Orbit Equivalence since Zimmer's Cocycle Superrigidity Theorem

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Orbit Structures of II_1 Group Actions

II_1 actions (X, μ, Γ):

- Γ – discrete countable group
- $(X, \mathcal{B}, \mu)$ – std prob space $\cong ([0,1], \mathrm{Borel}, \mathrm{Lebesgue})$
- $\Gamma \curvearrowright (X, \mu)$ – ergodic m.p. $(\gamma_* \mu = \mu, \quad \forall \gamma \in \Gamma)$

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Stable, or weak, OE: $\quad (X, \mu, \Gamma) \overset{sOE}{\sim} (Y, \nu, \Lambda)$

$$X \supset A \xrightarrow{T} B \subset Y \qquad T \times T(\mathcal{R}_{X, \Gamma}|_{A \times A}) \cong \mathcal{R}_{Y, \Lambda}|_{B \times B}$$

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$\Longrightarrow$

For free II_1actions of higher rank lattices $\Gamma < G$ any $\mathcal{R}_{X,\Gamma}$ remembers G !

Cocycles and Orbit Equivalence

Definition

$c : G \times X \to H$ is a measurable **cocycle** if for all $g_1, g_2 \in G$ a.e. on X

$$c(g_2 g_1, x) = c(g_2, g_1.x) \cdot c(g_1, x)$$

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Moreover, $T' : (X, \mu) \cong (Y, \nu)$ is a measure space iso and ρ is a group iso.

Zimmer's Cocycle Superrigidity Theorem

Theorem (Zimmer, 1981)

Let G, H be (semi)simple Lie groups with $\mathrm{rk}(G) \geq 2$, $G \curvearrowright (X, \mu)$ an (irr) erg. p.m.p. action, $\alpha : G \times X \to H$ a non-compact Zariski dense cocycle.

Then α is conjugate to a homomorphism $\rho : G \to H$.

Same for cocycles $\Gamma \curvearrowright (X, \mu)$ where $\Gamma < G$ is a lattice.

A generalization of

Theorem (Margulis, 1973)

Let G, H be (semi)simple Lie groups, $\mathrm{rk}(G) \geq 2$, $\Gamma < G$ an (irr) lattice, $\rho : \Gamma \to H$ a homomorphism with unbounded Zariski dense $\rho(\Gamma)$.

Then ρ extends to $G \to H$.

Measurable Group Theory

Definition (Gromov)

Let Γ_1, Γ_2 be two groups

1. **Topological Equivalence** is a loc cpt space Σ with cont action of $\Gamma_1 \times \Gamma_2$ with $\Gamma_i \curvearrowright \Sigma$ properly disc and cocompact.
2. **Measure Equivalence** is a measure space (Ω, m) with a m.p. action of $\Gamma_1 \times \Gamma_2$ s.t. $\Gamma_i \curvearrowright \Omega$ has a finite measure fundamental domain.

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Theorem

1. $\Gamma_1 \overset{TE}{\sim} \Gamma_2$ *if and only if* $\Gamma_1 \overset{qi}{\sim} \Gamma_2$ *(Gromov).*
2. $\Gamma_1 \overset{ME}{\sim} \Gamma_2$ *if and only if* $\exists$ *free* $(X_1, \Gamma_1) \overset{sOE}{\sim} (X_2, \Gamma_2)$.

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Example

1. **Uniform** lattices Γ_1, Γ_2 in a loc cpt group G: $\Gamma_1 \curvearrowright G \curvearrowleft \Gamma_2$
2. Arbitrary lattices Γ_1, Γ_2 in a loc cpt group G: $\Gamma_1 \curvearrowright G \curvearrowleft \Gamma_2$

Measure Equivalence and Higher Rank Lattices

Theorem (F. 1999)

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Other applications

Feldman-Moore question, computations of $\mathrm{Out}\,(\mathcal{R}_{X,\Gamma}) = \mathrm{Aut}\,/\mathrm{Inn}\,$,
Enveloping grps for lattices

Rigidity for Products of Hyperbolic-like groups

Theorem (Monod-Shalom 2005)

Let $\Gamma = \prod_i^n \Gamma_i \curvearrowright (X, \mu)$ *free* $n \geq 2$, *where* Γ_i *are "hyperbolic-like" and* $\Gamma_i \curvearrowright (X, \mu)$ *erg.* $i = 1, 2$.

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- $\mathcal{R}_{X,\Gamma}$ *remembers the number of factors:* n.
- *If* $\Lambda \curvearrowright (Y, \nu)$ *is any free and mildly mixing, and* $\mathcal{R}_{X,\Gamma} \sim \mathcal{R}_{Y,\Lambda}$ *then* $\Gamma \cong \Lambda$ *and* $\Gamma \curvearrowright X \cong \Lambda \curvearrowright Y$.

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Let $A = A_1 \times A_2$ *acts on* (X, μ) *with both* $A_i \curvearrowright (X, \mu)$ *erg* $i = 1, 2$
$\alpha : A \times X \to \Gamma$ *a non-elementary cocycle into a "hyperbolic-like"* Γ.

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Results of Kida 2006,+

For Γ Mapping Class Group

- Full rigidity: $\mathrm{ME}(\Gamma) = \{\Gamma\}$, OE=isom (up to finite)
- Γ is not a lattice in any loc comp G (except trivial)
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- $\mathrm{cost}(\mathcal{R}_{X,F_n}) = n, \ldots$ (Levitt, Gaboriau)
- $\beta_n^{(2)}(\Gamma), \quad \chi(\Gamma)$ (Gaboriau)

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Applications

- Descriptive Set Theory: Adams-Kechris, Hjorth, Thomas,...

Applications to QI of amenable groups

- Shalom, Sauer

New Cocycle Superrigidity (after Sorin Popa)

Theorem (Popa 2006)

Let Γ have (T) and $\Gamma \curvearrowright X = (X_0, \mu_0)^\Gamma$ be a Bernoulli action.
Λ any discrete, or cpt (or $\in \mathcal{U}_{\mathrm{fin}}$) group.

Then any cocycle $\alpha : \Gamma \times X \to \Lambda$ is conjugate in Λ to a homomorphism $\rho : \Gamma \to \Lambda$.

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Corollary

For $\Gamma \curvearrowright (X, \mu)$ *as above,* $\mathcal{R}_{X,\Gamma}$ *remembers* Γ *and* $\Gamma \curvearrowright X$.

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Then any cocycle $\alpha : \Gamma \times X \to \Lambda$ *is conjugate in* Λ *to a homomorphism* $\rho : \Gamma \to \Lambda$.

Corollary

For $\Gamma \curvearrowright (X, \mu)$ *as above,* $\mathcal{R}_{X,\Gamma}$ *remembers* Γ *and* $\Gamma \curvearrowright X$.

Theorem (Ioana 2007)

Let Γ *have* (T), $K = \lim \Gamma/\Gamma_i$ *be a profinite completion.*
$\alpha : \Gamma \times K \to \Lambda$ *any cocycle into any* $\Lambda \in \mathcal{U}_{\mathrm{fin}}$.

Then $\exists\ i$, *and* $\rho : \Gamma_i \to \Lambda$ *so that* $\alpha|_{\Gamma_i \times K_i}$ *is conjugate to* ρ, $K_i = \overline{\Gamma_i} < K$.

One Proof

Theorem (F. after Ioana, 2007)

Let Γ *have* (T), K *compact,* $\tau : \Gamma \to K$ *dense hom.*
$\alpha : \Gamma \times K \to \Lambda$ *any cocycle into any discrete group.*

Then $\exists$ *a hom* $\rho : \Gamma' \to \Lambda$ *from a fin ind* $\Gamma' < \Gamma$, *and a finite cover* $\hat{K}' \to K' = \overline{\tau(\Gamma')}$ *so that* $\alpha : \Gamma' \times \hat{K}' \to \Lambda$ *is conjugate to* ρ.

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Proof using deformation - rigidity ideas

Proposition (Local Rigidity, after Popa, Hjorth)

Let Γ *have* (T), Λ *discrete, and* II_1*action* $\Gamma \curvearrowright (X, \mu)$. *Then close cocycles are conjugate:*

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$f_{ts}(x) = f_t(xs^{-1}) f_s(x)$ on a set of meas > 0, hence a.e.

Try to propagate to $K' = \langle U \rangle$. May need to lift to a finite cover $\hat{K}' \to K'$.

On $\hat{K}'$ we have $f_t(x) = \phi(xt^{-1})^{-1} \phi(x)$

$$\phi(\gamma x t^{-1}) \alpha(\gamma, xt^{-1}) \phi(xt^{-1}) = \phi(\gamma x) \alpha(\gamma, x) \phi(x)^{-1} \quad = \rho(\gamma)$$

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