**Math 313 - Analysis I** **Spring 2009**

# HOMEWORK #6
# DUE NOON, FEBRUARY 27, 2009

(1) Let $f$ be a function such that $|f(u)-f(v)| \leq \sqrt{|u-v|}$ for all points $u$ and $v$ in an interval $[a, b]$. Prove that $f$ is continuous at each point of $[a, b]$. (This includes continuous from the right at $a$ and from the left at $b$.)

(2) Give an example of a function that is continuous at one point of an interval, and discontinuous at all other points of the interval, or prove that there is no such function.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that for all $\lambda, x \in \mathbb{R}$ we have $f(\lambda x) = \lambda f(x)$ (such a function $f$ is called *homogeneous*). Prove that $f$ is continuous at all points $a \in \mathbb{R}$.

(4) Let $(a_n)$ and $(b_n)$ be two sequences of real numbers ($n \in \mathbb{N}$). Suppose that $a_n > 0$ for all $n$ and that the series $\sum_{n=1}^{\infty} a_n$ converges, with sum $a$. For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x < b_n \\ a_n & \text{if } x \geq b_n \end{cases} .$$

Now define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove that:

(i) $f(x)$ is defined for all $x \in \mathbb{R}$.
(ii) $f$ is a nondecreasing function.
(iii) $f$ is discontinuous at all points in $A = \{b_n : n \in \mathbb{N}\}$.
(iv) $f$ is continuous at all points in $\mathbb{R} \setminus A$.