The space of metrics on Gromov hyperbolic groups

Alex Furman

University of Illinois at Chicago

Northwestern University, 2010-10-31

Negatively Curved Manifolds: Geometry

Setting

(M, g) where M - closed manifold, g - Riemannian metric with $K < 0$

Negatively Curved Manifolds: Geometry

Setting

(M, g) where M - closed manifold, g - Riemannian metric with $K < 0$

Marked Length Spectrum

- Free homotopy classes $[S^1; M] = \{S^1 \to M\}/\sim$.

Negatively Curved Manifolds: Geometry

Setting

(M, g) where M - closed manifold, g - Riemannian metric with $K < 0$

Marked Length Spectrum

- Free homotopy classes $[S^1; M] = \{S^1 \to M\} / \sim$.
- $\forall c_0 \neq c \in [S^1; M], \quad \exists!$ closed geodesic geo_c in c.

Negatively Curved Manifolds: Geometry

Setting

(M, g) where M - closed manifold, g - Riemannian metric with $K < 0$

Marked Length Spectrum

- Free homotopy classes $[S^1; M] = \{S^1 \to M\}/\sim$.
- $\forall c_0 \neq c \in [S^1; M], \quad \exists!$ closed geodesic geo_c in c.
- Marked Length Spectrum: $c \mapsto \ell_g(c) = \text{Length}_g(\text{geo}_c)$.

Negatively Curved Manifolds: Geometry

Setting

(M, g) where M - closed manifold, g - Riemannian metric with $K < 0$

Marked Length Spectrum

- Free homotopy classes $[S^1; M] = \{S^1 \to M\}/\sim$.
- $\forall c_0 \neq c \in [S^1; M]$, $\exists!$ closed geodesic geo_c in c.
- Marked Length Spectrum: $c \mapsto \ell_g(c) = \text{Length}_g(\text{geo}_c)$.

Marked Length Spectrum Rigidity

- Conjecture (Burns-Katok '85): ℓ_g determines g, up to $\text{Diff}(M)^0$

Negatively Curved Manifolds: Geometry

Setting

(M, g) where M - closed manifold, g - Riemannian metric with $K < 0$

Marked Length Spectrum

- Free homotopy classes $[S^1; M] = \{S^1 \to M\}/\sim$.
- $\forall c_0 \neq c \in [S^1; M]$, $\exists!$ closed geodesic geo_c in c.
- Marked Length Spectrum: $c \mapsto \ell_g(c) = \mathrm{Length}_g(\mathrm{geo}_c)$.

Marked Length Spectrum Rigidity

- Conjecture (Burns-Katok '85): ℓ_g determines g, up to $\mathrm{Diff}(M)^0$
- Deformation rigidity (Guillemin-Kazhdan '80)
- Surfaces (Otal '90, Croke '90)
- (M, g) loc. symmetric (Hamenstädt '99, using BCG)

Negatively Curved manifolds: Dynamics of (SM, ϕ^t)

- Topological entropy h_{top} of ϕ^t on SM

Negatively Curved manifolds: Dynamics of (SM, ϕ^t)

- Topological entropy h_{top} of ϕ^t on SM
- Stable/Unstable foliations

Negatively Curved manifolds: Dynamics of (SM, ϕ^t)

- Topological entropy h_{top} of ϕ^t on SM
- Stable/Unstable foliations
- Bowen-Margulis measure μ_{BM} on SM

Negatively Curved manifolds: Dynamics of (SM, ϕ^t)

- Topological entropy h_{top} of ϕ^t on SM
- Stable/Unstable foliations
- Bowen-Margulis measure μ_{BM} on SM which
  1. is the unique measure of maximal entropy:

$$\mathrm{Ent}(SM, \phi^t, \mu_{\text{BM}}) = h_{\text{top}}$$

Negatively Curved manifolds: Dynamics of (SM, ϕ^t)

- Topological entropy h_{top} of ϕ^t on SM
- Stable/Unstable foliations
- Bowen-Margulis measure μ_{BM} on SM which
  1. is the unique measure of maximal entropy:

$$\mathrm{Ent}(SM, \phi^t, \mu_{\text{BM}}) = h_{\text{top}}$$

  2. is weal limit of periodic orbits organized by length

$$\mu_{\text{BM}} = \lim_{T\to\infty} \frac{1}{\#\{c \mid \ell(c) < T\}} \cdot \sum_{\{c \mid \ell(c) < T\}} \lambda(\text{geo}_c)$$

Negatively Curved manifolds: Dynamics of (SM, ϕ^t)

- Topological entropy h_{top} of ϕ^t on SM
- Stable/Unstable foliations
- Bowen-Margulis measure μ_{BM} on SM which
  1. is the unique measure of maximal entropy:

$$\mathrm{Ent}(SM, \phi^t, \mu_{\text{BM}}) = h_{\text{top}}$$

  2. is weal limit of periodic orbits organized by length

$$\mu_{\text{BM}} = \lim_{T\to\infty} \frac{1}{\#\{c \mid \ell(c) < T\}} \cdot \sum_{\{c \mid \ell(c) < T\}} \lambda(\text{geo}_c)$$

  3. has conditionals on stable/unstable scaled by $e^{\pm ht}$ where $h = h_{\text{top}}$

$$d\phi^t_* \mu^{(s)}_{\text{BM}} = e^{-ht} \cdot d\mu^{(s)}_{\text{BM}}, \qquad d\phi^t_* \mu^{(u)}_{\text{BM}} = e^{+ht} \cdot d\mu^{(u)}_{\text{BM}}$$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes:

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \left\{ \langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e \right\}$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \{\langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e\}$
- g on M $\rightsquigarrow$ Γ-invariant metric $\tilde{g}$ on $\tilde{M}$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \big\{\langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e\big\}$
- g on M $\quad\rightsquigarrow\quad$ Γ-invariant metric $\tilde{g}$ on $\tilde{M}$

$$\ell_g(\langle\gamma\rangle) = \min_{x\in\tilde{M}} \mathrm{dist}_{\tilde{g}}(\gamma \cdot x, x)$$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \big\{\langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e\big\}$
- g on M $\quad\rightsquigarrow\quad$ Γ-invariant metric $\tilde{g}$ on $\tilde{M}$

$$\ell_g(\langle\gamma\rangle) = \min_{x\in\tilde{M}} \mathrm{dist}_{\tilde{g}}(\gamma\cdot x, x) = \lim_{n\to\infty} \frac{1}{n}\mathrm{dist}_{\tilde{g}}(\gamma^n y, y)$$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \{\langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e\}$
- g on M $\rightsquigarrow$ Γ-invariant metric $\tilde{g}$ on $\tilde{M}$

$$\ell_g(\langle\gamma\rangle) = \min_{x\in\tilde{M}} \mathrm{dist}_{\tilde{g}}(\gamma \cdot x, x) = \lim_{n\to\infty} \frac{1}{n} \mathrm{dist}_{\tilde{g}}(\gamma^n y, y)$$

- Top entropy = volume entropy = Γ-orbit growth

$$h_{\mathrm{top}} = \lim_{R\to\infty} \frac{1}{R} \log \mathrm{vol}_{\tilde{g}}(B_{x,R})$$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \{\langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e\}$
- g on M $\rightsquigarrow$ Γ-invariant metric $\tilde{g}$ on $\tilde{M}$

$$\ell_g(\langle\gamma\rangle) = \min_{x\in\tilde{M}} \mathrm{dist}_{\tilde{g}}(\gamma \cdot x, x) = \lim_{n\to\infty} \frac{1}{n} \mathrm{dist}_{\tilde{g}}(\gamma^n y, y)$$

- Top entropy = volume entropy = Γ-orbit growth

$$h_{\mathrm{top}} = \lim_{R\to\infty} \frac{1}{R} \log \mathrm{vol}_{\tilde{g}}(B_{x,R}) = \lim_{R\to\infty} \frac{1}{R} \log \#(\Gamma \cdot y \cap B_{x,R})$$

Negatively Curved manifolds Inside-Out

- Instead of M think of $\widetilde{M}$ or better $\Gamma = \pi_1(M, x)$
- F.h.c. $[S^1; M]$ are conj classes: $C_\Gamma = \{\langle\gamma\rangle = \{a\gamma a^{-1}\}_{a\in\Gamma} \mid \gamma \neq e\}$
- g on M $\rightsquigarrow$ Γ-invariant metric $\tilde{g}$ on $\tilde{M}$

$$\ell_g(\langle\gamma\rangle) = \min_{x\in\tilde{M}} \mathrm{dist}_{\tilde{g}}(\gamma\cdot x, x) = \lim_{n\to\infty} \frac{1}{n}\mathrm{dist}_{\tilde{g}}(\gamma^n y, y)$$

- Top entropy = volume entropy = Γ-orbit growth

$$h_{\mathrm{top}} = \lim_{R\to\infty} \frac{1}{R}\log \mathrm{vol}_{\tilde{g}}(B_{x,R}) = \lim_{R\to\infty}\frac{1}{R}\log \#(\Gamma\cdot y \cap B_{x,R})$$

- Bowen-Margulis measure μ_{BM} vs. Patterson-Sullivan current m_{PS}

$$\mathrm{Meas}(SM)^{\phi^t} \quad\leftrightarrow\quad \mathrm{Meas}(S\tilde{M})^{\phi^t\times\Gamma} \quad\leftrightarrow\quad \mathrm{Meas}(\partial\widetilde{M}\times\partial\tilde{M})^{\Gamma}$$

Metrics on Negatively Curved Groups

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{\text{left invariant metrics on } \Gamma \text{ q.i. to a word metric}\}/\sim$

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{\text{left invariant metrics on } \Gamma \text{ q.i. to a word metric}\}/\sim$
  where $d_1 \sim d_2$ if $|d_1 - d_2|$ is bounded

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{$left invariant metrics on Γ q.i. to a word metric$\}/\sim$
  where $d_1 \sim d_2$ if $|d_1 - d_2|$ is bounded

Examples

1. $\Gamma = \pi_1(M, x)$ with $[d_g]$ where $d_{g,x}(\gamma_1, \gamma_2) = \mathrm{dist}_{\tilde{g}}(\gamma_1.x, \gamma_2.x)$

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{\text{left invariant metrics on } \Gamma \text{ q.i. to a word metric}\}/\sim$
  where $d_1 \sim d_2$ if $|d_1 - d_2|$ is bounded

Examples

1. $\Gamma = \pi_1(M, x)$ with $[d_g]$ where $d_{g,x}(\gamma_1, \gamma_2) = \mathrm{dist}_{\tilde{g}}(\gamma_1.x, \gamma_2.x)$
   Note: $d_{g,x} \sim d_{g,y}$ because $|d_{g,x} - d_{g,y}| \leq \mathrm{dist}_{\tilde{g}}(\Gamma.x, \Gamma.y) \leq \mathrm{diam}(M, g)$.

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{\text{left invariant metrics on } \Gamma \text{ q.i. to a word metric}\}/\sim$
  where $d_1 \sim d_2$ if $|d_1 - d_2|$ is bounded

Examples

1. $\Gamma = \pi_1(M, x)$ with $[d_g]$ where $d_{g,x}(\gamma_1, \gamma_2) = \mathrm{dist}_{\tilde{g}}(\gamma_1.x, \gamma_2.x)$
   Note: $d_{g,x} \sim d_{g,y}$ because $|d_{g,x} - d_{g,y}| \leq \mathrm{dist}_{\tilde{g}}(\Gamma.x, \Gamma.y) \leq \mathrm{diam}(M, g)$.
2. $\Gamma \to \mathrm{Isom}(X)$ where X is CAT(-1) space, and Γ is convex cocompact

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{$left invariant metrics on Γ q.i. to a word metric$\}/\sim$ where $d_1 \sim d_2$ if $|d_1 - d_2|$ is bounded

Examples

1. $\Gamma = \pi_1(M, x)$ with $[d_g]$ where $d_{g,x}(\gamma_1, \gamma_2) = \mathrm{dist}_{\tilde{g}}(\gamma_1.x, \gamma_2.x)$
   Note: $d_{g,x} \sim d_{g,y}$ because $|d_{g,x} - d_{g,y}| \leq \mathrm{dist}_{\tilde{g}}(\Gamma.x, \Gamma.y) \leq \mathrm{diam}(M, g)$.
2. $\Gamma \to \mathrm{Isom}(X)$ where X is CAT(-1) space, and Γ is convex cocompact
3. Γ - Gromov hyperbolic, $[d]$ where d - a word metric

Metrics on Negatively Curved Groups

General Setting

- Γ torsion free Gromov-hyperbolic group
- $D_\Gamma = \{\text{left invariant metrics on } \Gamma \text{ q.i. to a word metric}\}/\sim$
  where $d_1 \sim d_2$ if $|d_1 - d_2|$ is bounded

Examples

1. $\Gamma = \pi_1(M, x)$ with $[d_g]$ where $d_{g,x}(\gamma_1, \gamma_2) = \mathrm{dist}_{\tilde{g}}(\gamma_1.x, \gamma_2.x)$
   Note: $d_{g,x} \sim d_{g,y}$ because $|d_{g,x} - d_{g,y}| \leq \mathrm{dist}_{\tilde{g}}(\Gamma.x, \Gamma.y) \leq \mathrm{diam}(M, g)$.
2. $\Gamma \to \mathrm{Isom}(X)$ where X is CAT(-1) space, and Γ is convex cocompact
3. Γ - Gromov hyperbolic, $[d]$ where d - a word metric
4. ...

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Marked Length $\qquad \ell_{[d]}(\langle \gamma \rangle) = \lim_{n \to \infty} \frac{1}{n} d(\gamma^n, e)$

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Marked Length $\qquad \ell_{[d]}(\langle\gamma\rangle) = \lim_{n\to\infty} \frac{1}{n} d(\gamma^n, e)$

Growth/Entropy $\qquad h_{[d]} = \lim_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(\gamma, e) < R\}$

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Marked Length $\qquad \ell_{[d]}(\langle\gamma\rangle) = \lim_{n\to\infty} \frac{1}{n} d(\gamma^n, e)$

Growth/Entropy $\qquad h_{[d]} = \lim_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(\gamma, e) < R\}$

Theorem

Given $[d] \in D_\Gamma$ *there is a Radon measure* $m_{[d]}$ *on* $\partial^{(2)}\Gamma = \partial\Gamma \times \partial\Gamma \setminus \Delta$

- $m_{[d]}$ *is* Γ*-invariant and ergodic*
- $dm_{[d]}(x, y) = e^{2h_{[d]}\cdot F(x,y)}\, d\nu(x)\, d\nu(y)$ *where*

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Marked Length $\qquad \ell_{[d]}(\langle\gamma\rangle) = \lim_{n\to\infty} \frac{1}{n} d(\gamma^n, e)$

Growth/Entropy $\qquad h_{[d]} = \lim_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(\gamma, e) < R\}$

Theorem

Given $[d] \in D_\Gamma$ there is a Radon measure $m_{[d]}$ on $\partial^{(2)}\Gamma = \partial\Gamma \times \partial\Gamma \setminus \Delta$

- *$m_{[d]}$ is Γ-invariant and ergodic*
- *$dm_{[d]}(x, y) = e^{2h_{[d]} \cdot F(x,y)}\, d\nu(x)\, d\nu(y)$ where*
  1. $\nu \in \mathrm{Prob}(\partial\Gamma)$ *with* $\frac{d\gamma_*\nu}{d\nu}(x) = e^{h_{[d]}(d(e,x) - d(\gamma,x)) + O(1)}$
  2. *F measurable, bdd away from $(x \mid y)_e$, or $d(e, \mathrm{geo}_{x,y})$*

Generalizing Geometric concepts

For Gromov hyperbolic Γ and $[d] \in D_\Gamma$ define

Marked Length $\qquad \ell_{[d]}(\langle\gamma\rangle) = \lim_{n\to\infty} \frac{1}{n} d(\gamma^n, e)$

Growth/Entropy $\qquad h_{[d]} = \lim_{R\to\infty} \frac{1}{R} \log \#\{\gamma \in \Gamma \mid d(\gamma, e) < R\}$

Theorem

Given $[d] \in D_\Gamma$ there is a Radon measure $m_{[d]}$ on $\partial^{(2)}\Gamma = \partial\Gamma \times \partial\Gamma \setminus \Delta$

- *$m_{[d]}$ is Γ-invariant and ergodic*
- *$dm_{[d]}(x, y) = e^{2h_{[d]}\cdot F(x,y)}\, d\nu(x)\, d\nu(y)$ where*
  1. $\nu \in \mathrm{Prob}(\partial\Gamma)$ *with* $\frac{d\gamma_*\nu}{d\nu}(x) = e^{h_{[d]}(d(e,x)-d(\gamma,x))+O(1)}$
  2. *F measurable, bdd away from $(x \mid y)_e$, or $d(e, \mathrm{geo}_{x,y})$*

Based on Coornaert's Patterson-Sullivan theory for Gromov hyperbolic groups, and If $c : \Gamma \times X \to \mathbb{R}$ cocycle with $|c(-, x)| \leq M(x)$, then $c(\gamma, z) = b(\gamma \cdot z) - b(z)$.

Relating D_Γ, $\mathrm{Meas}(\partial^{(2)}\Gamma)$, and $\mathbb{R}_+^{C_\Gamma}$

Relating D_Γ, Meas$(\partial^{(2)}\Gamma)$, and $\mathbb{R}_+^{C_\Gamma}$

Theorem

For $d_1, d_2 \in D_\Gamma$ the following are equivalent

1. $d_1 \sim c \cdot d_2$ *so* $c = h_{[d_2]}/h_{[d_1]}$,
2. $\ell_{[d_1]} = c \cdot \ell_{[d_2]}$,
3. $m_{[d_1]} \not\perp m_{[d_2]}$,
4. $m_{[d_1]} = m_{[d_2]}$.

Relating D_Γ, Meas($\partial^{(2)}\Gamma$), and $\mathbb{R}_+^{C_\Gamma}$

Theorem

For $d_1, d_2 \in D_\Gamma$ the following are equivalent

1. $d_1 \sim c \cdot d_2$ *so* $c = h_{[d_2]}/h_{[d_1]}$,
2. $\ell_{[d_1]} = c \cdot \ell_{[d_2]}$,
3. $m_{[d_1]} \not\perp m_{[d_2]}$,
4. $m_{[d_1]} = m_{[d_2]}$.

Corollary

For $\Gamma = \pi_1(M, p)$ *one has* $\quad \mathrm{Riem}_{<0}(M) \longrightarrow \mathrm{Riem}_{<0}^{\mathrm{MLS}}(M) \hookrightarrow D_\Gamma$.

Relating D_Γ, Meas$(\partial^{(2)}\Gamma)$, and $\mathbb{R}_+^{C_\Gamma}$

Theorem

For $d_1, d_2 \in D_\Gamma$ the following are equivalent

1. $d_1 \sim c \cdot d_2$ *so* $c = h_{[d_2]}/h_{[d_1]}$,
2. $\ell_{[d_1]} = c \cdot \ell_{[d_2]}$,
3. $m_{[d_1]} \not\perp m_{[d_2]}$,
4. $m_{[d_1]} = m_{[d_2]}$.

Corollary

For $\Gamma = \pi_1(M, p)$ *one has* $\quad \mathrm{Riem}_{<0}(M) \longrightarrow \mathrm{Riem}^{\mathrm{MLS}}_{<0}(M) \hookrightarrow D_\Gamma$.

(1) $\implies$ (2) by construction, (3) $\implies$ (4) from ergodicity. (4) $\implies$ (1) ...
(2) $\implies$ (3) is proved using an analogue of Bowen's construction - weak limits of

$$\frac{1}{\#\{\langle\gamma\rangle \in C_\Gamma \mid \ell_{[d]}(\langle\gamma\rangle) < R\}} \cdot \sum_{\{\gamma\in\Gamma \mid \ell_{[d]}(\gamma)<R\}} \delta_{(\gamma_-,\gamma_+)}.$$

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

1. Teich$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{R}))/\,\mathsf{PSL}_2(\mathbb{R})$

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

1. Teich$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{R}))/\,\mathsf{PSL}_2(\mathbb{R})$
2. QF$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{C}))/\,\mathsf{PSL}_2(\mathbb{C})$

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

1. $\mathsf{Teich}(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{R}))/\,\mathsf{PSL}_2(\mathbb{R})$
2. $\mathsf{QF}(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{C}))/\,\mathsf{PSL}_2(\mathbb{C})$
3. $\mathsf{Riem}_{<0}(\Sigma) =$ Riemannian metrics with $K < 0$ mod $\mathsf{Diff}(\Sigma)^0$

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

1. Teich$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{R}))/\,\mathsf{PSL}_2(\mathbb{R})$
2. QF$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{C}))/\,\mathsf{PSL}_2(\mathbb{C})$
3. Riem$_{<0}(\Sigma)$ = Riemannian metrics with $K < 0$ mod $\mathsf{Diff}(\Sigma)^0$
4. Word metrics d_S

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

1. Teich$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{R}))/\,\mathsf{PSL}_2(\mathbb{R})$
2. QF$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{C}))/\,\mathsf{PSL}_2(\mathbb{C})$
3. $\mathrm{Riem}_{<0}(\Sigma)$ = Riemannian metrics with $K < 0$ mod $\mathrm{Diff}(\Sigma)^0$
4. Word metrics d_S

Theorem

- *The maps* $\mathrm{Riem}_{<0}(\Sigma) \hookrightarrow D_\Gamma$ *and* $\mathrm{QF}(\Sigma) \hookrightarrow D_\Gamma$ *are injective.*

Distinguishing metrics on surface groups

Examples

Metrics on $\Gamma = \pi_1(\Sigma)$ where Σ higher genus closed surface

1. Teich$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{R}))/\,\mathsf{PSL}_2(\mathbb{R})$
2. QF$(\Sigma) = \mathrm{Hom}_{cc}(\Gamma, \mathsf{PSL}_2(\mathbb{C}))/\,\mathsf{PSL}_2(\mathbb{C})$
3. Riem$_{<0}(\Sigma)$ = Riemannian metrics with $K < 0$ mod Diff$(\Sigma)^0$
4. Word metrics d_S

Theorem

- *The maps* Riem$_{<0}(\Sigma) \hookrightarrow D_\Gamma$ *and* QF$(\Sigma) \hookrightarrow D_\Gamma$ *are injective.*
- *The spaces*
  1. Teich(Σ)
  2. QF$(\Sigma) \setminus$ Teich(Σ)
  3. Riem$_{<0}(\Sigma) \setminus$ Teich(Σ)
  4. *Word metrics on* $\Gamma = \pi_1(\Sigma)$

  have disjoint images in D_Γ.

Hidden symmetries of a metric

Goal

Define and describe the group of hidden/rough symmetries of $(\Gamma, [d])$
When is this group richer than Γ?

Hidden symmetries of a metric

Goal

Define and describe the group of hidden/rough symmetries of $(\Gamma, [d])$
When is this group richer than Γ?

Definition

Given $[d] \in D_\Gamma$ define $$H_{[d]} = \{h \in \mathrm{Homeo}(\partial\Gamma) \mid (h \times h)_* m_{[d]} = m_{[d]}\}.$$

Hidden symmetries of a metric

Goal

Define and describe the group of hidden/rough symmetries of $(\Gamma, [d])$
When is this group richer than Γ?

Definition

Given $[d] \in D_\Gamma$ define $$H_{[d]} = \left\{ h \in \mathrm{Homeo}(\partial\Gamma) \mid (h \times h)_* m_{[d]} = m_{[d]} \right\}.$$

Theorem

For any $[d] \in D_\Gamma$ the group $H_{[d]}$ is a locally compact group
$\Gamma < H_{[d]}$ is a cocompact lattice.

Hidden symmetries of a metric

Goal

Define and describe the group of hidden/rough symmetries of $(\Gamma, [d])$
When is this group richer than Γ?

Definition

Given $[d] \in D_\Gamma$ define $$H_{[d]} = \{h \in \mathrm{Homeo}(\partial\Gamma) \mid (h \times h)_* m_{[d]} = m_{[d]}\}.$$

Theorem

For any $[d] \in D_\Gamma$ the group $H_{[d]}$ is a locally compact group
$\Gamma < H_{[d]}$ is a cocompact lattice.

Examples

1. $\Gamma = F_n$ with d word metric $\rightsquigarrow$ $H_{[d]} = \mathrm{Aut}(T_{2n})$

Hidden symmetries of a metric

Goal

Define and describe the group of hidden/rough symmetries of $(\Gamma, [d])$
When is this group richer than Γ?

Definition

Given $[d] \in D_\Gamma$ define $$H_{[d]} = \left\{ h \in \mathrm{Homeo}(\partial\Gamma) \mid (h \times h)_* m_{[d]} = m_{[d]} \right\}.$$

Theorem

For any $[d] \in D_\Gamma$ the group $H_{[d]}$ is a locally compact group
$\Gamma < H_{[d]}$ is a cocompact lattice.

Examples

1. $\Gamma = F_n$ with d word metric $\rightsquigarrow$ $H_{[d]} = \mathrm{Aut}(T_{2n})$
2. $\Gamma < \mathrm{Isom}(\mathbf{H}^n_K)$ with $d = \mathrm{dist}_{\mathbf{H}^n_K}$ $\rightsquigarrow$ $H_{[d]} = \mathrm{Isom}(\mathbf{H}^n_K)$ with $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$

The most symmetric groups and metrics

Theorem

Let $\Gamma = \pi_1(M)$ *where* M *admits n.c. metric. Then*

- *either* $H_{[d]}$ *is discrete and* $[H_{[d]} : \Gamma] < \infty$,
- *or* Γ *is a uniform lattice in* $\mathrm{Isom}(\mathbf{H}^n_K)$ *where* $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$
  $M = \Gamma \backslash \mathbf{H}^n_K$ *and* $d \sim c \cdot \mathrm{dist}_{\mathbf{H}^n_K}$ *and* $H_{[d]} \simeq \mathrm{Isom}(\mathbf{H}^n_K)$.

The most symmetric groups and metrics

Theorem

Let $\Gamma = \pi_1(M)$ where M admits n.c. metric. Then

- *either $H_{[d]}$ is discrete and $[H_{[d]} : \Gamma] < \infty$,*
- *or Γ is a uniform lattice in $\mathrm{Isom}(\mathbf{H}^n_K)$ where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$*
  $M = \Gamma \backslash \mathbf{H}^n_K$ and $d \sim c \cdot \mathrm{dist}_{\mathbf{H}^n_K}$ and $H_{[d]} \simeq \mathrm{Isom}(\mathbf{H}^n_K)$.

This uses recent results of Mahan Mj on Hilbert-Smith conjecture.

The most symmetric groups and metrics

Theorem

Let $\Gamma = \pi_1(M)$ where M admits n.c. metric. Then

- *either $H_{[d]}$ is discrete and $[H_{[d]} : \Gamma] < \infty$,*
- *or Γ is a uniform lattice in $\mathrm{Isom}(\mathbf{H}^n_K)$ where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ $M = \Gamma \backslash \mathbf{H}^n_K$ and $d \sim c \cdot \mathrm{dist}_{\mathbf{H}^n_K}$ and $H_{[d]} \simeq \mathrm{Isom}(\mathbf{H}^n_K)$.*

This uses recent results of Mahan Mj on Hilbert-Smith conjecture.

Theorem

Let $\Gamma = F_n$ and $[d] \in D_\Gamma$. Then

- *either $H_{[d]}$ is discrete and $[H_{[d]} : F_n] < \infty$,*
- *or $d \sim d_S$ – word metric; in which case $H_{[d]} \simeq \mathrm{Aut}(T)$.*

Related results on general lattices

Theorem (Bader-Furman-Sauer)

Let H be a lcsc group containing $\Gamma = F_n$ as a lattice.
Then, up to finite index and compact kernel

- *either $H \simeq \Gamma$ (trivial lattice),*
- *or $H \simeq \mathsf{PSL}_2(\mathbb{R})$ (non-uniform lattice),*
- *or H is a non-discrete closed subgroup of $\mathsf{Aut}(\mathrm{Tree})$ (uniform lattice).*

Related results on general lattices

Theorem (Bader-Furman-Sauer)

Let H be a lcsc group containing $\Gamma = F_n$ as a lattice.
Then, up to finite index and compact kernel

- *either $H \simeq \Gamma$ (trivial lattice),*
- *or $H \simeq \mathrm{PSL}_2(\mathbb{R})$ (non-uniform lattice),*
- *or H is a non-discrete closed subgroup of $\mathrm{Aut}(\mathrm{Tree})$ (uniform lattice).*

Theorem (Bader-Furman-Sauer)

Let Γ be a Gromov-hyperbolic PD-group, H a lcsc group, $\Gamma < H$ lattice.
Then, up to finite index and compact kernel

- *either $H \simeq \Gamma$,*
- *or Γ is a cocompact rank one lattice and $H \simeq \mathrm{Isom}(\mathbf{H}^n_K)$.*

The last result uses recent results of Mahan Mj on Hilbert-Smith conjecture.

Minimal entropy characterization

Minimal entropy characterization

Definition

Let Γ be Gromov-hyperbolic group, $[d] \in D_\Gamma$. Let

$$\kappa_{[d]} = \inf\{\kappa > 0 \mid \exists \text{ rough isometric embedding } (\Gamma, \kappa \cdot d) \to \mathbf{H}^{\infty}_{\mathbb{R}}\}$$

After Bonk - Schramm.

Minimal entropy characterization

Definition

Let Γ be Gromov-hyperbolic group, $[d] \in D_\Gamma$. Let

$$\kappa_{[d]} = \inf\{\kappa > 0 \mid \exists \text{ rough isometric embedding } (\Gamma, \kappa \cdot d) \to \mathbf{H}^\infty_{\mathbb{R}}\}$$

After Bonk - Schramm.

Theorem (after Bourdon)

Let $M = \Gamma \backslash \mathbf{H}^n_K$ where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Then

$$\frac{h_{[d]}}{\kappa_{[d]}} \geq kn + k - 2, \qquad k = \dim_{\mathbb{R}} K$$

with equality attained iff $d \sim c \cdot \mathrm{dist}_{\mathbf{H}^n_K}$.

Thanks

Thank you.

Thanks

Thank you.

Applause to the Organizers!

Keith Burns, John Franks, Bryna Kra,
Clark Robinson, Amie Wilkinson, Jeff Xia

Thank you!