Math 313 - Analysis I **Spring 2009**

HOMEWORK #7
DUE NOON, MARCH 6, 2009

(1) Suppose that $f : [a, b] \to \mathbb{R}$ is a function, and that $c \in (a, b)$. Prove that if the limit
$$\lim_{x \to c} \frac{f(x) - f(c)}{(x - c)^2}$$
exists then f is continuous at c.

(2) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function so that for all $p \in \mathbb{Q} \cap [a, b]$ $f(p) = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

(3) (a) Suppose that $f : [a, b] \to [a, b]$ is a function so that for all $x, y \in [a, b]$ with $x \neq y$ we have
$$|f(x) - f(y)| < |x - y|$$
Prove that f is continuous on $[a, b]$.

(b) Let f be as in Part (a). Prove that there is a point $c \in [a, b]$ so that $f(c) = c$. (Such a point c is called a *fixed point* of f.)

(c) Let f be as in Part (a). Prove that f has exactly one fixed point.

(d) Give an example of a function $f : [0, 1] \to [0, 1]$ so that for all $x, y \in [0, 1]$ we have
$$|f(x) - f(y)| \leq |x - y|$$
but so that f has more than one fixed point.

(4) Let $f(x) = \frac{1}{x^2}$. Prove that

(a) f is continuous on (a, ∞) if $a \geq 0$.

(b) If $a > 0$ then f is uniformly continuous on (a, ∞).

(c) f is not uniformly continuous on $(0, \infty)$.