HOMEWORK #6
WORKED SOLUTIONS

(1) Let f be a function such that $|f(u)-f(v)| \leq \sqrt{|u-v|}$ for all points u and v in an interval $[a,b]$. Prove that f is continuous at each point of $[a,b]$. (This includes continuous from the right at a and from the left at b.)

Solution: Suppose that $c \in [a,b]$. Let $\epsilon > 0$ be arbitrary, and let $\delta = \epsilon^2$. Then, for any $x \in \text{Dom}(f)$ so that $|x-c| < \delta$ we have

$$|f(x) - f(c)| < \sqrt{|x-c|} < \sqrt{\epsilon^2} = \epsilon.$$

Thus, if $c = a$, we have proved

$$\lim_{x \to a^+} f(x) = f(a).$$

If $c = b$ we have proved

$$\lim_{x \to b^-} f(x) = f(b).$$

Finally, if $x \in (a,b)$ we have proved

$$\lim_{x \to c} f(x) = f(c).$$

In all of these cases, we have proved that f is continuous at c, as required.

(NOTE: The statement '$x \in \text{Dom}(f)$ and $|x-c| < \delta$' is what allows us to deal with the cases $c = a$ and $c = b$ at the same time as the case $c \in (a,b)$.)

(2) Give an example of a function that is continuous at one point of an interval, and discontinuous at all other points of the interval, or prove that there is no such function.

Solution: See Example 3.8 on pages 83 and 84 of Howie.

[Of course, I expected *you* to write something for this question.]

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that for all $\lambda, x \in \mathbb{R}$ we have $f(\lambda x) = \lambda f(x)$ (such a function f is called *homogeneous*). Prove that f is continuous at all points $a \in \mathbb{R}$.

Solution: Let $a = f(1)$. Then, for any $x \in \mathbb{R}$ we have

$$f(x) = f(x.1) = x.f(1) = ax,$$

so f has the form $f(x) = ax$ for a constant $a \in \mathbb{R}$.

Now, the function $g(x) = a$ is continuous at all points of $\mathbb{R}$, by work in class, as is the function $h(x) = x$. Therefore, by Theorem 3.11(iii), the function $f = g.h$ is continuous at all points of $\mathbb{R}$.

(4) Let (a_n) and (b_n) be two sequences of real numbers ($n \in \mathbb{N}$). Suppose that $a_n > 0$ for all n and that the series $\sum_{n=1}^{\infty} a_n$ converges, with sum a. For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x < b_n \\ a_n & \text{if } x \geq b_n \end{cases}.$$

Now define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove that:

(i) $f(x)$ is defined for all $x \in \mathbb{R}$.
(ii) f is a nondecreasing function.
(iii) f is discontinuous at all points in $A = \{b_n : n \in \mathbb{N}\}$.
(iv) f is continuous at all points in $\mathbb{R} \setminus A$.

Solution:
(i) For any $x \in \mathbb{R}$, define the sequence (c_n^x) as follows:

$$c_n^x = \begin{cases} a_n & \text{if } x \geq b_n \\ 0 & \text{if } x < b_n. \end{cases}$$

Then, for any $x \in \mathbb{R}$ we have $f(x) = \sum_{n=1}^{\infty} c_n^x$.
Now, we have $0 \leq c_n^x \leq a_n$, so by the Comparison Theorem we know that

$$\sum_{n=1}^{\infty} c_n^x$$

exists.
That is to say that $f(x)$ is well-defined.
(ii) Certainly, if $x < y$ then $c_n^x \leq x_n^y$ for all n. This means that

$$\sum_{n=1}^{\infty} c_n^x \leq \sum_{n=1}^{\infty} c_n^y.$$

This means that $f(x) \leq f(y)$ is $x < y$, as required.
(iii) Let $n \in \mathbb{N}$ and consider the function f at the point b_n.
Let $\epsilon = a_n$. Then, if $y < b_n$ we have $c_n^{b_n} = a_n$ and $c_n^y = 0$. Therefore, $f(b_n) \geq f(y) + a_n$. Let $\delta > 0$ be arbitrary. There is $y \in (b_n - \delta, b_n)$, and for such a y we have

$$|f(b_n) - f(y)| \geq a_n.$$

This proves that $\lim_{x \to b_n} f(x) \neq f(b_n)$, which is to say that f is not continuous at b_n.
(iv) Now, suppose that $d \notin A$.
If $y \neq d$ then define a sequence $(e_n^{d,y})$ by

$$e_n^{d,y} = \begin{cases} a_n & \text{if } b_n \in (d, y] \\ 0 & \text{otherwise.} \end{cases}$$

Then if $y > d$ we have $f(y) \geq f(d)$ and $f(y) - f(d) = \sum_{n=1}^{\infty} e_n^{d,y}$ (note that $e_n^{d,y}$ measures when c_n^d is different to c_n^y so the sum measures the difference between the sum for $f(d)$ and the sum for $f(y)$).

If $d > y$ then $f(d) \geq f(y)$ and $f(d) - f(y) = \sum_{n=1}^{\infty} e_n^{y,d}$.

Now, fix $\epsilon > 0$. Because the series $\sum_{n=1}^{\infty} a_n$ converges, there is an $N \in \mathbb{N}$ so that for all $m \geq N$ we have

$$a - \sum_{n=1}^{m} a_n < \epsilon.$$

(Note that $a > \sum_{n=1}^{m} a_n$ since all a_n are greater than 0.)

Now, $a - \sum_{n=1}^{m} a_n < \epsilon$ if and only if

$$\sum_{n=m+1}^{\infty} a_n < \epsilon.$$

Fix $m \geq N$. Since the set $\{b_1, \ldots, b_m\}$ is finite, there is a $\delta > 0$ so that

$$\{b_1, \ldots, b_m\} \cap (d - \delta, d + \delta) = \emptyset.$$

(just take δ to be the minimum of the distances from d to b_i, with $i \in \{1, \ldots, m\}$. Since $d \notin A$ all of these distances are positive).

Then, if $y \in (d, d + \delta)$ we have $e_i^{d,y} = 0$ for all $i \in \{1, \ldots, m\}$ and

$$|f(y) - f(d)| = f(y) - f(d) = \sum_{n=1}^{\infty} e_n^{d,y} \leq \sum_{n=m+1}^{\infty} a_n < \epsilon.$$

Similarly, if $y \in (d - \delta, d)$ we have $e_i^{y,d} = 0$ for all $i \in \{1, \ldots, m\}$ and

$$|f(y) - f(d)| = f(d) - f(y) = \sum_{n=1}^{\infty} e_n^{y,d} \leq \sum_{n=m+1}^{\infty} a_n < \epsilon.$$

Thus we have proved that for all $y \in (d - \delta, d + \delta) \setminus \{d\}$ we have

$$|f(y) - f(d)| < \epsilon,$$

which is to say that $\lim_{x \to d} f(x) = f(d)$, so f is continuous at d, as required.