HOMEWORK #11
DUE NOON, APRIL 24, 2009

(1) Suppose that $f, g : [a, b] \to \mathbb{R}$, and suppose that f is integrable on $[a, b]$. Suppose also that there are finitely many points $c_1, \ldots, c_k \in [a, b]$ so that for all $y \in [a, b] \setminus \{c_1, \ldots, c_k\}$ we have $f(y) = g(y)$. Prove that g is integrable on $[a, b]$ and that

$$\int_a^b f = \int_a^b g.$$

(2) Use the Riemann integral to evaluate the following limit (where $p = \frac{1}{2}$ and where $p \in \mathbb{N}$ is arbitrary):

$$\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}.$$

(3) For $n \in \mathbb{N}$, define a function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(x) = \frac{nx^{n-1}}{1 + x}.$$

(a) Show that f_n is integrable on $[0, 1]$.

(b) Prove that for all $y \in (0, 1)$ we have $\lim\limits_{n\to\infty} f_n(y) = 0$.

(c) Prove that $\lim\limits_{n\to\infty} \int_0^1 f_n \neq 0$.

(4) For $n \in \mathbb{N}$, let $f_n : [a, b] \to \mathbb{R}$ be functions integrable on $[a, b]$. Suppose that there is a (bounded) function $f : [a, b] \to \mathbb{R}$ so that f is integrable on $[a, b]$ and so that for all $x \in [a, b]$ we have

$$\lim_{n\to\infty} \left[\sup_{x\in[a,b]} \{|f_n(x) - f(x)|\} \right] = 0.$$

Prove that

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$