Superrigidity and Measure Equivalence, Part I

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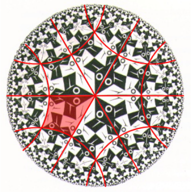
Poincaré disc and surfaces



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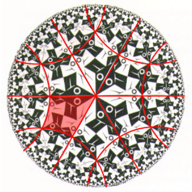

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- $SL_2(\mathbb{R})$
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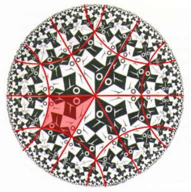

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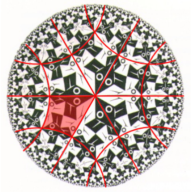

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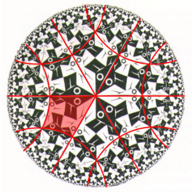

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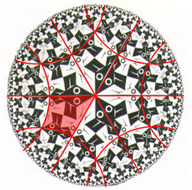

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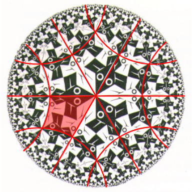

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Defn: Teichmüller space = moduli of hyperbolic metrics on Σ

$$\mathrm{Teich}(\Sigma) = \{\text{lattice embeddings} \quad \rho : \Gamma \to G\}/G$$

Flexibility of lattices in $SL_2(\mathbb{R})$

Theorem (Riemann ?, Poincaré, Teichmüller ?)

For a closed surface of genus $g \geq 2$ one has

$$\mathrm{Teich}(\Sigma) \cong \mathbb{R}^{6 \cdot g - 6}$$

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$$\Gamma \to G = \mathrm{PSL}_2(\mathbb{R})$$

where

$$\Gamma = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

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Remarks

- $\forall\ \rho_1, \rho_2 : \Gamma \to PSL_2(\mathbb{R})$ lattice embeddings

$$\exists! f \in \mathrm{Homeo}(S^1 = \partial \mathbf{H}^2) \qquad \rho_2(\gamma) = f^{-1} \circ \rho_1(\gamma) \circ f$$

- Similar results apply to non-uniform lattices.

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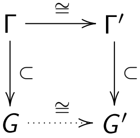

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$G = \mathrm{Isom}(\mathbf{H})$, $G' = \mathrm{Isom}(\mathbf{H}')$ where $\mathbf{H}, \mathbf{H}' \in \{\mathbf{H}^n, \mathbf{H}^n_{\mathbb{C}}, \mathbf{H}^n_{\mathbb{H}}, \mathbf{H}^2_{\mathbb{O}}\} \setminus \mathbf{H}^2$.
Let $\Gamma < G$, $\Gamma' < G'$ be uniform lattices and $j : \Gamma \cong \Gamma'$ an isomorphism.
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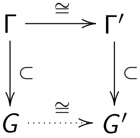

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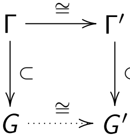
Theorem 3 (Mostow '73)

Same for any (semi)-simple G, $G' \not\simeq \mathrm{SL}_2(\mathbb{R})$
and uniform (irreducible) lattices $\Gamma < G$, $\Gamma' < G'$.



Sketch of Mostow's proof of Theorem 1

Given:

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- f is j-equivariant

More on Mostow rigidity

Theorem (Mostow's strong rigidity for non-uniform lattices)

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G - semi-simple Lie group, H - general locally compact.
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- Furman ('01): simple $\mathrm{rk}(G) \geq 2$, or $G = \mathrm{Isom}(\mathbf{H}^n_K)$ and H/Γ' compact.
- Bader-Furman-Sauer ('12): all cases (including $\mathrm{SL}_2(\mathbb{R})$) and more...

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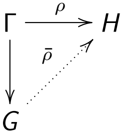

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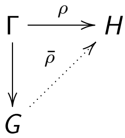

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Let $G = \prod G_i$ semi-simple, $\sum \mathrm{rk}(G_i) \geq 2$; H - simple, center-free.
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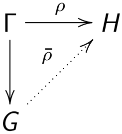

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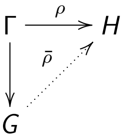

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Arith lattice ?

Something like
$\mathrm{SL}_n(\mathbb{Z}) < \mathrm{SL}_n(\mathbb{R})$

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Take $L < G \times H$ be the Z-closure of the graph of ρ

$$\Lambda_\rho = \{(\gamma, \rho(\gamma)) \in G \times H \mid \gamma \in \Gamma\}, \qquad L = \overline{\Lambda}^Z$$

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$\Gamma < G$ lattice, G' - simple, $\rho : \Gamma \to G'$ hom with Z-dense unbounded image.
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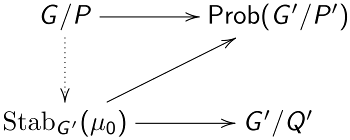

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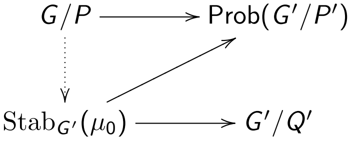

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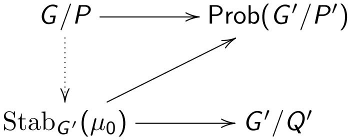

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Theorem 1 can be strengthened to

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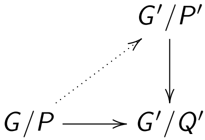

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Special case of Margulis' superrigidity

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Further superrigidity phenomena (long list of names...)

- **Other** H: CAT(-1), Gromov-hyp, Homeo(S^1), $\mathrm{MCG}(\Sigma)$, $\mathcal{C}_{reg}$, $\mathcal{S}$,...
- **Other** G: products $G = G_1 \times \cdots \times G_n$, $n \geq 2$, of general lcsc grps, $\tilde{A}_2$ groups

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* Everything is measurable, taken up to null sets !

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- $\alpha : G \times G/\Gamma \to H$ $\quad \rightsquigarrow \quad$ $\rho_x(\gamma) = \alpha(\sigma(x)\gamma\, \sigma(x)^{-1}, x)$ is a hom.

Zimmer's cocycle superrigidity

Cocycle Superrigidity Theorem (Zimmer '81)

Let G (semi)-simple, H be simple Lie groups, $\mathrm{rk}(G) \geq 2$
$G \curvearrowright (X, \mu)$ (irred) ergodic p.m.p. $c : G \times X \to H$ cocycle
where c is Zariski-dense, not compact.
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Strategy of the proof

- Boundary map: $f : X \times G/P \to H/Q$ s.t. $f_{g.x}(g\xi) = c(g, x) f_x(\xi)$.
- Ergodicity vs. smoothness of algebraic actions
- Regularity as in Margulis' proof.

Cocycles in nature

(stable) Orbit Equivalence

$\Gamma \curvearrowright (X,\mu)$ and $\Lambda \curvearrowright (Y,\nu)$ freely, and $T : X \cong Y$ with $T(\Gamma.x) = \Lambda.T(x)$
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G connected and simply connected $\curvearrowright M \quad \rightsquigarrow \quad$ a cocycle $c : G \times M \to \pi_1(M)$.

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Popa's cocycle superrigidity

Invest in the action $\Gamma \curvearrowright (X,\mu)$ rather than in Γs and Gs
(program in flux - follow the arXiv closely...)