RIGIDITY OF GROUP ACTIONS

II. Orbit Equivalence in Ergodic Theory

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Ergodic Theory of II_1 Group Actions

II_1 Systems:

- Γ – discrete countable group
- $(X, \mathcal{B}, \mu)$ – std prob space $\cong ([0,1], \mathrm{Borel}, \mathrm{Lebesgue})$
- $\Gamma \curvearrowright (X, \mu)$ – ergodic m.p. $(\gamma_* \mu = \mu, \quad \forall \gamma \in \Gamma)$

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- $T : X \to Y$ with $T_* \mu = \nu$ and $T(\gamma.x) = \gamma.T(x)$ a.e. $(\gamma \in \Gamma)$

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Standing Convention:

- Everything is measurable and considered modulo null sets

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equivalently, an iso $T : \mathcal{R}_{\Gamma,X} \cong \mathcal{R}_{\Lambda,Y}$ of **orbit relations**.

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Equivalence Relations (Feldman-Moore, 1977)

- Axiomatization; types: I_n , II_1 , II_∞ , III_λ , $0 \leq \lambda \leq 1$.
- Axiomatization of $L^\infty(X) \hookrightarrow \mathrm{vN}(\mathcal{R})$ ...
- Every $\mathcal{R} = \mathcal{R}_{\Gamma,X}$ for some countable $\Gamma \curvearrowright (X,\mu)$

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Example

$\mathrm{SL}_n(\mathbf{Z}) \curvearrowright \mathbf{T}^n$ are pairwise non-OE for $n \geq 3$.

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(Weak) Morphisms $\mathcal{R}_1 \to \mathcal{R}_2$ in/sur/bi-jective morphisms

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Let $\Gamma < G$ *simple* $rk(G) \geq 2$. *Then any* $\mathcal{R}_{X,\Gamma}$ *remembers* $Lie(G)$ *and* $G \curvearrowright G \times_\Gamma X$.

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Idea: Cripple a very rigid relation:
e.g. $\mathcal{R} := \mathcal{R}_{\mathrm{SL}_3(\mathbf{Z}),\mathbf{T}^3}|_A$ where $A \subset \mathbf{T}^3$ with $\mu(A) \notin \mathbf{Q}$

Rigidity for Products of Hyperboli-like groups

Theorem (Monod-Shalom 2005)

Let $\Gamma = \prod_i^n \Gamma_i \curvearrowright (X, \mu)$ *free* $n \geq 2$*, where* Γ_i *are "hyperbolic-like" and* $\Gamma_i \curvearrowright (X, \mu)$ *erg.*

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Let $\Gamma = \prod_i^n \Gamma_i \curvearrowright (X, \mu)$ *free* $n \geq 2$, *where* Γ_i *are "hyperbolic-like" and* $\Gamma_i \curvearrowright (X, \mu)$ *erg. Then*

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Theorem (Monod-Shalom 2005, Hjorth-Kechris 2004)

Let $\Gamma = \Gamma_1 \times \Gamma_2$ acts on (X, μ) with both Γ_1, Γ_2 erg,
$\alpha : \Gamma \times X \to \Gamma'$ a non-elementary cocycle into a "hyp-like" Γ'.
Then α is cohom to a hom $\rho : \Gamma \to \Gamma_i \to \Gamma'$ for $i = 1$ or 2.

Computations of $\mathrm{Out}\,(\mathcal{R}) = \mathrm{Aut}\,(\mathcal{R})/\mathrm{Inn}\,(\mathcal{R})$

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- $\Gamma \curvearrowright G/\Gamma$ gives $\mathrm{Out}(\mathcal{R}_{\Gamma,G/\Gamma}) \cong \mathbf{Z}/2\mathbf{Z}$
- $\Gamma \to K$ with K cpct connected, $\mathrm{Out}(\mathcal{R}_{\Gamma,K}) \simeq K$
- $\Gamma \to K/K_0$ with K cpct connected, $\mathrm{Out}(\mathcal{R}_{\Gamma,K}) \simeq N_K(K_0)/K_0$
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Ingredients Rigidity: $\operatorname{Out}(\mathcal{R}_{\Gamma,X})$ comes from
$\operatorname{Aut}(\Gamma \curvearrowright X)$ and quotients $X \to G/\Gamma$
uses Ratner's theorem...

Cost, L^2-Betti numbers etc. after D.Gaboriau

Definition (From groups to II_1 relations, or groupoids...)

- cost($\mathcal{R}$) in analogy to $d(\Gamma) = \min\{n \mid F_n \to \Gamma\}$ (Levitt)

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- $\mathrm{cost}(\mathcal{R}|_A) = \frac{\mu(B)}{\mu(A)} \cdot \mathrm{cost}(\mathcal{R}|_B)$

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Theorem (Gaboriau 1998)

For free actions of free groups: $\mathrm{cost}(\mathcal{R}_{F_n,X}) = n$.

Corollary

F_n and F_k with $n \neq k$ do not have free OE actions.

Cost, L^2-Betti numbers etc. after D.Gaboriau (cont.)

If $\mathrm{cost}(\mathcal{R}_{\Gamma,X})$ depends only on Γ set $=:$ **price**(Γ).

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If $\mathrm{cost}(\mathcal{R}_{\Gamma,X})$ depends only on Γ set $=:$ **price**(Γ). Examples:

- price > 1: for F_n, surface groups, ...
- price $= 1$: for amenable, $\Gamma_1 \times \Gamma_2$, $\mathrm{SL}_n(\mathbf{Z})$ ($n \geq 3$)...

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For $\Gamma \curvearrowright (X, \mu)$ free erg: $\quad \beta_i^{(2)}(\Gamma) = \beta_i^{(2)}(\mathcal{R}_{\Gamma,X})$

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For free actions $\mathcal{R}_{\Gamma,X}$ remembers the Euler characteristic $\chi(\Gamma)$

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Theorem (Gaboriau)

For f.g. Γ with an infinite normal amenable subgroup $\beta_1^{(2)}(\Gamma) = 0$

After S.Popa...

Theorem (S.Popa 2006)

Let Γ *have* (T) *and* $\Gamma \curvearrowright X = (X_0, \mu_0)^\Gamma$ *be a Bernoulli action.* *Then for any discrete* Λ *every cocycle* $\alpha : \Gamma \times X \to \Lambda$ *is cohomologous to a homomorphism* $\rho : \Gamma \to \Lambda$.

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A gold mine of applications/related results (Popa, Popa-Sasyk, Popa-Vaes,...):

- Rigidity: $\mathcal{R}_{\Gamma,X}$ determines Γ and $\Gamma \curvearrowright X$
- $\mathcal{R}$ not generated by free actions
- Prescribed countable $\mathcal{F}(\mathcal{R})$
- Prescribed $H^1(\mathcal{R}, \mathbf{T})$
- Prescribed $\mathrm{Out}(\mathcal{R})$ – any Countable$\times$Compact
- Many non-OE actions for many groups

von Neumann rigidity ! $\mathcal{F}$(factor), ... !

Many Orbit Structures for non-amenable groups

Question

Given Γ how big is $\mathrm{OrbStr}_\Gamma = \{\mathcal{R}_{\Gamma,X} \mid \Gamma \curvearrowright (X,\mu) \text{ free erg}\}$?

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Theorem (A.Ionna 2007)

The Conjecture is true for Γ containing F_2.