Superrigidity revisited

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Margulis' Higher rank Superrigidity

Theorem (Margulis, 1974)

G, H (semi)simple Lie groups, $\mathrm{rk}(G) \geq 2$, H adjoint, $\Gamma < G$ an irred lattice, $\rho : \Gamma \to H$ a homomorphism with unbounded Zariski dense image $\rho(\Gamma)$.

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Cocycle Superrigidity, Zimmer 1981

$G \curvearrowright (\Omega, \mu)$ erg (irred), $c : G \times \Omega \to H ...,\ \Longrightarrow\ c(g, x) = f(gx)\rho(g)f(x)^{-1}$.

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5. $G = G_1 \times \cdots \times G_n$ – general product, H – hyperbolic-like,
   Monod-Shalom (JDG 2004), Mineyev-Monod-Shalom (Top. 2004), Hjorth-Kechris (Mem.AMS 2005).

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Superrigidity results (Bader-Furman, Bader-Furman-Shaker)

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H is a **convergence grp**, if $\exists$ $H \to \mathrm{Homeo}(M)$ with $H \curvearrowright M^3 \setminus \Delta$ is proper

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Next: consider quotients $f : B \to B'$ instead of automorphisms of B ...

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$\{e\}$	$\{e, (2,3)\}$	$\{e, (1,3)\}$	S_3	i.e. $W_Q < W$

Boundaries and generalized Weyl groups

G-boundary (Burger-Monod)

For a lcsc G a measure space (B, ν) is G-**boundary** if

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- $G = \prod^n G_i$ with non-amenble factors, W contains $(\mathbb{Z}/2\mathbb{Z})^n$

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$\square$

For $G = G_1 \times \cdots \times G_n$ take $B = B_1 \times \cdots \times B_n$ and deduce: f factors through $B \longrightarrow B_i \xrightarrow{\bar{f}} M$ for some $i \in \{1, \ldots, n\}$

The End

Thank you!