

HOMEWORK #5 SOLUTIONS

- (1) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = \lfloor x \rfloor$, so f is the function which takes a real number x to the greatest integer n so that $n \leq x$. Let $g : [1, \infty) \rightarrow \mathbb{R}$ be defined by $g(x) = \frac{1}{f(x)} = \frac{1}{\lfloor x \rfloor}$. Find $\lim_{x \rightarrow \infty} g(x)$ and prove that your answer is correct.

Solution:

Note that for any $x \in \mathbb{R}$ there is a single integer in $(x - 1, x]$. Let n be this integer. Then $\lfloor x \rfloor = n$. Thus, $0 \leq x - 1 < \lfloor x \rfloor$, for all $x \in [1, \infty)$.

We claim that $\lim_{x \rightarrow \infty} g(x) = 0$.

Let $\epsilon > 0$ be arbitrary, and let $M = \frac{1}{\epsilon} + 1$.

Then, if $x > M$, we have $\lfloor x \rfloor > x - 1 > \frac{1}{\epsilon} > 0$, and we have

$$\begin{aligned} |g(x) - 0| &= \frac{1}{\lfloor x \rfloor} \\ &< \frac{1}{x - 1} \\ &< \epsilon, \end{aligned}$$

as required. Thus we have proved that $\lim_{x \rightarrow \infty} g(x) = 0$.

- (2) From the definition of limit, prove that

$$\lim_{x \rightarrow 2} \frac{x - 1}{x + 4} = \frac{1}{6}.$$

Solution:

Suppose that $\epsilon > 0$.

Let $\delta = \min\{1, 6\epsilon\}$.

If $|x - 2| < \delta$, then $x \in (1, 3)$ and $x + 4 \in (5, 7)$, so $|x + 4| = x + 4 > 5$.

Now, if $0 < |x - 2| < \delta$, then

$$\begin{aligned} \left| \frac{x - 1}{x + 4} - \frac{1}{6} \right| &= \left| \frac{6(x - 1) - (x + 4)}{6(x + 4)} \right| \\ &= \left| \frac{5(x - 2)}{6(x + 4)} \right| \\ &= \frac{5}{6} \cdot \frac{|x - 2|}{|x + 4|} \\ &< \frac{5}{6} \cdot \frac{6\epsilon}{5} = \epsilon, \end{aligned}$$

since $|x - 2| < \delta \leq 6\epsilon$, and $|x + 4| > 5$.

Therefore, $\lim_{x \rightarrow 2} \frac{x-1}{x+4} = \frac{1}{6}$, as required.

(3) Define $h : (-1, 1) \rightarrow \mathbb{R}$ as follows: If $x = \frac{1}{n}$ for some integer $n \neq 0$, let $h(x) = 1$. For all other x , let $h(x) = 0$.

(a) If $a = \frac{1}{n}$ for some integer $n \neq 0$, show that $\lim_{x \rightarrow a} h(x) = 0$, despite the fact that $h(a) = 1$.

(b) Prove that if $a \neq \frac{1}{n}$ for any integer n , and $a \neq 0$, then we still have $\lim_{x \rightarrow a} h(x) = 0$.

Solution:

Part (a): Suppose that $a = \frac{1}{n}$, for some $n \in \mathbb{Z}$.

Let $d = \min\{|\frac{1}{n} - \frac{1}{n-1}|, |\frac{1}{n} - \frac{1}{n+1}|\}$. (Which one of these is smaller depends on whether n is positive or negative.)

Then $(\frac{1}{n} - d, \frac{1}{n} + d) \subseteq (\frac{1}{n-1}, \frac{1}{n+1})$, so the only real number x which is such that $x \in (\frac{1}{n} - d, \frac{1}{n} + d)$ and $x = \frac{1}{j}$ for some $j \in \mathbb{Z}$ is n itself.

Now, let $\epsilon > 0$ and let $\delta = d = \min\{|\frac{1}{n} - \frac{1}{n-1}|, |\frac{1}{n} - \frac{1}{n+1}|\}$. Then if $0 < |x - a| < \delta$ then $h(x) = 0$ (since $x \in (\frac{1}{n} - d, \frac{1}{n} + d) \setminus \{\frac{1}{n}\}$, so x is not of the form $\frac{1}{j}$ for any $j \in \mathbb{Z}$.) Therefore

$$|h(x) - 0| = 0 < \epsilon.$$

Thus we have proved that $\lim_{x \rightarrow a} h(x) = 0$, as required.

Part (b): Now suppose that $a \neq 0$ and that $\frac{1}{a} \notin \mathbb{Z}$, but that $a \in (-1, 1)$. Then there is some integer $n \neq 0, 1$ so that $\frac{1}{a} \in (n-1, n)$. This means that $a \in (\frac{1}{n}, \frac{1}{n-1})$.

Let $d_1 = \min\{|a - \frac{1}{n}|, |a - \frac{1}{n-1}|\}$. Then $d_1 > 0$, and if $x \in (a - d_1, a + d_1)$ then $h(x) = 0$.

Now let $\epsilon > 0$ be arbitrary, and let $\delta = d_1$. If $0 < |x - a| < \delta$ then $h(x) = 0$, as noted above. For such an x , we have

$$|h(x) - 0| = |0 - 0| = 0 < \epsilon.$$

Therefore we have proved that $\lim_{x \rightarrow a} h(x) = 0$, as required.

(4) Let h be the function defined in Question 3 above. Prove that $\lim_{x \rightarrow 0} h(x)$ does not exist.

Solution:

Let $L \in \mathbb{R}$ be arbitrary. We prove that $\lim_{x \rightarrow 0} h(x) \neq L$. This will prove that this limit does not exist.

Let $\epsilon = \frac{1}{3}$, and let $\delta > 0$ be arbitrary. There are $x_1, x_2 \in (-\delta, \delta) \setminus \{0\}$ so that

- (i) $x_1 = \frac{1}{n}$ for some $n \in \mathbb{Z}$ (take an integer n so that $n > \frac{1}{\delta}$, and let $x_1 = \frac{1}{n}$); and
- (ii) x_2 is not of the form $\frac{1}{m}$ for any $m \in \mathbb{Z}$ (for example, take an integer i so that $i > \frac{1}{\delta^2}$ and so that i is not a perfect square, and let $x_2 = \frac{1}{\sqrt{i}}$).

Now, $h(x_1) = 1$ and $h(x_2) = 0$. Then either

$$|h(x_1) - L| > \frac{1}{3}, \text{ or } |h(x_2) - L| > \frac{1}{3}.$$

(This is because $1 = |h(x_1) - h(x_2)| \geq |h(x_1) - L| + |h(x_2) - L|$, so it can't be that both of the quantities are bounded above by $\frac{1}{3}$.)

This proves that $\lim_{x \rightarrow 0} h(x) \neq L$, as required.