Lattice Envelopes

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Problem

Given Γ, describe all lattice envelopes: groups G with a lattice embedding $\Gamma \xrightarrow{i} G$.

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- If $\Lambda < H$ is a lattice imbedding, then $\Gamma \times \Lambda < G \times H$ is a lattice imbedding.

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Theorem

Let $F_n \longrightarrow G$ be a lattice embedding (uniform or non-uniform). Then

- *either $K \longrightarrow G \longrightarrow \mathrm{PSL}_2(\mathbb{R})$ or $\mathrm{PGL}_2(\mathbb{R})$*
- *or $K \longrightarrow G \longrightarrow H$ where $H < \mathrm{Aut}(T)$ cocompact action on a bdd deg tree*

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The uniform case uses a result of Mosher - Sageev - Whyte.

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Theorem (Rigidity of Classical lattices, extends [F. 2001])

Let $F_n \not\leq \Gamma < H$ *be (irred.) lattice in a conn, center free, (semi)-simple real Lie group* H *w/o compact factors. Let* $\Gamma \longrightarrow G$ *be a lattice imbedding.*

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Theorem (Rigidity of S-arithmetic lattices)

Let $\Gamma < H = H^{(\infty)} \times H^{(\mathrm{fin})}$ be an S-arithmetic lattice $\mathbf{H}(k(S)) < \prod_{\nu \in S} \mathbf{H}(k_\nu)$. Let $\Gamma \to G$ be a lattice imbedding. Then up to fin ind and compact kernel

- *either G is $H^{(\infty)} \times H^{(\mathrm{fin}),*}$, where $H^{(\mathrm{fin})} < H^{(\mathrm{fin}),*} < \mathrm{Aut}(X_{\mathrm{B-T}})$*
- *or G is Γ.*

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- *a uniform lattice in a totally disconnected group $H < \mathrm{Homeo}(M)$. If Γ is a PD hyperbolic group, then $H \simeq \Gamma$ (after M.Mj).*

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1. *either a classical (irreducible) lattice in a semi-simple real Lie group*
2. *or an S-arithmetic lattice*
3. *or a lattice in a totally disconnected group (uniform if Γ is torsion free).*

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For any lcsc G the quotient $G/\mathrm{Rad}_{\mathrm{am}}(G)$ has $L \times D$ as fin ind subgroup, where

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Proposition (using Breuillard-Gelander)

Assume Γ has no infinite amenable commensurated subgroups.
Let $\Gamma \longrightarrow G$ be lattice imbedding. Then $\mathrm{Rad}_{\mathrm{am}}(G)$ is compact.

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The proof uses Margulis' commensurator superrigidity and arithmeticity theorems.

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2. The sequence $1 \to N \to D' \to D'_1 \to 1$ splits as $D' \simeq D_1 \times M = H^{(\mathrm{fin})} \times M$ where $N < M$ is a lattice
3. Moreover $D \simeq H^{(\mathrm{fin}),*} \times M$

The proof uses a property of class C and Margulis' normal subgroup theorem, and quasi-isometric rigidity.

Proposition

The sequence $1 \to \Gamma_1 \to \Gamma \to \Gamma_0 \to 1$ splits as $\Gamma \simeq \Gamma_1 \times \Gamma_0 \simeq \Lambda \times N \times \Gamma_0$
The group $G/\mathrm{Rad}_{\mathrm{am}}(G) \simeq L \times D$ splits as $L_0 \times (L_1 \times D_1) \times M$

Step 3: reconstructing Γ

Proposition

1. The sequence $1 \to N \to \Gamma_1 \to \Lambda \to 1$ splits as $\Gamma_1 \simeq \Lambda \times N$
2. The sequence $1 \to N \to D' \to D'_1 \to 1$ splits as
   $D' \simeq D_1 \times M = H^{(\mathrm{fin})} \times M$
   where $N < M$ is a lattice
3. Moreover $D \simeq H^{(\mathrm{fin}),*} \times M$

The proof uses a property of class C and Margulis' normal subgroup theorem, and quasi-isometric rigidity.

Proposition

The sequence $1 \to \Gamma_1 \to \Gamma \to \Gamma_0 \to 1$ splits as $\Gamma \simeq \Gamma_1 \times \Gamma_0 \simeq \Lambda \times N \times \Gamma_0$
The group $G/\mathrm{Rad}_{\mathrm{am}}(G) \simeq L \times D$ splits as $L_0 \times (L_1 \times D_1) \times M$

1. $\Gamma_0 < L_0$ product of classical lattices
2. $\Lambda < L_1 \times D_1$ product of S-arithmetic lattices
3. $N < M$ lattice in a totally disconnected lc group.

The end

Thank you !