HOMEWORK #10
DUE NOON, APRIL 17, 2009

(1) Prove that the function $f(x) = x^3$ is (Riemann) integrable on $[0, 1]$ and show that
$$\int_0^1 x^3 dx = \frac{1}{4}.$$
(Without using formulae for integration that you learnt in previous calculus classes...)

You may use the identity $\sum_{i=1}^{n} i^3 = \frac{1}{4}(n^4 + 2n^3 + n^2)$.

(2) Suppose that $g(x)$ is a continuous function on an interval $[a, b]$ such that $g(x) > 0$ for all x. Show that
$$\int_a^b g(x)dx > 0.$$

(3) Let $f : [1, 3] \to \mathbb{R}$ be defined by:
$$f(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ 0 & \text{if } x \in (2, 3] \cap \mathbb{Q} \\ 1 & \text{if } x \in (2, 3] \setminus \mathbb{Q} \end{cases}$$
Prove that f is not Riemann integrable.

(4) Define $p : [0, 2] \to \mathbb{R}$ as follows:
$$p(x) = \begin{cases} x, & \text{if } x \leq 1 \\ 1 & \text{if } x > 1 \end{cases}$$
Prove that $p(x)$ is Riemann integrable on $[0, 2]$ and determine
$$\int_0^2 p(x)dx.$$

(5) Suppose that $f : [a, b] \to \mathbb{R}$ is (Riemann) integrable on $[a, b]$. Prove that
$$\int_a^b f(x)dx = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f\left(a + i\frac{b-a}{n}\right).$$