The G-representation π_0 on $L^2_0(C, m)$ has no non-trivial invariant vectors.
(b) The operator $\pi_0(\mu)$ has no non-trivial fixed vectors in $L^2_0(X, m)$.
(c) m is the unique G-invariant *mean* on $L^p(X, m)$ for $1 \le p < \infty$.

(Recall that a *mean* on $L^p(X, m)$ for $1 \le p \le \infty$ is a positive, *finitely additive* functional, normalized by $m(\mathbf{1}) = 1$.)

We shall say that a probability measure μ on a locally compact group G is *aperiodic* if it is not supported on a coset of a proper closed subgroup of G, equivalently if $\mathsf{grp}(\check{\mu} * \mu) = G$. Observe that if μ is aperiodic on G, then condition (b) above can be replaced by

(b') The semi-contraction $\pi_0(\mu)$ has no eigenvalues on the unit circle.

The following result describes natural strengthennings of conditions (a), (b'), (c). We shall formulate it for actions of discrete countable groups Γ.

Theorem 3.5 (Schmidt [62], Furman-Shalom [10]). *Let (X, m, Γ) be a measure-preserving system, where Γ is a discrete countable group, and let μ be an aperiodic probability measure on Γ. Then the following conditions are equivalent:*

(a) $\|\pi_0(\mu)\| < 1$.
(b) *The Γ-representation π_0 on $L^2_0(X, m)$ does not have almost invariant vectors.*

(c) *m is the unique Γ-invariant mean on $L^\infty(X,m)$.*

If the above conditions are satisfied the system (X,m,Γ) will be called *strongly ergodic.*

Remarks 3.6. (i) Theorem 3.5 has a partial generalization to actions of locally compact groups, in which case the implications (c) $\Rightarrow$ (b) $\Rightarrow$ (a) hold assuming certain regularity of μ (see [10]), but the equivalence of these conditions ceases to hold in general.

(ii) Strong ergodicity is a feature of non-amenability. In fact, a group Γ is amenable iff Γ has no strongly ergodic actions. On the other hand a group Γ has Kazhdan's property (T) iff all its ergodic actions are strongly ergodic (Connes-Weiss [5]).

(iii) There are many natural examples of strongly ergodic actions beyond those of Kazhdan groups. For example, any subgroup $\Gamma \subseteq \mathrm{SL}_n(\mathbb{Z})$, which is not virtually abelian and acts irreducibly on $\mathbb{R}^n$, acts strongly ergodically on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (see Furman-Shalom [10] for details and further examples).

It is well known that in the classical ergodic theorems for $\mathbb{Z}$ or $\mathbb{N}$, von Neumann's mean ergodic theorem or Birkhoff's pointwise ergodic theorem, no uniform rate of convergence can be achieved for all L^2 or even L^∞ functions in any non-trivial action. However, the following result shows that strongly ergodic actions exhibit certain universal rate of convergence in the context of Kakutani's Random Ergodic Theorem:

Theorem 3.7 (Furman-Shalom [10])**.** *Let (X,m,G) be a measure-preserving ergodic system and μ a probability measure on G so that the unitary G-representation on $L^2_0(X,m)$ has a spectral gap $\|\pi_0(\mu)\| < 1$. Given an arbitrary measurable function f on (X,m) form the sequence of partial sums*

$$F_n(f,\omega,x) = \sum_{k=1}^{n} f(S_k(\omega)\cdot x) \qquad \text{where} \qquad S_k(\omega) = \omega_k \cdots \omega_1$$

Then, for any $f \in L^2(X,m)$ and any $\epsilon > 0$, for P-a.e. $\omega \in \Omega$ one has

Random Mean Ergodic Theorem:

$$\left\| \frac{1}{n} F_n(f,\omega,\cdot) - \int f\,dm \right\|_{L^2(X,m)} = o\left(\frac{\log^{1/2+\epsilon} n}{\sqrt{n}}\right) \tag{3.2}$$

Random Pointwise Ergodic Theorem: *for m-a.e. $x \in X$*

$$\left| \frac{1}{n} F_n(f,\omega,x) - \int f\,dm \right| = o\left(\frac{\log^{3/2+\epsilon} n}{\sqrt{n}}\right)$$

Functional CLT: *If $f \in L^p(X,m)$ for some $p > 2$, and f is not a constant function, then there exists a $\sigma > 0$ so that the linear approximation $\Sigma_n(\omega,x,t)$, $t \in [0,1]$,* *determined by the discrete values*

$$\Sigma_n(\omega, x, k/n) = \frac{1}{\sigma\sqrt{n}}\left(F_k(f,\omega,x) - k\int_X f\,dm\right), \qquad (k = 0,\dots,n)$$

converge in distribution (w.r.to $\mathsf{P} \times m$*) to the canonical Wiener measure on* $\mathcal{C}([0,1])$.

Remarks 3.8. (i) The spectral gap assumption in the Theorem is essential: if π_0 has almost invariant vectors in $L^2_0(X,m)$ then for every sequence $a_n \to 0$ one can find an $f \in L^2(X,m)$ so that $\|F_n(f,\omega,\cdot) - \int f\,dm\| \geq a_n$ infinitely often with P-probability one.

(ii) Recall that for a measure-preserving system (X,m,G) to be *weakly mixing* (i.e. not to have finite dimensional G-invariant subspaces in $L^2(X,m)$) is equivalent to ergodicity of the diagonal action $(X\times X, m\times m, G)$. If the latter diagonal action has a spectral gap $\|\pi_0 \otimes \pi_0(\mu)\| < 1$ then random walks on G have *exponentially fast* mixing properties for the original G-action on (X,m) (see [10]).

3.3. **Entropy for random transformations.** Let Φ be a family of invertible measurable transformations of a probability measure space (X,ν), μ - a probability measure on Φ so that ν is a μ-*stationary* measure. As before we denote by G the group generated by Φ, $(\Omega,\mathsf{P}) = (\Phi^{\mathbb{N}}, \mu^{\mathbb{N}})$ and denote by $S_n = \omega_n \cdots \omega_1$ the product (actually composition) of μ-distributed independent Φ-valued variables, which we call μ-random transformations.

In [41] Kifer introduced the following notion of entropy for this setting of random transformations. Given a finite measurable partition P of (X,ν), define

$$\tilde{h}^\mu(P,G) = \lim_{n\to\infty}\frac{1}{n}\int_\Omega H(S_{n-1}^{-1}P \vee \cdots \vee S_1^{-1}P \vee P)\,d\mathsf{P}$$

and let $\tilde{h}^\mu(X,G) = \sup_P \tilde{h}^\mu(P,G)$ over all finite measurable partitions P of (X,ν). One can verify that the limit exists. In fact, $\tilde{h}^\mu(X,\Phi)$ is precisely the *relative*, or *fiber*, entropy introduced by Abramov and Rohlin for the computation of an entropy of a skew-product relative to the base system. In our setup the skew-product is $(\Omega\times X, \mathsf{P}\times\nu, T)$, $T : (\omega,x) \mapsto (\theta\omega, \omega_1(x))$ and the base is $(\Omega,\mathsf{P},\theta)$. So one has

$$h(\Omega\times X, T) = h(\Omega,\theta) + \tilde{h}^\mu(X,G)$$

but $\tilde{h}^\mu(X,G)$ is well defined (finite or infinite) even when the other two entropies are infinite.

It is clear from the definition that $\tilde{h}^\mu(X,G)$ is an invariant with respect G-equivariant measurable isomorphisms, and it does not increase when (X,ν) is replaced by a measurable equivariant factor. One also has

$$\tilde{h}^{(\mu^p)}(X,G) = p\cdot\tilde{h}^\mu(X,G) \tag{3.3}$$

for every convolution power μ^p of μ.

If ν is not only stationary, but rather invariant measure for μ-a.e. $g \in G$, then for every finite partition P the functions

$$f_n(\omega) = H(S_{n-1}^{-1}(\omega)P \vee \cdots \vee S_1(\omega)^{-1}P \vee P)$$

satisfy

$$\begin{aligned} f_{n+m}(\omega) &= H(\vee_{i=n}^{n+m-1} S_i(\omega)^{-1}P \vee \vee_{j=0}^{n-1} S_j(\omega)^{-1}P) \\ &\leq H(\vee_{i=n}^{n+m-1} S_i(\omega)^{-1}P) + H(\vee_{j=0}^{n-1} S_j(\omega)^{-1}P) \\ &\leq H(S_n(\omega)^{-1} \vee_{i=0}^{m-1} S_i(\theta^n\omega)^{-1}P) + H(\vee_{j=0}^{n-1} S_j(\omega)^{-1}P) \\ &= f_m(\theta^n\omega) + f_n(\omega) \end{aligned}$$

Thus $\{f_n(\omega)\}$ form a bounded ($f_n \leq H(P) < \infty$) subadditive cocycle over the ergodic system $(\Omega, \mathsf{P}, \theta)$, and by Kingman's subadditive ergodic theorem for P-a.e. ω

$$\lim_{n\to\infty} \frac{1}{n} H(S_{n-1}(\omega)^{-1}P \vee \cdots \vee P) = \tilde{h}^{\mu}(P, G) \tag{3.4}$$

In this case one also has $\tilde{h}^{\check{\mu}}(X, G) = \tilde{h}^{\mu}(X, G)$, where $\check{\mu}$ denotes the reflected measure $d\check{\mu}(g) = d\mu(g^{-1})$.

In some cases $\tilde{h}^{\mu}(X, G)$ can be related to the usual entropy. For example, if μ is a distribution on powers $\{T^n, n \in \mathbb{Z}\}$ of a single measure-preserving transformation T (or on an $\mathbb{R}$-flow T^t) of a measures space (X, ν), then assuming that μ has finite first moment

$$\int |x|\, d\mu(x) < \infty$$

one can prove that

$$\tilde{h}^{\mu}(X, \mathbb{Z}) = |\mathsf{bar}(\mu)| \cdot h(X, T)$$

where $h(X,T)$ is the Kolmogorov-Sinai entropy and $\mathsf{bar}(\mu) \in \mathbb{R}$ is the barycenter of of μ, defined by $\mathsf{bar}(\mu) = \int x\, d\mu(x)$. In particular, in the case of a measurable action of $\Gamma = \mathbb{Z}$ (or $\mathbb{R}$) the entropy of μ-random transformations vanishes for all symmetric μ with finite first moment.

3.4. **Lyapunov exponents and non-random filtration.** Let (X, ν) be a probability measure and G a group of measurable transformations endowed with a Borel structure and a probability measure μ on G so that ν is μ-stationary. Let $D : G \times X \to \mathrm{GL}_k(\mathbb{R})$ be a measurable cocycle, i.e. $D(gh, x) = D(g, h\cdot x)D(h, x)$ for ν-a.e. x and μ-a.e. $g, h \in G$. Impose the following integrability condition on D

$$\int_G \int_X \left(\log^+ \|D(g,x)\| + \log^+ \|D(g,x)^{-1}\|\right) d\nu(x)\, d\mu(g) < \infty \tag{3.5}$$

where $\log^+ t = \max\{0, \log t\}$.

Consider the skew-product $(\Omega \times X, \mathsf{P} \times \nu, T)$ and the matrix-valued measurable function $A : \Omega \times X \to \mathrm{GL}_k(\mathbb{R})$ defined by $A(\omega, x) = D(\omega_1, x)$. For $n \geq 1$ set

$$A_n(\omega, x) = (A \circ T^{n-1} \cdots A \circ T \cdot A)(\omega, x) = D(\omega_n \cdots \omega_2\omega_1, x) = D(S_n(\omega), x)$$

Then $A_{n+m} = A_n \circ T^m A_m$ for all $m, n \in \mathbb{N}$. Finite first moment condition (3.5) means that both $\log^+ \|A\|$ and $\log^+ \|A^{-1}\|$ are in $L^1(\mathsf{P} \times \nu)$.

Assume for convenience that the skew-product transformation T is *ergodic* (see Remark 3.2.(b)). Then the matrix-valued function $A : \Omega \times X \to \mathrm{GL}_k(\mathbb{R})$ gives rise to the Lyapunov exponents $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_k$ defined inductively by

$$\begin{aligned} \lambda_1 + \cdots + \lambda_p &= \lim_{n\to\infty} \frac{1}{n} \int_{\Omega \times X} \log \| \wedge^p A_n(\omega, x)\| \, d\mathsf{P} \times \nu \\ &= \lim_{n\to\infty} \frac{1}{n} \int_G \int_X \log \| \wedge^p D(g, x)\| \, d\nu(x) \, d\mu^n(g) \end{aligned}$$

for $p = 1, 2, \ldots, k$. Since these exponents are associated to the cocycle $D : G \times X \to \mathrm{GL}_k(\mathbb{R})$ and the measure μ, we shall denote them by

$$\lambda_1(D, \mu) \geq \cdots \geq \lambda_k(D, \mu)$$

By Oseledec theorem there exists a measurable filtration

$$\{0\} = E_{s+1}(\omega, x) \subset E_s(\omega, x) \subset \cdots \subset E_2(\omega, x) \subset E_1(\omega, x) = \mathbb{R}^n$$

of $\mathbb{R}^n$ by nested subspaces of dimensions $\dim E_i(\omega, x) = \tau_i$, $i = 1, \ldots, s$, where τ_i are precisely the indices of strict inequalities $\lambda_{\tau_i - 1}(D, \mu) > \lambda_{\tau_i}(D, \mu)$, and $E_i(\omega, x)$ are characterized by the property that $u \in E_i(\omega, x) \setminus E_{i+1}(\omega, x)$ iff

$$\lim_{n\to\infty} \frac{1}{n} \log \|A_n(\omega, x) u\| = \lim_{n\to\infty} \frac{1}{n} \log \|D(S_n(\omega), x) u\| = \lambda_{\tau_i}(D, \mu)$$

Moreover the subspaces $E_i(\omega, x)$ are related by $A(\omega, x) E_i(\omega, x) = E_i(T(\omega, x))$, i.e.

$$D(\omega_1, x) E_i(\omega, x) = E_i(\theta\omega, \omega_1 \cdot x)$$

However, besides the Oseledec filtration, which depends on both $x \in X$ and $\omega \in \Omega$, there is a *non-random filtration* analogous to the one described in section 1.4, which describes the P-a.e. constant growth rate $n^{-1} \log \|D(S_m(\omega), x)\, u\|$ in terms of x and $u \in \mathbb{R}^k$ only. More precisely

Theorem 3.9 (Kifer, [41], chapter III)**.** *Let $G, \mu, (X, \nu)$ be as above and let $D : G \times X \to \mathrm{GL}_k(\mathbb{R})$ be a measurable cocycle satisfying (3.5). Assume that ν is an ergodic μ-stationary measure. Then there exist integers $k = k_1 > k_2 > \cdots > k_r > k_{r+1} = 0$, constants*

$$\tilde{\lambda}_1(D, \mu) > \tilde{\lambda}_2(D, \mu) > \cdots > \tilde{\lambda}_r(D, \mu) \qquad \text{with} \qquad \tilde{\lambda}_1(D, \mu) = \lambda_1(D, \mu)$$

and a measurable map $X \to \mathcal{F}_{(k_r, \ldots, k_1)}$ which assigns to ν-a.e. $x \in X$ a flag

$$\{0\} = L_{r+1}(x) \subset \cdots \subset L_2(x) \subset L_1(x) = \mathbb{R}^k \qquad \dim E_i(x) = k_i$$

so that for each $i = 1, \ldots, r$ one has for ν-a.e. $x \in X$

$$D(g, x) E_i(x) = E_i(g \cdot x) \qquad \text{for} \qquad \mu - \text{a.e. } g$$

and $u \in E_i(x) \setminus E_{i+1}(x)$ *iff for* P*-a.e.* ω

$$\lim_{n\to\infty} \frac{1}{n} \log \|D(S_n(\omega), x)u\| = \tilde{\lambda}_i(D,\mu)$$

Remarks 3.10. (a) This non-random filtration can also be described in terms of the supports of μ-stationary measures on $X \times \mathbb{P}^{k-1}$ which project onto ν, where G acts on $X \times \mathbb{P}^{k-1}$ by $g : (x, \bar{u}) \mapsto (g \cdot x, D(g,x) \cdot \bar{u})$. See Kifer [41] for the precise statements which generalize the ones in Theorem 1.11, where the latter corresponds to the case of trivial $X = \{pt\}$.

(b) As in Theorem 1.11 let us point out that the exponents $\tilde{\lambda}_1(D,\mu), \dots, \tilde{\lambda}_r(D,\mu)$ form a subset of the Lyapunov exponents $\lambda_1(D,\mu), \dots, \lambda_k(D,\mu)$, and typically the former is a *proper* subset of the latter.

(c) If the cocycle $D : G \times X \to \mathrm{GL}_k(\mathbb{R})$ is measurably *irreducible* in the sense that there does not exist a measurable $B : X \to \mathrm{GL}_k(\mathbb{R})$ (it is sufficient to consider $B : X \to \mathrm{SO}(k)$) so that the cocycle $D_1(g,x) = B(g \cdot x)^{-1} D(g,x)\, B(x)$ takes values in a reducible subgroup of $\mathrm{GL}_k(\mathbb{R})$, then $r = 1$ and *every* $u \neq 0$ satisfies

$$\frac{1}{n} \log \|D(S_n(\omega), x)u\| = \tilde{\lambda}_1(D,\mu) = \lambda_1(D,\mu)$$

for P-a.e. ω and ν-a.e. $x \in X$.

(d) One of the applications of Theorem 3.9 is in the study of *regularity* of $\lambda_1(D,\mu)$ as μ varies (compare Theorem 1.46). For example, one can show that if ν is an *invariant measure* and $\mu_n \to \mu$ are measures on a fixed, say locally compact group G, so that μ is generating, μ_n satisfy a uniform first bound condition (3.5) and D is measurably irreducible, then $\lambda_1(D,\mu_n) \to \lambda_1(D,\mu)$. See Kifer [41] for further results.

(e) If ν is μ-stationary but not ergodic, then one can apply the theorem to the ergodic components to obtain a non-random filtration where r, the dimensions $k_i = \dim E_i$, and the corresponding exponents $\tilde{\lambda}_i$ might depend on x but not on ω.

One of the natural situations where a cocycle $G : G \times X \to \mathrm{GL}_k(\mathbb{R})$ appears naturally is when X is a (compact) Riemannian manifold and G is a group of diffeomorphisms of X, with D representing the derivative cocycle (under some measurable trivialization of the tangent bundle TX). The non-random filtration, described in Theorem 3.9 gives in this context a measurable family of G-invariant subbundles of TX, which describe the (non-random) growth of the tangent vectors to X under P-a.e. random product of diffeomorphisms from G.

3.5. **Entropy for random diffeomorphisms.** Let X be a compact Riemannian k-manifold and μ be a probability measure on $G \subset \mathrm{Diff}^2(X)$, considered with the Borel structure defined by the C^2-topology. Let ν be a μ-stationary measure on X. Choosing a measurable trivialization of the tangent bundle TX one can consider the derivative cocycle

$$D : G \times X \to \mathrm{GL}_k(\mathbb{R}) \qquad D(gh, x) = D(g, h \cdot x)D(h, x)$$

Assume that μ on G has finite first moment in the sense of (3.5), and consider the case where μ-stationary measure ν is in the Lebesgue class on the manifold X.

Theorem 3.11 (Kifer, [41] p. 160). *With the notations as above, assume that the μ-stationary measure ν is in the Lebesgue class on X. Then*

$$\sum_{i=1}^{k} \lambda_i(D, \mu) \leq 0$$

and the equality holds if and only if ν is invariant, i.e. $g\nu = g$ for μ-a.e. $g \in G$.

The next result requires a technical assumption of integrability of the C^2-norms $|g|_{c^2}$ of the elements $g \in G \subset \mathrm{Diff}^2(X)$

$$\int_G \log^+ |g|_{C^2} \, d\mu(g) + \log^+ |g^{-1}|_{C^2} \, d\mu(g) < \infty$$

Theorem 3.12 (Ledrappier-Young, [51]). *Let G, μ, (X, ν) be as above where the μ-stationary measure ν on X is in Lebesgue class. Then*

$$\tilde{h}^\mu(X, G) = \sum_{\lambda_i(D,\mu) \geq 0} \lambda_i(D, \mu)$$

This is a random version of Pesin's formula for a single diffeomorphisms preserving a smooth measure. In the case of ν being invariant it can be deduced from the relative version of Pesin's formula on the skew-product. However, in Theorem 3.12 the individual diffeomorphisms are not assumed to preserve a smooth measure. Ledrappier and Young [51] further prove that if $\lambda_1(D, \mu) > 0$ then the limit measures $\nu_\omega = \omega_1 \cdots \omega_n \nu$ are SRB measures with respect to a random (i.e. ω-dependent) stable foliation.

3.6. **Measure-preserving K^μ-property.** Let us return to a general measurable setting, focusing on *measure-preserving* actions. Consider, say locally compact, group G acting ergodically by measure-preserving transformations on a probability space (X, m) (we use m for the measure to emphasize that it is invariant). Take a generating probability measure μ on G. We shall consider μ and μ-random walks on G as an auxiliary tool for the study of the G-action on (X, m). The entropy $\tilde{h}^\mu(X, G)$ is an invariant of the action (X, m, G) which depends on μ.

The definition of entropy for μ-random transformations, is quite similar to the Kolmogorov-Sinai definition of the entropy of a single transformation. In particular, one has a natural random (i.e. path-dependent) analogues of the time direction, past and future, which allow to generalize many classical results to the random entropy setup. For example, the theory of K-automorphisms for actions of $\mathbb{Z}$ admits a generalization to actions of general groups G in the the random walk setting.

Theorem 3.13 ([9]). *Let (X, m, G) be an ergodic measure preserving system, and let μ be a generating probability measure on G. The following conditions are equivalent:*

(a) *Any non-trivial measurable G-equivariant factor system (X_0, m_0, G) has positive μ-entropy $\tilde{h}^\mu(X_0, G) > 0$.*

(b) *For any finite partition P of (X, m) the tail partition*
$$P^\infty(\omega) = \wedge_{n=1}^\infty \vee_{k=n}^\infty S_k(\omega)^{-1} P$$
is measurably trivial for P-a.e. ω.

(c) *For any non-trivial partition P for P-a.e. $\omega \in \Omega$ one has*
$$\lim_{n\to\infty} \sup_{A \in P,\, B \in \vee_{k=n}^\infty S_k(\omega)^{-1} P} |m(A \cap B) - m(A) \cdot m(B)| = 0$$

A system (X, m, G) satisfying the above conditions can be called K^μ-system.

Remark 3.14. Condition (c) describes a phenomenon of strong (uniform) mixing *along μ-random walks*, taking place in μ-K-systems. However, this type of mixing, in general, does not imply that the Γ-action on (X, m) is mixing in the usual sense.

Example 3.15. Let $\Gamma \subseteq \mathrm{SL}_k(\mathbb{Z})$ act on the k-torus $\mathbb{T}^k = \mathbb{R}^k/\mathbb{Z}^k$, whose action on $\mathbb{R}^k$ is strongly irreducible. Then $(\mathbb{T}^n, \Gamma)$ is a K^μ-system for any generating probability measure μ on Γ with finite first moment.

3.7. **Cocycle growth along μ-random walks.** Let (X, m, G) be an ergodic measure-preserving system. The value of the numeric invariant $\tilde{h}^\mu(X, G)$ of the action typically depends on the auxiliary generating measure μ on G (observe the relation (3.3)). It is natural, however, to ask to what extent the *positivity* of $\tilde{h}^\mu(X, G)$ depends on μ ?

In the smooth setup, where X is a compact Riemannian k-manifold with the Riemannian volume m and $G \subset \mathrm{Diff}^2_m(X)$ is a group of volume preserving diffeomorphisms, positivity of the μ-entropy $\tilde{h}^\mu(X, G) > 0$ is equivalent, by Theorem 3.12, to the condition
$$\lambda_1(D, \mu) > 0$$
where $D : G \times X \to \mathrm{SL'}_k(\mathbb{R})$ is the derivative cocycle (the target group is $\mathrm{SL'}_k(\mathbb{R})$ rather than $\mathrm{GL}_k(\mathbb{R})$ because we assume that elements of G preserves the volume m). So the question becomes: under what conditions on (X, m, G) and $D : G \times X \to \mathrm{SL'}_k(\mathbb{R})$ the alternative $\lambda_1(D, \mu) > 0$ or $\lambda_1(D, \mu) = 0$ does not depend on an auxiliary measure μ ?

One can consider the following general setting: G is a group acting ergodically by measure-preserving transformations on an abstract *probability space* (X, m), and $D : G \times X \to \mathrm{SL'}_k(\mathbb{R})$ is a measurable cocycle with
$$\int_X \log \|D(g, x)\| \, dm(x) < \infty \qquad \forall\, g \in G$$
(this is automatically satisfied in the context of diffeomorphisms). Let μ be a generating probability measure on G and assume that μ has *finite first moment* in the

following sense

$$\int_G \int_X \log \|D(g,x)\| \, dm(x) \, d\mu(g) < \infty \tag{3.6}$$

Let $\lambda_1(D,\mu) \geq \cdots \geq \lambda_k(D,\mu)$ denote the corresponding Lyapunov exponents.

Observe that if (X,m) is the trivial space (single point) then any cocycle $D : G \times \{pt\} \to \mathrm{SL}'_k(\mathbb{R})$ is just a homomorphism $D : G \to \mathrm{SL}'_k(\mathbb{R})$. Condition (3.6) in this case is equivalent to the finite first moment condition for the push-forward measure $D_*\mu$ on $\mathrm{SL}'_k(\mathbb{R})$, and $\lambda_p(D,\mu) = \lambda_p(D_*\mu)$, $p = 1, 2, \ldots, k$. Recall that Furstenberg's condition 1.14 for $\lambda_1(D_*\mu) > 0$ is stated in terms of the group $\mathsf{grp}(D_*\mu) \subseteq \mathrm{SL}'_k(\mathbb{R})$ rather than the measure $D_*\mu$ itself. For every generating μ on G one has $\mathsf{grp}(D_*\mu) = \overline{D(G)}$ and assuming that the latter is not compact, and is strongly irreducible one has $\lambda_1(D,\mu) = \lambda_1(D_*\mu) > 0$.

In the framework of measurable cocycles $D : G \times X \to \mathrm{SL}'_k(\mathbb{R})$ we shall use the following:

Definition 3.16. Let G be a group acting by measure-preserving transformations on a probability space (X,m). Two cocycles $C, D : G \times X \to L = \mathrm{SL}'_k(\mathbb{R})$ are said to be (measurably) *cohomologous* if there exists a measurable function $B : X \to L$ so that $D(g,x) = B(g \cdot x)^{-1} C(g,x) B(x)$ for every $g \in G$ and m-a.e. $x \in X$. A cocycle $D : G \times X \to L$ is said to be *non-compact/strongly irreducible/Zariski dense* respectively if it is not measurably cohomologous to a cocycle $C : G \times X \to L_0 \subset L$, taking values in a compact/virtually reducible/proper algebraic subgroup L_0 in $L = \mathrm{SL}'_k(\mathbb{R})$, respectively.

With these definitions we can formulate a cocycle generalization of Furstenberg's condition for non-triviality of the Lyapunov spectrum, for measurable cocycles over *strongly ergodic* systems. To avoid certain technicalities assume that G is a discrete countable group.

Theorem 3.17 ([9])**.** *Let (X,m,G) be a strongly ergodic system and $D : G \times X \to L = \mathrm{SL}'_k(\mathbb{R})$ be a measurable cocycle. Assume that the cocycle D is strongly irreducible and non-compact. Then for any generating measure μ on G with finite first moment (3.6) one has $\lambda_1(D,\mu) > 0$.*

We postpone the proof of this result to the end of the section.

Remarks 3.18. (a) The assumption that (X,m,G) is *strongly* ergodic is used in the proof below, but it might be unnecessary for the result.

(b) It is conceivable that Theorems 1.23 and 1.25 can also be extended to the cocycle setting. In particular, it is plausible that the cocycle Lyapunov spectrum is simple, i.e. $\lambda_1(D,\mu) > \lambda_2(D,\mu) > \cdots > \lambda_k(D,\mu)$, provided $D : \Gamma \times X \to \mathrm{SL}'_k(\mathbb{R})$ is Zariski dense (and possibly assuming the action (X,m,Γ) is strongly ergodic).

Theorem 3.17 can be used in the analysis of not strongly-irreducible cocycles as well (for that one needs to consider certain finite extensions of the original G-action and reducible cocycles over these extensions). For a general cocycle $D : G \times X \to \mathrm{SL'}_k(\mathbb{R})$ over a strongly ergodic system (X, m, G) it can be proved that for all *symmetric* generating measures μ on G satisfying finite first moment condition either $\lambda_1(D, \mu) = 0$ or $\lambda_1(D, \mu) > 0$ for all symmetric measures μ with finite first moment (the first case corresponds to the algebraic hull of the cocycle D being amenable). This gives the following

Corollary 3.19. *Let X be a compact Riemannian manifold, $\Gamma \subset \mathrm{Diff}^2_m(X)$ a countable group of volume-preserving diffeomorphisms such that the Γ-action on (X, m) is strongly ergodic. Then the entropy $\tilde{h}^\mu(X, \Gamma)$ either vanishes or is strictly positive simultaneously for all symmetric generating measures μ on Γ with finite first moment (3.6).*

In other words, under the above assumptions of smoothness and strong ergodicity the positivity of random entropies $\tilde{h}^\mu(X, \Gamma)$ is an invariant of the action (X, m, Γ) which is independent of the choice of the symmetric generating μ on Γ.

In the special case of G being a discrete group Γ with Kazhdan's property T, a sharper dichotomy can be proved, namely

Theorem 3.20 ([9], [71])**.** *Let Γ be a discrete group with Kazhdan property T acting by C^∞ diffeomorphisms on a compact manifold X preserving a smooth measure m. Then either $\tilde{h}^\mu(X, \Gamma) > 0$ for all aperiodic probability measures μ on Γ, or the system (X, m, Γ) has a discrete spectrum, i.e. the Γ-action on (X, m) is measure-theoretically isomorphic to an action of Γ on K/K_0 given by $\gamma : kK_0 \mapsto \rho(\gamma)kK_0$ where $\rho : \Gamma \to K$ is a homomorphism into a compact group K and $K_0 \subseteq K$ is a closed subgroup.*

The last theorem combines two results. The first ([9]) states that for groups Γ with property T any measurable cocycle $D : \Gamma \times X \to \mathrm{SL'}_k(\mathbb{R})$ over an ergodic system (X, m, Γ) either has $\lambda_1(D, \mu) \geq \epsilon(\mu) > 0$, where $\epsilon(\mu) > 0$ depends only on Γ, or D is a compact cocycle. The second result, due to Zimmer [71], states that for C^∞-smooth actions of groups Γ with Kazhdan's property T if the derivative cocycle is compact (equivalently Γ preserves a *measurable* Riemannian structure on the manifold) then the underlying system (X, m, Γ) has discrete spectrum. Note that this last implication does not hold for $\mathbb{Z}$-actions (see Gunesch and Katok [29]; the appendix to this paper grew out of only partially successful attempt of the author to generalize Zimmer's result [71] to a group $G \subset \mathrm{Diff}^1(X)$ which acts strongly ergodically on (X, m)).

Let us also point out that for higher rank lattices Γ one can deduce the above results (and further information on entropies of individual elements of Γ) using superrigidity for measurable cocycles (Zimmer [72]).

Sketch of the Proof of Theorem 3.17. We shall use a cocycle analogue of the unitary representation approach discussed in section 1.6. Consider the product measure space $(\tilde{X}, \tilde{m}) = (X \times \mathbb{P}^{k-1}, m \times \nu_0)$, where ν_0 denotes the Lebesgue measure on the projective space $\mathbb{P}^{k-1}$. The measurable G-action defined by $g:(x,\bar{u})\mapsto(g\cdot x, D(g,x)\cdot\bar{u})$ leaves the measure $\tilde{m}$ quasi-invariant, and therefore defines a quasi-regular Γ-representation $\tilde{\pi}$ on $L^2(\tilde{X},\tilde{m})$ by

$$\tilde{\pi}(g)F(x,\bar{u})=\sqrt{\rho(g,x,\bar{u})}\,F(g^{-1}\cdot x,\,D(g^{-1},x)\cdot\bar{u})$$

where ρ is the multiplicative Radon-Nikodym cocycle

$$\rho(g,x,\bar{u})=\frac{dg\tilde{m}}{d\tilde{m}}(x,\bar{u})=\frac{d\,D(g,x)\nu_0}{d\nu_0}(\bar{u})$$

We claim that under the assumptions of the theorem there is a spectral gap $\|\tilde{\pi}(\mu)\|_{\mathrm{sp}}<1$.

Assume that $\|\tilde{\pi}(\mu)\|_{\mathrm{sp}}=1$. Then (see analogous argument in the proof of Proposition 1.17) for $T=\tilde{\pi}(\mu)$ or for $T^*=\tilde{\pi}(\check{\mu})$ there exists a complex number z with $|z|=1$ which is either an eigenvalue for T, or an approximate eigenvalue for T, or $\bar{z}$ is an eigenvalue for T^*. In any case, taking $\mu'=\mu$ or $\check{\mu}$, and $z'=z$ or $\bar{z}$ one can find a sequence of unit vectors $F_n\in L^2(\tilde{X},\tilde{m})$ so that

$$\|\tilde{\pi}(\mu')F_n-z'F_n\|\to 0 \tag{3.7}$$

Consider the measures $\tilde{m}_n$ on $\tilde{X}$ defined by $d\tilde{m}_n(x,\bar{u})=|F_n(x,\bar{u})|^2\,dm(x)\,d\nu_0(\bar{u})$ and their disintegrations $\tilde{m}_n=\int_X\eta_{n,x}\,dm(x)$ with respect to (X,m). The total masses of $|\eta_{n,x}|=\eta_{n,x}(\mathbb{P}^{k-1})$ of the positive measures $\eta_{n,x}$ are given by the non-negative functions

$$f_n(x)=\eta_{n,x}(\mathbb{P}^{k-1})=\|F_n(x,\cdot)\|^2_{L^2(\mathbb{P}^{k-1},\nu_0)}$$

which are unit vectors in $L^1(X,m)$ because $\|f_n\|_1=\|F_n\|_2^2=1$. Passing to a subsequence, one can assume that f_n weakly converge to some ϕ - a positive normalized functional on $L^\infty(X,m)$, i.e. a *mean* on $L^\infty(X,m)$. Cauchy-Schwartz inequality gives for every $g\in G$

$$\begin{aligned}
&\|f_n\circ g^{-1}-f_n\|_1\\
=\ &\int_X\left|\int_{\mathbb{P}^{k-1}}\left(|F_n(g^{-1}\cdot x,\bar{u})|^2\frac{dD(g,x)\nu_0}{d\nu_0}(\bar{u})-|F_n(x,\bar{u})|^2\right)\nu_0(\bar{u})\right|\,dm(x)\\
\le\ &\int_X\int_{\mathbb{P}^{k-1}}|\tilde{\pi}(g)F_n-z'F_n|\cdot|\tilde{\pi}(g)F_n+z'F_n|\,d\nu_0\,dm\\
\le\ &2\cdot\|\tilde{\pi}(g)F_n-z'F_n\|_2
\end{aligned}$$

which, in view of (3.7) converge μ'-a.e. to 0. Thus the mean ϕ is g-invariant for μ'-a.e. $g\in G$, so that ϕ is G by continuity and therefore $\phi=m$. Thus $f_n(x)=|\eta_{n,x}|\to 1$ for m-a.e. x.

Passing, if necessary to a further subsequence, one can assume that $\eta_{n,x}$ converge weakly to probability measures: $\eta_{n,x}\to\eta_x\in\mathcal{P}(\mathbb{P}^{k-1})$, and (3.7) and Cauchy-Schwartz inequality can be used to show that that the measure $\int_X \delta_x \times \eta_x \, dm(x)$ on $X \times \mathbb{P}^{k-1}$ is G-invariant, which means that

$$D(g,x) \cdot \eta_x = \eta_{g \cdot x} \tag{3.8}$$

for m-a.e. $x \in X$.

The following argument is known as Zimmer's *"Cocycle Reduction Lemma"*. The natural action of $L = \mathrm{SL'}_k(\mathbb{R})$ on the space of probability measures $\mathcal{P}(\mathbb{P}^{k-1})$ is *tame*, which (by definition) means that there exists a countable collection $\{M_i \subset \mathcal{P}(\mathbb{P}^{k-1})\}$ of L-invariant Borel sets, which separates points of $\mathcal{P}(\mathbb{P}^{k-1})$. Let $X_i = \{x \in X \mid \eta_x \in M_i\}$. By (3.8) each X_i is a G-invariant set, and by ergodicity it has m-measure 0 or 1. Intersecting those X_i which have $m(X_i) = 1$ and the complements of the others, gives a measurable subset $X' \subseteq X$ for which $\{\eta_x \mid x \in X'\}$ lie in a *single* L-orbit $L \cdot \eta_0$ on $\mathcal{P}(\mathbb{P}^{k-1})$, i.e. $\eta_x = A(x) \cdot \eta_0$ for some measurable $A : X' \to L$ and $\eta_0 \in \mathcal{P}(\mathbb{P}^{k-1})$. Let L_0 denote the stabilizer of η_0 in L, then the cocycle

$$C(g,x) = A(g,x)^{-1} D(g,x) A(x)$$

satisfies $C(g,x) \cdot \eta_0 = \eta_0$, i.e. takes values in $L_0 \subset L$. If η_0 is a proper measure, then L_0 is a compact subgroup in $L = \mathrm{SL'}_k(\mathbb{R})$; while if η_0 is not proper then L_0 is not strongly irreducible. Both cases contradict the assumption on D. Hence there is a spectral gap $\|\tilde{\pi}(\mu)\| < 1$.

Recall that for $A \in \mathrm{SL'}_k(\mathbb{R})$ one has

$$\log \|A\| \geq -\frac{2}{k} \log \left(\int_{\mathbb{P}^{k-1}} \sqrt{\frac{dA^{-1}\nu_0}{d\nu_0}(\bar{u})} \, d\nu_0(\bar{u}) \right)$$

which gives, using the convexity of $-\log()$ the following estimate

$$\begin{aligned}
\frac{1}{n} \int_G \int_X \quad & \log \quad \|D(g,x)\| \, dm(x) \, d\mu^n(g) \\
\geq \quad & \frac{1}{n} \int_G \int_X -\frac{2}{k} \log \left(\int_{\mathbb{P}^{k-1}} \sqrt{\rho(g,x,\bar{u})} \, d\nu_0(\bar{u}) \right) \, dm(x) \, d\mu^n(g) \\
\geq \quad & -\frac{2}{nk} \log \left(\int_G \int_X \int_{\mathbb{P}^{k-1}} \sqrt{\rho(g,x,\bar{u})} \, d\nu_0(\bar{u}) \, dm(x) \, d\mu^n(g) \right) \\
= \quad & -\frac{2}{nk} \log \langle \tilde{\pi}(\mu^n) \mathbf{1}, \, \mathbf{1} \rangle \geq \frac{2}{k} \log \frac{1}{\|\tilde{\pi}(\mu^n)\|^{1/n}}
\end{aligned}$$

so that $\lambda_1(D,\mu) \geq 2/k \log \left(1/ \|\tilde{\pi}(\mu)\|_{\mathrm{sp}} \right) > 0$. □

4. Selected Proofs

This section contains the proofs of the statements marked by a star in the paper. We use the notations of the corresponding statements.

Proof of Lemma 1.7. Suppose that μ is a probability measure on $G = \mathrm{SL}'_k(\mathbb{R})$ and ν a μ-stationary measure on $\mathbb{P}^{k-1}$. Assume that ν is not proper, and let $\{0\} \subset W \subset \mathbb{R}^k$ be a proper subspace of *minimal dimension*, such that $\nu(\overline{W}) > 0$. The function $f : G \to [0,1]$ defined by $f(g) = g\nu(W) = \nu(\overline{g}^{-1}\overline{W})$ satisfies

$$\begin{aligned}\int_G f(gh)\, d\mu(h) &= \int_G gh\nu(\overline{W})\, d\mu(h)\\ &= g\,\mu * \nu(\overline{W}) = g\nu(\overline{W}) = f(g)\end{aligned}$$

By minimality of $\dim W$, one has $\nu(\overline{g}\overline{W} \cap \overline{g}'\overline{W}) = 0$, whenever $gW \neq g'W$. Hence the set of gW with $f(g) = \nu(\overline{g}\overline{W}) > \epsilon$ is finite for any $\epsilon > 0$. Thus f reaches its maximum value $f(g_0) = \max\{f(g) \mid g \in G\}$, while the relation above yields that $f(g_0 h) = f(g_0)$ for μ-a.e. $h \in G$. Thus there exists a finite collection E of proper subspaces, s.t. $h^{-1}g_0^{-1}W$ belongs to E for μ-a.e. h, and therefore the whole group $G_\mu = \mathsf{grp}(\mu)$ permutes the finite collection E of proper subspaces. Consequently, μ is not strongly irreducible.

On the other hand, assuming that μ is not strongly irreducible there is a finite collection $\overline{E} = \overline{W}_1 \cup \cdots \cup \overline{W}_r \subset \mathbb{P}^{k-1}$ of proper projective subspaces, which is invariant under μ-a.e. $g \in G$. Being a compact invariant subset of $\mathbb{P}^{k-1}$ the set $\overline{E}$ supports μ-stationary measures ν which are not proper. □

Proof of Proposition 1.17. Let X be a compact metric space, H a locally compact group acting continuously on X, and μ - a generating probability measure on H. Let $\{\nu_n\}$ be a sequence of H-quasi-invariant probability measures on X, such that the unitary quasi-regular H-representations π_{ν_n} on $L^2(X, \nu_n)$ defined by

$$\pi_n(h)f(x) = \sqrt{\frac{dh\nu_n}{d\nu_n}(x)}\, f(h^{-1}\cdot x)$$

satisfy $\|\pi_{\nu_n}(\mu)\|_{\mathrm{sp}} \to 1$. Denote by $T_n = \pi_{\nu_n}(\mu)$ the (semi)contractions on the Hilbert spaces $\mathcal{H}_n = L^2(X, \nu_n)$.

For a bounded operator T on a Hilbert space $\mathcal{H}$, $\|T\|_{\mathrm{sp}} = \max_{\rho \in \sigma(T)} |\rho|$, where $\sigma(T)$ denotes the spectrum of T. Recall that $\rho \in \sigma(T)$ if (i) ρ is an eigenvalue for T, or (ii) ρ is an approximate eigenvalue for T, i.e.

$$\exists f_k \in \mathcal{H} : \qquad \|f_k\| = 1 \qquad \text{and} \qquad \|Tf_k - \rho f_k\| \to 0$$

or (iii) $\mathrm{Im}\,(T - \rho)$ is not dense in $\mathcal{H}$, in which case $\mathrm{Im}\,(T-\rho)^\perp = \mathrm{Ker}(T-\rho)^* = \mathrm{Ker}(T^* - \bar{\rho}) \neq \{0\}$, i.e. $\bar{\rho}$ is an eigenvalue for T^*.

Therefore there always exists a complex number ρ with $|\rho| = \|T\|_{\mathrm{sp}}$ and unit vectors f_k in $\mathcal{H}$ such that $\|Tf_k - \rho f_k\| \to 0$ or $\|T^* f_k - \rho f_k\| \to 0$.

Recall that if $T_n = \pi_{\nu_n}(\mu)$ then $T_n^* = \pi_{\nu_n}(\check{\mu})$. Using the above argument about the spectrum, and assuming $\|T_n\|_{\mathrm{sp}} \to 1$ one concludes that for $\mu' = \mu$ or $\check{\mu}$ and an appropriate subsequence of ν_n, one can find functions $f_n \in L^2(X, \nu_n)$, and numbers ρ_n with $|\rho_n| = \|\{\|_{\mathrm{sp}} T_n\} \to 1$, so that

$$\|f_n\|_2 = 1 \qquad \text{and} \qquad \|\pi_{\nu_n}(\mu')f_n - \rho_n f_n\|_2 \to 0$$

Passing to a further subsequence, one can assume that $\|\pi(h)_{\nu_n} f_n - \rho_n f_n\| \to 0$ for μ'-a.e. $h \in H$.

Now consider the measures η_n on X defined by $d\eta_n(x) = |f_n(x)|^2\, d\nu_n(x)$. These are probability measures because $\|f_n\|_2^2 = 1$. Since $\mathcal{P}(X)$ is compact, passing to a subsequence one can assume $\eta_n \to \eta \in \mathcal{P}(X)$ in weak-* topology. Let us verify that η is H-invariant. Fix $\phi \in \mathcal{C}(X)$ and $\epsilon > 0$, then for $n \geq n(\phi, \epsilon)$ for every $h \in H$ one has

$$\begin{aligned} \left|\int_X \phi\, dh\eta - \int_X \phi\, \eta\right| &\leq \left|\int_X \phi\, dh\eta_n - \int_X \phi\, \eta_n\right| + \epsilon \\ =& \quad \left|\int_X \phi(x) \cdot \left(|f_n(h^{-1}x)|^2 \frac{dh^{-1}\nu_n}{d\nu_n}(x) - |f_n(x)|^2\right) d\nu_n(x)\right| + \epsilon \\ \leq& \quad \|\phi\|_\infty \cdot \|\pi_{\nu_n}(h)f_n - \rho_n f_n\|_2 \cdot \|\pi_{\nu_n}(h)f_n + \rho_n f_n\|_2 + \epsilon \\ \leq& \quad 2\|\phi\|_\infty \cdot \|\pi_{\nu_n}(h)f_n - \rho_n f_n\|_2 + \epsilon \end{aligned}$$

Hence for μ'-a.e. $h \in H$ one has $h \cdot \eta = \eta$, and by continuity, η is invariant under $\mathsf{grp}(\mu') = H$. □

Proof of Lemma 1.18. Take $\epsilon > 0$ and consider the non-negative series $h_\epsilon(\omega) = \sum_{n=0}^{\infty} (\|\pi(\mu)\| + \epsilon)^{-n} \langle \pi(S_n(\omega))f_1,\, f_2\rangle$. Term-by-term integration gives

$$\int_\Omega h_\epsilon\, d\mathsf{P} = \sum_{n=0}^{\infty} (\|\pi(\mu)\| + \epsilon)^{-n} \langle \pi(\mu^n)f_1,\, f_2\rangle \leq \|f_1\| \cdot \|f_2\| \cdot \sum_{n=0}^{\infty} \frac{\|\pi(\mu)\|^n}{(\|\pi(\mu)\| + \epsilon)^n} < \infty$$

Hence for P-a.e. ω

$$\limsup_{n\infty} \frac{1}{n} \log \langle \pi(S_n(\omega))f_1,\, f_2\rangle \leq \log(\|\pi(\mu)\| + \epsilon) \tag{4.1}$$

and taking $\epsilon \to 0$ one obtains $\log \|\pi(\mu)\|$ as an upper bound for the LHS for P-a.e. ω. Repeating this argument with the convolution power μ^p one has P-a.e.

$$\limsup_{n\infty} \frac{1}{n} \log \langle \pi(S_n(\omega))f_1,\, f_2\rangle \geq \frac{1}{p} \log \|\pi(\mu^p)\| = \log \|\pi(\mu)^p\|^{1/p} \to \log \|\pi(\mu)\|_{\mathrm{sp}}$$

as $p \to \infty$. This completes the proof of the Lemma. $\square$

Proof of Lemma 1.33. Given any $\phi \in \mathcal{C}(Q)$ consider the bounded continuous function F_ϕ on G defined by $F_\phi(g) = \int \phi \, dg\nu$. Note that F_ϕ satisfies the following μ-mean value property:

$$\begin{aligned}\int_G F_\phi(gh)\, d\mu(h) &= \int_G \int_Q \phi(gh \cdot u)\, d\nu(u)\, d\mu(h) \\ &= \int \phi(g \cdot v)\, d\mu * \nu(u) = \int_Q \phi\, d\nu = F_\phi(g)\end{aligned}$$

so that the sequence of random variables $w_n = F_\phi(Y_1 \cdots Y_n)$ on (Ω, P) forms a bounded martingale, i.e. the conditional expectations satisfy

$$E(w_{n+1} \mid Y_1, \ldots, Y_n) = w_n$$

The martingale convergence theorem implies that with probability one w_n converge to some value

$$w(\phi, \omega) = \lim_{n \to \infty} \int_Q \phi\, dY_1 \cdots Y_n \nu$$

Repeating this argument for a countable dense family of functions $\phi_i \in \mathcal{C}(Q)$, one obtains for P-a.e. ω a positive normalized functional $\phi_i \mapsto w(\phi_i, \omega)$, which can be extended to all of $\mathcal{C}(Q)$, and is therefore given by a probability measure, which we denote by ν_ω. The fact that ν is the average of ν_ω follows from Lebesgue dominated convergence theorem.

Denote $Z_n = Y_1 \cdots Y_n$. We need to show $Z_n g\, \nu \to \nu_\omega$ for μ_*-a.e. g, and it suffices to show that for any $\phi \in \mathcal{C}(Q)$ for all $k \geq 0$ one has $\mathsf{P} \times \mu^k$-a.e. $|F_\phi(Z_n g) - F_\phi(Z_n)| \to 0$. Note that

$$\begin{aligned}\int |F_\phi(Z_n g) &- F_\phi(Z_n)|^2\, d\mu^k(g)\, d\mathsf{P}(\omega) \\ &= \int F_\phi^2\, d\mu^{n+k} + \int F_\phi^2\, d\mu^n - 2\int F_\phi(hg) F_\phi(h)\, d\mu^n(h)\, d\mu^k(g) \\ &= \int F_\phi^2\, d\mu^{n+k} - \int F_\phi^2\, d\mu^n\end{aligned}$$

where the last equality uses the μ-mean value property of F_ϕ. Hence

$$\int_G \int_\Omega \sum_{n=0}^{p} |F_\phi(Z_n g) - F_\phi(Z_n)|^2\, d\mathsf{P}(\omega)\, d\mu^k(g) \leq 2k \|\phi\|_\infty$$

for every $k \geq 0$. Hence the series $\sum_n |F_\phi(Z_n g) - F_\phi(Z_n)|^2$ converge $\mathsf{P} \times \mu^k$-a.e., and in particular $|F_\phi(Z_n g) - F_\phi(Z_n)| \to 0$, and the proof is completed. $\square$

Proof of Proposition 1.36.
The first statement follows directly from Lemma 1.35 by realizing the real-valued stationary process $\{\sigma(Z_n^\theta)\}$ as values $\{f(T^n x)\}$ of a measurable function f on a measure-preserving system (X, m, T) along a random orbit $T^n x$.

To prove the second statement consider the set $S_P = \{\theta \in \mathcal{P}(M) \mid P^*\theta = \theta\}$ and let $\gamma = \sup\{\int_M \sigma\, d\theta \mid \theta \in S_P\}$. By compactness the sup is attained by some $\theta_0 \in S_P$, and the first assertion gives $\gamma < 0$. Consider the subadditive sequence

$$u_n = \sup_{x\in M} \sum_{k=1}^{n} P^k \sigma(x)$$

and for each n denote by $x_n \in M$ a point which attains this sup. Any cluster point θ of the sequence $\eta_n = n^{-1}\sum_{k=1}^{n}(P^*)^k \delta_{x_n}$ of probability measures is easily seen to be P-stationary. Since u_n are subadditive, u_n/n converges and one has

$$\frac{u_n}{n} = \int_M \sigma\, d\eta_n \to \int_M \sigma\, d\theta \le \gamma < 0$$

□

Proof of Claim 1.40. Let $\{B_i : L_i \to \mathbb{R}^k\}_{i=0}^r$ represent the QP-transformation b. By Lemma 1.29 one has $M_1(b) = \overline{L_1}$ and $M_0(b) = \overline{\operatorname{Im} B_0}$. The square b^2 is also in $\overline{T}^{qp}$ (Lemma 1.39). Assuming the claim is wrong, one has that $\operatorname{Im} B_0$ does not contain $L_1 = \operatorname{Ker} B_0$, but $\dim(\operatorname{Im} B_0 \cap \operatorname{Ker} B_0) > 0$, so that $B_0^2 \neq 0$ and $\dim B_0^2 < \dim B_0$. This would imply that $\dim M_0(b^2) = \dim \overline{\operatorname{Im} B_0^2} < \dim M_0(b) = d$, contrary to the assumption that $b \in Q$. □

Proof of Claim 1.41. Observe that $TQ = Q$ because $TQ \subseteq \overline{T}$ and for every projective transformation $\bar{g}$ one has $M_1(\bar{g}b) = M_1(b)$ while $\dim M_0(\bar{g}b) = \dim \bar{g}M_0(b) = \dim M_0(b)$. Since T is (strongly) irreducible, for every $b \in Q$ with $M_0(b) \subseteq M_1(b)$, one can find a $g \in T$ so that

$$M_0(\bar{g}b) \not\subseteq M_1(b) = M_1(\bar{g}b)$$

so that claim (1) implies $M_0(\bar{g}b) \cap M_1(\bar{g}b) = \emptyset$. □

Proof of Claim 1.43. Given a sequence $\phi_n = b\bar{g}_n$ in Φ_0 one needs to find a convergent subsequence for the projective transformations $\beta(\phi_n) \subseteq \mathrm{PGL}(V)$. Passing to a subsequence one can assume that ϕ_n converge to a QP-limit $\phi \in \overline{T}^{qp}$ on $\mathbb{R}^{k-1}$. Since $M_0(b\bar{g}_n) = M_0(b)$ the QP-limit ϕ is in Q. Repeating the argument of Claim 1.40 one shows that there is a $g \in T$ such that $\phi' = \phi\bar{g} \in \overline{T}^{qp}$ satisfies $M_0(\phi') \cap M_1(\phi') = \emptyset$.

Note that $\phi_n' = \phi_n \bar{g}$ converge to ϕ' pointwise in $\mathbb{P}^{k-1}$ (because $\phi_n \to \phi$), and since

$$\overline{V} = M_0(\phi') \subseteq \mathbb{P}^{k-1} \setminus M_1(\phi')$$

for every $\bar{u} \in \mathbb{P}(V)$, one has $\phi_n'\bar{u} \to \phi'\bar{u}$. In other words $\beta(\phi_n') \to \beta(\phi') \in \mathrm{PGL}(V)$ is a convergent sequence with a projective (rather than quasi-projective) limit. Hence $\beta(\phi_n) = \beta(\phi_n' g^{-1})$ is also a convergent sequence in $\mathrm{PGL}(V)$, and the claim is proved. $\square$

Proof of Remark 2.29.

Define $A_n = \{g \in \Gamma \mid \log \mu^n(g) > (-h - \epsilon)n\}$. Since $\mu^n(A_n) \leq 1$ we have the bound $\log |A_n| \leq (h + \epsilon)n$, while pointwise convergence in Theorem 2.28 implies that for P-a.e. ω one has $g_n(\omega) \in A_n$ for all but finitely many n-s. This proves (a).

Now suppose that $B_n \subset \Gamma$ are sets of size $\log |B_n| \leq (h - \epsilon)n$. Let $\Omega_n = \{w \mid \log \mu^n(g_n(\omega)) < (h - \epsilon/2)n\}$. The pointwise convergence in Theorem 2.28 implies that P-a.e. ω belongs to Ω_n for all but finitely many n-s. Observe that

$$\mathsf{P}\{\omega \in \Omega_n \mid g_n(\omega) \in B_n\} \leq e^{(h-\epsilon)n} \cdot e^{(-h+\epsilon/2)n} = e^{-\epsilon/2n}$$

The easy direction of Borel-Cantelli lemma implies that for P-a.e. ω the event $\{\omega \in \Omega_n \mid g_n(\omega) \in B_n\}$ occurs only finitely many times, proving (b). $\square$

Proof of Proposition 2.32.

Let $S_k = \{g \in \Gamma \mid d(g,e) = k\}$, $k \geq 0$, denote the k-sphere in Γ, $p_k = \mu(S_k)$ and $\mu_k = p_k^{-1} \cdot \mu|_{S_k}$ denote the normalized restrictions of μ to S_k. If α denotes the partition of (Γ, μ) into elements and $\beta \prec \alpha$ the partition into spheres, then the standard formula $H(\alpha) = H(\alpha) + H(\alpha \mid \beta)$ gives

$$\begin{aligned} H(\mu) &= \sum_{k=0}^{\infty} -p_k \log p_k + \sum_{k=0}^{\infty} p_k \cdot H(\mu_k) \\ &\leq \sum_{k=0}^{\infty} p_k \cdot \max\{\sqrt{k}, -\log p_k\} + \sum_{k=0}^{\infty} p_k \cdot \log |S_k| \end{aligned}$$

As the function $-t\log(t)$ is increasing on $(0, e^{-1})$, one has for $k \geq 1$

$$\sum_1^{\infty} p_k \cdot \max\{\sqrt{k}, -\log p_k\} \leq \sum_1^{\infty} \sqrt{k}\, p_k + \sqrt{k}\, e^{-\sqrt{k}}$$

Since $\sum \sqrt{k} e^{-\sqrt{k}} = C < \infty$, $\log |S_k|^{1/k} \to \delta = \delta(\Gamma, d)$ and

$$\sum \sqrt{k}\, p_k \leq \sum k\, p_k = \sum_g d(g,e)\, \mu(g)$$

one concludes that $H(\mu)$ is finite, provided μ has finite first moment.

Next take an $\epsilon > 0$ and choose K large enough so that for all $k > K$

$$\log|S_k|^{1/k} \le (\delta+\epsilon) \qquad \text{and} \qquad \sqrt{k} < \epsilon k$$

Since the finite set $B_K = \{g \mid d(g,e) \le K\}$ contributes at most $\log|B_K|$ to the entropy of any measure on Γ one has

$$\begin{aligned} h(\Gamma,\mu) &= \lim_{n\to\infty} \frac{1}{n} H(\mu^n) \\ &\le (\delta+2\epsilon) \lim_{n\to\infty} \left(\frac{1}{n} \sum d(g,e)\, \mu^n(g) + \frac{C+\log|B_K|}{n} \right) \\ &= (\delta+2\epsilon)) \cdot \lambda^{(d)} \end{aligned}$$

and since $\epsilon > 0$ can be taken arbitrarily small the proof is completed. □

Proof of Theorem 3.3.

We need to show that there is a set $\Omega' \subseteq \Omega$ of full P-measure, so that for every $\omega \in \Omega'$, every $x \in X$ and any non-empty open $U \subseteq X$, one has $S_n(\omega)\cdot x \in U$ for some $n \ge 0$, or equivalently, for every $\omega \in \Omega'$ and any open non-empty U

$$\cup_{n=0}^{\infty} S_n(\omega)^{-1} U = X$$

Since X has a countable base of open sets, it suffices to show that for any fixed set U the event

$$E_U = \{\omega \in \Omega \mid \cup_{n=0}^{\infty} S_n(\omega)^{-1} U = X\}$$

has P-probability one. The minimality of the original G-action implies that $X = \cup_{g\in G} gU$, and since X is compact there exists a *finite* sub-cover $g_0'U \cup \cdots \cup g_N'U = X$. By continuity of the G-action on X there exists a small neighborhood V of the identity in G, so that

$$g_0 U \cup \ldots g_N U = X \tag{4.2}$$

for any $g_0, \ldots, g_N$ satisfying $g_i \in V_i$, where $V_i = g_i' V$, $i = 0, \ldots, N$. In fact, for (4.2) it suffices to have $g_i \in gV_i$, $0 = 1, \ldots, N$, for any fixed $g \in G$; in particular, (4.2) is satisfied for an arbitrary g_0 provided

$$g_i g_{i-1}^{-1} \in W_i = V_{i+1} V_i^{-1} \qquad \text{for} \qquad i = 1, 2, \ldots, N \tag{4.3}$$

Therefore to show that $\mathsf{P}(E_U) = 1$ it suffices to show that for P-a.e. ω the sequence $S_n(\omega) = \omega_n \cdots \omega_1$ contains an N-long subsequence satisfying (4.3). Since $G = \mathsf{sgr}(\mu)$ and W_i are open non-empty subsets of G, there are integers $k_i > 0$ so that $\mu^{k_i}(W_i) > 0$. Denoting $n_i = k_1 + \cdots + k_i$ one observes that $g_i = S_{n_i}(\omega)$, $i = 1, \ldots, N$, satisfies (4.3) with positive probability, and therefore with P-probability one a sequence $g_i = S_{t+n_i}(\omega)$, $i = 1, \ldots, N$, satisfies (4.3) for some $t \in \mathbb{N}$. This proves $\mathsf{P}(E_U) = 1$ and the proof is complete. □

References

[1] H. Abels, G. A. Margulis, G. A. Soifer, Semigroups containing proximal linear maps, *Israel J. Math.* **91** (1995), 1–30.
[2] A. Avez, Théorème de Choquet-Deny pour les groupes à croissance non exponentielle, *C. R. Acad. Sci. Paris Sér. A* **279** (1974), 25–28.
[3] R. Azencott, *Espaces de Poisson des groupes localements compacts*, Springer Lecture Notes in Math. **148**, Springer, Berlin (1970).
[4] P. Bougerol, J. Lacroix, *Products of random matrices with applications to Schrodinger operators*, Progress in Probability and Statistics **vol 8**, Birkhauser, Boston-Basel-Stuttgart, (1985).
[5] A. Connes, B. Weiss, Property (T) and asymptotically invariant sequences, *Israel J. Math.* **37** no. 3 (1980), 209–210.
[6] Y. Derriennic, Quelques applications du théorème ergodique sous-additif, Conference on Random Walks (Kleebach, 1979), pp. 183–201, 4, *Astérisque* **74**, Soc. Math. France, Paris, 1980.
[7] Y. Derriennic, Y. Guivarc'h, Théorème de renouvellement pour les groupes non moyennables, *C. R. Acad. Sci. Paris Sér. A-B* **277** (1973), A613–A615.
[8] E. B. Dynkin, M. B. Maljutov, Random walks on groups with a finite number of generators, *Soviet Math. Dokl.* **2** (1961), 399–402.
[9] A. Furman, Random entropy and cocycle growth, *preprint.*
[10] A. Furman, Y. Shalom, Sharp ergodic theorems and strong ergodicity, *Ergod. Th. Dynam. Syst.* (1999) **19**, 1037–1061.
[11] A. Furman, Y. Shalom, Random walks in Hilbert spaces and Lyapunov exponents, *unpublished.*
[12] H. Furtsenberg, A Poisson formula for semi-simple Lie groups, *Ann. of Math.* (2) **77** (1963), 335–386.
[13] H. Furtsenberg, Noncommuting random products, *Trans. Am. Math. Soc.* **108** (1963), 377–428.
[14] H. Furstenberg, Boundary theory and stochastic processes on homogeneous spaces, *Harmonic analysis on homogeneous spaces*, Proc. Sympos. Pure Math., Vol. **XXVI**, 193–229. Amer. Math. Soc., Providence, R.I., 1973.
[15] H. Furstenberg, Random walks and discrete subgroups of Lie groups, *Advances in Probability and Related Topics* (1971) **1**, 1–63.
[16] H. Furstenberg, A note on Borel's density theorem, *Proc. Amer. Math. Soc.* **55** (1976), no. 1, 209–212.
[17] H. Furstenberg, Rigidity and cocycles for ergodic actions of semi-simple Lie groups [after G. A. Margulis and R. Zimmer], *Séminaire Bourbaki*, **559** (1980), 273–292.
[18] H. Furstenberg, H. Kesten, Products of random matrices, *Ann. Math. Stat.* **31** (1960), 457–469.
[19] H. Furtsenberg, Y. Kifer, Random matrix products, *Israel J. Math.* **46** (1983), no. 1, 12–32.
[20] S. Glasner, *Proximal flows*, Lecture Notes in Math. **517**, Springer-Verlag, Berlin-New York, (1976).
[21] I. Ya. Goldsheid, Lyapunov exponents and asymptotic behaviour of the product of random matrices, *Lyapunov exponents (Oberwolfach, 1990), Lecture Notes in Math.*, **1486**, Springer, Berlin, (1991), 23–37.
[22] I. Ya. Goldsheid, Y. Guivarc'h, Zariski closure and the dimension of the Gaussian law of the product of random matrices I, *Probab. Th. Related Fields* **105** (1996), no. 1, 109–142.

[23] I. Ya. Goldsheid, G. A. Margulis, Lyapunov exponents of a product of random matrices, *Russian Math. Surveys* **44** (1989), no. 5, 11–71.

[24] Y. Guivarc'h, Produits de matrices aléatoires et applications aux propriétéie géométriques des sous-groupes du groupe linéaire, *Ergod. Th. Dynam. Syst.* **10** (1990), no. 3, 483–512.

[25] Y. Guivarc'h, L. Ji, J. C. Taylor, *Compactifications of symmetric spaces*, Progress in Mathematics **156**, Birkäuser Boston, 1998.

[26] Y. Guivarc'h, A. Raugi, Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence, *Z. Whr.* **67** (1985), 265–278.

[27] Y. Guivarc'h, A. Raugi, Products of random matrices: convergence theorems. Random matrices and their applications (Brunswick, Maine, 1984), 31–54, *Contemp. Math.*, **50**, Amer. Math. Soc., Providence.

[28] Y. Guivarc'h, A. Raugi, Propriétés de contraction d'un semi-groupe de matrices inversibles. Coefficients de Liapunoff d'un produit de matrices aléatoires indépendantes. *Israel J. Math.* **65** (1989), no. 2, 165–196.

[29] R. Gunesch, A. Katok, Construction of weakly mixing diffeomorphisms preserving measurable Riemannian metric and smooth measure (with an appendix by Alex Furman), *Discrete Contin. Dynam. Systems* **6** (2000), no. 1, 61–88.

[30] W. Jaworski, A Poisson formula for solvable Lie groups, *J. Anal. Math.* **68** (1996), 183–208.

[31] W. Jaworski, Random walks on almost connected locally compact groups: boundary and convergence, *J. Anal. Math.* **74** (1998), 235–273.

[32] V. A. Kaimanovich, An entropy condition for maximality of the boundary of random walks on discrete groups, *Soviet Math. Dokl.* **31** (1985), 193–197.

[33] V. A. Kaimanovich, Boundaries of invariant Markov operators: the identification problem, *Ergodic theory of* $\mathbb{Z}^d$ *actions* (Warwick, 1993–1994), 127–176, London Math. Soc. Lecture Note Ser., **228**, Cambridge Univ. Press, Cambridge, 1996.

[34] V. A. Kaimanovich, The Poisson formula for groups with hyperbolic properties, to appear in *Ann. of Math.*.

[35] V. A. Kaimanovich, H. Masur, The Poisson boundary of the mapping class group, *Invent. Math.* **125** (1996), 221–264.

[36] V. A. Kaimanovich, A. M. Vershik, Random walks on discrete groups: boundary and entropy, *Ann. Prob.* **11** (1983), 457–490.

[37] S. Kakutani, Random ergodic theorems and Markoff processes with a stable distribution, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, University of California Press, Berkeley and Los Angeles (1951), 247–261.

[38] H. Kesten, Full Banach mean values on countable groups, *Math. Scand.* **7**, (1959), 146–156.

[39] H. Kesten, Symmetric random walks on groups, *Trans. Amer. Math. Soc.* **92** (1959), 336–354.

[40] H. Kesten, Sums of stationary sequences cannot grow slower than linearly. *Proc. Amer. Math. Soc.* **49** (1975), 205–211.

[41] Y. Kifer, *Ergodic theory of random transformations*, Progress in Probability and Statistics **10**, Birkhuser Boston, Inc., Boston, MA, 1986.

[42] Y. Kifer, *Random perturbations of dynamical systems*, Progress in Probability and Statistics **16**, Birkhuser Boston, Inc., Boston, MA, 1988.

[43] Y. Kifer, Random dynamics and its applications, *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, Doc. Math. 1998, Extra Vol. II, 809–818.

[44] Y. Kifer, "Random" random matrix products, *J. D'Analise Math.* to appear.

[45] E. LePage, Théorèmes de la limite centrale pour certains produits de matrices aléatoires, *C. R. Acad. Sci. Paris Sr. I Math.* **292** (1981), 379–382.

[46] E. LePage, Théorèmes limites pour les produits de matrices aléatoires, Probability measures on groups (Oberwolfach, 1981), *Lecture Notes in Math.* **928**, Springer, Berlin-New York (1982), 258–303.
[47] F. Ledrappier, Poisson boundaries of discrete groups of matrices, *Isr. J. Math.* **50** (1985) no. 4, 319–336.
[48] F. Ledrappier, Some asymptotic properties of random walks on free groups, *preprint.*
[49] F. Ledrappier, Sharp estimates for the entropy, in *Harmonic analysis and discrete potential theory (Frascati, 1991)*, 281–288, Plenum, New York, 1992.
[50] F. Ledrappier, Profil d'entropie dans le cas continu, Hommage ‘a P. A. Meyer et J. Neveu, *Astérisque* No. **236**, (1996), 189–198.
[51] F. Ledrappier, L. S. Young, Entropy formula for random transformations, *Probab. Theory Related Fields* **80** (1988), no. 2, 217–240.
[52] T. Lyons, D. Sullivan, Function theory, random paths and covering spaces, *J. Diff. Geom.* **19** (1984), 299–323.
[53] G. A. Margulis, Positive harmonic functions on nilpotent groups, *Soviet Math. Dokl.* **166** (1966), 241–244.
[54] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, Berlin, 1991.
[55] A. Nevo, R. Zimmer, Rigidity of Furstenberg entropy for semisimple Lie group actions, to appear in *Ann. Sci. École Normale Supérieure.*
[56] A. Nevo, R. Zimmer, A structure theorem for actions of semisimple Lie groups, *preprint* (2000).
[57] Y. Peres, Analytic dependence of Lyapunov exponents on transition probabilities, *Lyapunov exponents (Oberwolfach, 1990), Lecture Notes in Math.*, **1486**, Springer, Berlin, (1991), 64–80.
[58] Ch. Pittet, L. Saloff-Coste, On stability of the behavior of random walks on groups, *preprint* (1998).
[59] A. Raugi, Fonctions harmoniques sur les groupes localement compacts à base dénombrable, *Bull. Soc. Math. France Mém.* No. **54**, (1977), 5–118.
[60] J. Rosenblatt, Ergodic and mixing random walks on locally compact groups, *Math. Ann.* **257** (1981), no. 1, 31–42.
[61] D. Ruelle, Analiticity properties of the characteristic exponents of random matrices, *Advances in Math.* **32** (1979), 68–80.
[62] K. Schmidt, Amenability, Kazhdan's property T, strong ergodicity and invariant means for ergodic group-actions. *Ergodic Th. Dynam. Syst.* **1** (1981), no. 2, 223–236.
[63] Y. Shalom, Random ergodic theorems, invariant means and unitary representation, *Lie groups and ergodic theory (Mumbai, 1996)*, 273–314, Tata Inst. Fund. Res. Stud. Math., 14, Tata Inst. Fund. Res., Bombay, 1998.
[64] V. N. Tutubalin, The central limit theorem for products of random matrices and some of its applications, *Symposia Math.* **21** (1977), 101–116.
[65] N. T. Varapoulos, Long range estimates for Markov chains, *Bull. Sci. Math.* **109** (1985), 225–252.
[66] N. T. Varapoulos, Information theory and harmonic functions, *Bull. Sci. Math.* **110** (1986), 347–389.
[67] A. D. Virtzer, On products of random matrices and operators, *Theory Prob. Appl.* **24** no. 2 (1979), 367–377.
[68] A. D. Virtzer, On the simplicity of the spectrum of the Lyapunov characteristic indices of a product of random matrices, *Theory Prob. Appl.* **28** no. 1 (1983), 122-135.
[69] W. Woess, *Random walks on infinite graphs and groups*, Cambridge Tracts in Mathematics **138**, Cambridge University Press, Cambridge, 2000.

[70] R. J. Zimmer, Amenable ergodic group actions and an application to Poisson boundaries of random walks, *J. Func. Anal.* **27** (1978), 350–372.

[71] R. J. Zimmer, Spectrum, entropy and geometric structures for smooth actions of Kazhdan groups, *Israel. J. Math.* **75** (1991), 65–80.

[72] R. J. Zimmer, Amenable ergodic group actions and an application to Poisson boundaries of random walks, *J. Functional Analysis* **27** (1978), 350–372.

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