Equidistribution for groups of toral automorphisms

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Basic dynamical questions

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$T : X \to X$ homeomorphism of a compact space X
Understand the *distribution* of $x, Tx, \ldots, T^N x$ as $N \to \infty$.

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- **Closed Invariant sets**

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Standard Example

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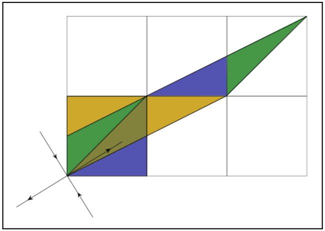

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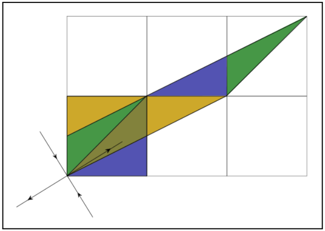

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Periodic points $= \left\{ \left(\frac{p_1}{q}, \ldots, \frac{p_d}{q} \right) + \mathbb{Z}^d : \gcd(p_1, \ldots, p_d, q) = 1 \right\}$

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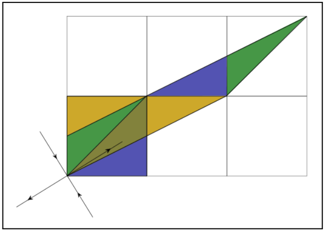

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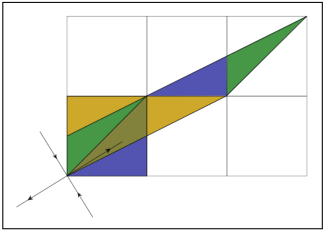

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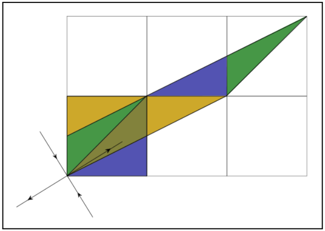

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3. No **equidistribution**

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- $\mathcal{P}_\nu(X) \neq \emptyset$, any closed invariant set supports ν-stationary measures

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4. **Equidistribution** (in fact, **quantitative**!)
   BLFM (07, 10).

[BFLM] *Stationary measures and equidistribution for orbits of non-abelian semi-groups on the torus*, JAMS to appear.

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$$\|x - \frac{\mathbf{p}}{q}\| < e^{-\lambda n} \qquad \text{with} \qquad q < \left(\frac{2\|a\|}{t}\right)^C$$

- *where $c > 0, \lambda > 0, C$ depend only on ν,*
- $\widehat{\mu}(a) = \int_{\mathbb{T}^d} e^{2\pi i \langle a,x \rangle} \, d\mu(x)$ *for* $a \in \mathbb{Z}^d$.

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Theorem (M. Burger)

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- $\mathrm{Density}(|\widehat{\mu}|^2) \geq t^2 \cdot \mathrm{Density}(\Gamma^{tr}.a) > 0$

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- $\mathrm{Density}(|\widehat{\mu}|^2) \geq t^2 \cdot \mathrm{Density}(\Gamma^{tr}.a) > 0$
- $\Rightarrow \quad \mu$ has atoms (by Wiener)

"Baby case"

Theorem (M. Burger)

Let $\mu \in \mathcal{P}(\mathbb{T}^d)$ be **invariant** *under a* **finite index** *subgroup $\Gamma < \mathrm{SL}_d(\mathbb{Z})$. Then μ is a convex combination of Leb and atomic on finite orbits.*

Proof

1. $\widehat{g_*\mu}(a) = \int_{\mathbb{T}^d} e^{2\pi i \langle a, gx\rangle}\, d\mu(x) = \int_{\mathbb{T}^d} e^{2\pi i \langle g^{tr} a, x\rangle}\, d\mu(x) = \widehat{\mu}(g^{tr} a)$
2. Wiener's Lemma: $\sum_{x \in \mathbb{T}^d} |\mu(\{x\})|^2 = \lim \frac{1}{|B_n|} \sum_{a \in B_n} |\widehat{\mu}(a)|^2$

Assume μ is Γ-invariant and $\mu \neq \mathrm{Leb}$.

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- Atoms of a Γ-inv prob measure belong to finite orbits.

First glance at the problem

Make the proof for the following effective

If $\mu = \nu * \mu$ has $|\widehat{\mu}(a)| = t > 0$ for some $a \in \mathbb{Z}^d \setminus \{0\}$. Then μ has atoms.

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2. This is 99% of the work !

General strategy

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$$\mathcal{N}_M(A_{c_1(t)} \cap [-N, N]^d) > c_2(t) \cdot \left(\frac{N}{M}\right)^d$$

where $\mathcal{N}_M(S)$ - minimal number of M-cubes needed to cover S

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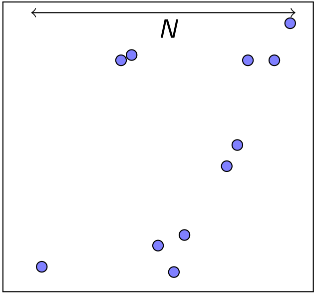

$|S| = 12$

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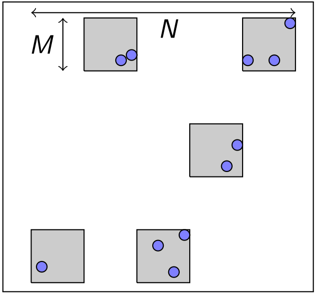

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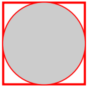
Products of random matrices

$G = KA^{+}K$: $\boxed{g = k' \,\mathrm{diag}[e^{t_1}, \ldots, e^{t_d}]\, k}$ $k, k' \in \mathrm{SO}(d)$, $t_1 \geq \cdots \geq t_d$



A

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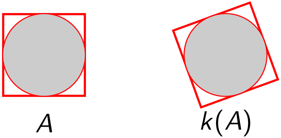

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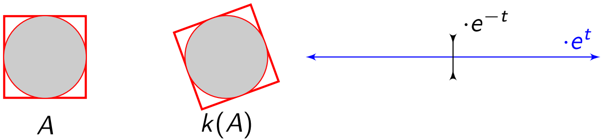

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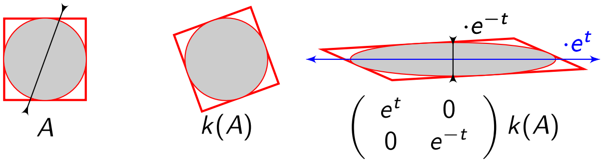

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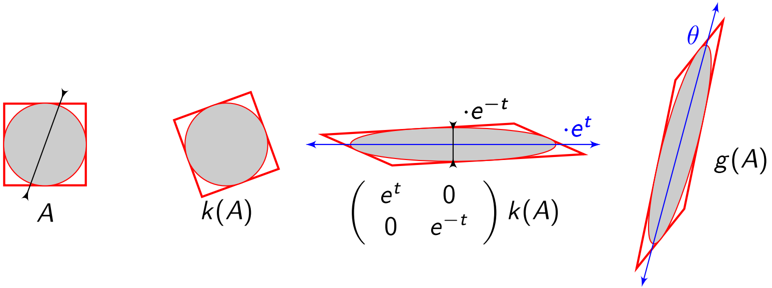

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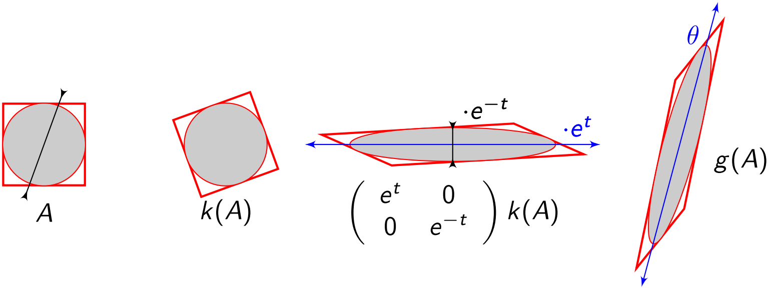

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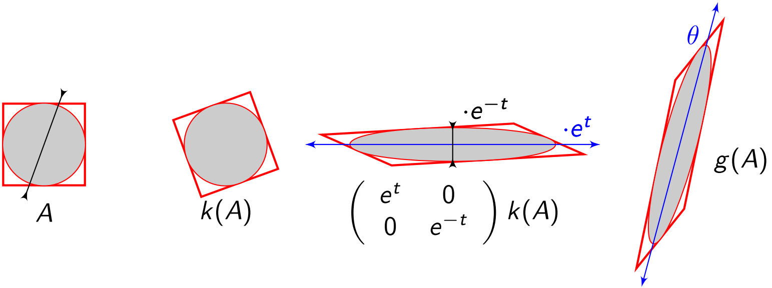

$\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} k(A)$

Furstenberg, Guivarc'h, Raugi, LaPage, Goldsheid-Margulis,...

- $\nu^{*n}\left\{ g = k' \begin{pmatrix} e^{(\lambda \pm \epsilon)n} & 0 \\ 0 & e^{(-\lambda \pm \epsilon)n} \end{pmatrix} k \right\} > 1 - e^{-cn}$

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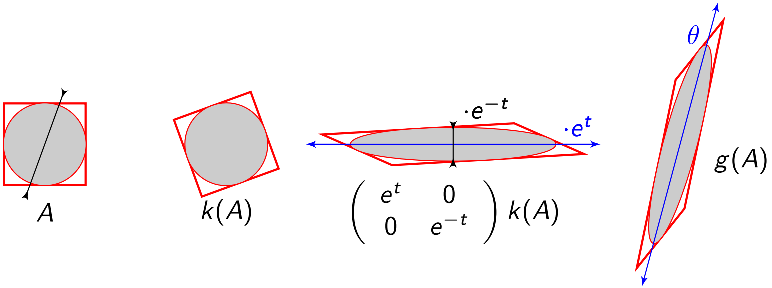

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Large scale dimension

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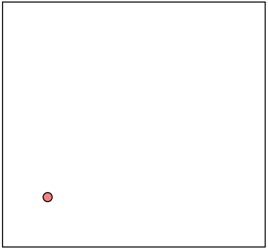
- **Given** $|\widehat{\mu}(a_0)| = t_0 > 0$



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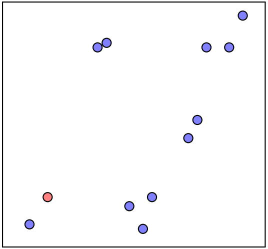

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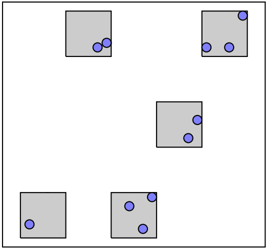

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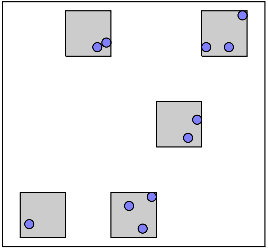

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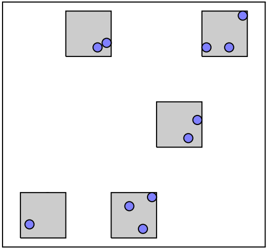

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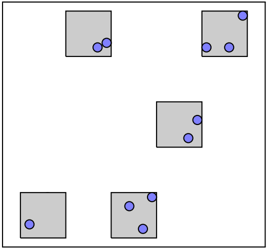

Need to improve the dimension $\alpha = \alpha_0$ to $\alpha = d$ in steps $\alpha_i \to \alpha_{i+1}$

$$\mathcal{N}_{M_i}(A_{t_i} \cap B_{0,N_i}) > \tilde{c}_i(t)\left(\frac{N_i}{M_i}\right)^{\alpha_i}$$

Additive structure of Fourier coefficients

Lemma

$$\frac{1}{|A|^2} \sum_{a,b\in A} \widehat{\mu}(a-b) \geq \left| \frac{1}{|A|} \sum_{a\in A} \widehat{\mu}(a) \right|^2$$

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$$=\int_{\mathbb{T}^d}\left(\frac{1}{|A|}\sum_{a\in A}e^{2\pi i\langle a,x\rangle}\right)\cdot\left(\overline{\frac{1}{|A|}\sum_{b\in A}e^{2\pi i\langle b,x\rangle}}\right)\,d\mu(x)$$

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□

Bourgain's Projection Theorem (informal)

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$A \subset \mathbb{R}^d$, $\dim(A) = \alpha$, $\quad \Theta \subset \mathbb{P}^{d-1}$, $\dim(\Theta) \geq \gamma$

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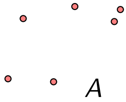


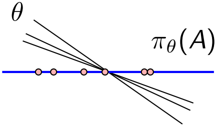
Theorem (Bourgain)

$\forall \beta, \gamma > 0$, $\exists \delta > 0$ *so that* $\forall \alpha \in [\beta, d-\beta]$

- $\eta \in \mathcal{P}(\mathbb{P}^{d-1})$ *with* $\eta(B_{\xi,r}) < r^\gamma$
- $A \subset B_{0,1}$ *with* $\mathcal{N}_r(A) \geq r^{-\alpha}$
- *A, η not too degenerate*

Then for η-most $\theta \in \mathbb{P}^{d-1}$ s.t.

$$\mathcal{N}_r(\pi_\theta(A)) > r^{-\frac{\alpha+\delta}{d}}$$



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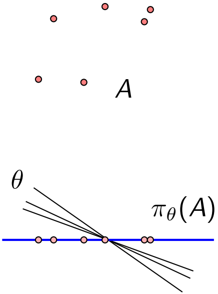

Theorem (Marstrand, Falconer)

$\dim A + \dim \eta > d \quad \rightsquigarrow \quad \exists \theta, \quad \mathrm{Leb}(\pi_\theta(A)) > 0$

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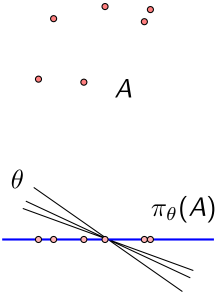
$$\exists \theta \in \Theta \qquad \dim(\pi_\theta(A)) > \frac{\alpha+\delta}{d}$$

Theorem (Bourgain)

$\forall \beta, \gamma > 0,\ \exists \delta > 0$ *so that* $\forall \alpha \in [\beta, d-\beta]$

- $\eta \in \mathcal{P}(\mathbb{P}^{d-1})$ *with* $\eta(B_{\xi,r}) < r^\gamma$
- $A \subset B_{0,1}$ *with* $\mathcal{N}_r(A) \geq r^{-\alpha}$
- *A, η not too degenerate*

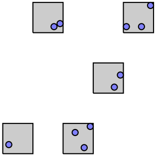
Then for η-most $\theta \in \mathbb{P}^{d-1}$ s.t.
$\quad \mathcal{N}_r(\pi_\theta(A)) > r^{-\frac{\alpha+\delta}{d}}$



Theorem (Marstrand, Falconer)

$\dim A + \dim \eta > d \quad \rightsquigarrow \quad \exists \theta, \quad \mathrm{Leb}(\pi_\theta(A)) > 0$
$\alpha + \gamma > d \quad \rightsquigarrow \quad \exists \theta, \quad \mathcal{N}_r(\pi_\theta(A)) > cr^{-1}$

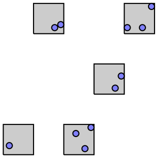
Amplifying the dimension



$A = A_{t_i} \cap B_{0,N_i}$

$\dim = \alpha_i$

Amplifying the dimension

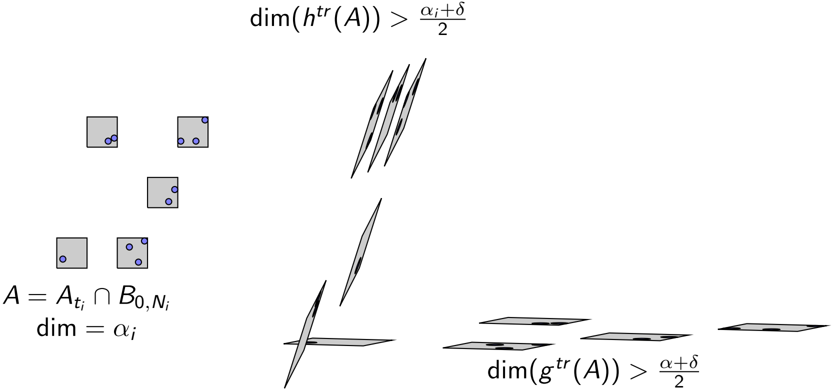


$A = A_{t_i} \cap B_{0,N_i}$

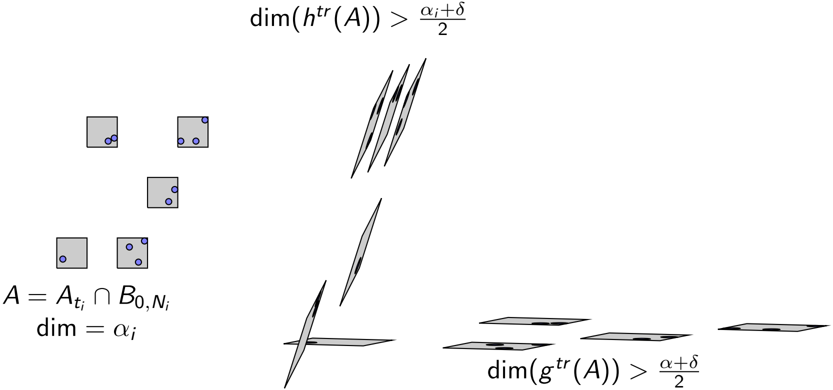
$\dim = \alpha_i$

$\dim(g^{tr}(A)) > \frac{\alpha+\delta}{2}$

Amplifying the dimension

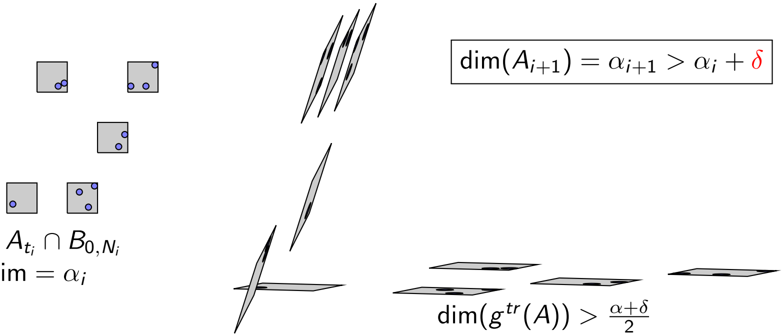


Amplifying the dimension



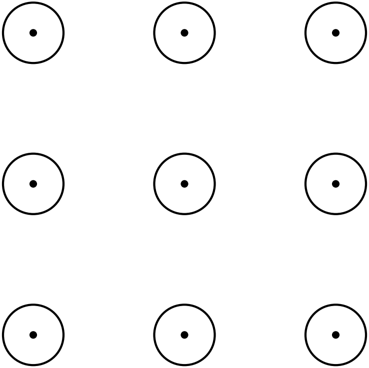
$$\exists g, h : \qquad \frac{1}{|A|^2} \sum_{a,b} |\widehat{\mu}(g^{tr} a - h^{tr} b)| \geq \left| \sum_g \nu(g) \cdot \frac{1}{|A|} \sum_a \widehat{\mu}(g^{tr} a) \right|^2$$

Amplifying the dimension

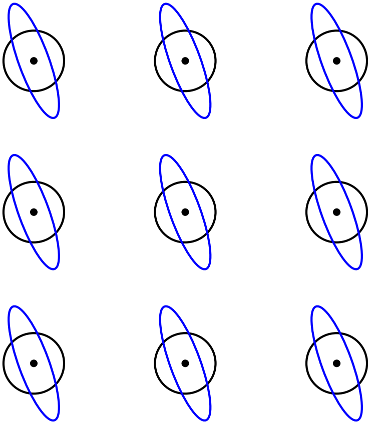


$$\exists g, h: \qquad \frac{1}{|A|^2}\sum_{a,b} |\widehat{\mu}(g^{tr}a - h^{tr}b)| \geq \left|\sum_g \nu(g)\cdot\frac{1}{|A|}\sum_a \widehat{\mu}(g^{tr}a)\right|^2$$

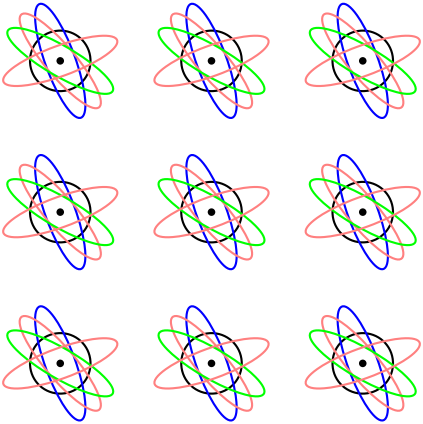
Self packing of dense balls



Self packing of dense balls



Self packing of dense balls



Self packing of dense balls