Math 313 - Analysis I **Spring 2009**

HOMEWORK #7
WORKED SOLUTIONS

(1) Suppose that $f : [a, b] \to \mathbb{R}$ is a function, and that $c \in (a, b)$. Prove that if the limit

$$\lim_{x\to c} \frac{f(x) - f(c)}{(x - c)^2}$$

exists then f is continuous at c.

Solution:

Let $\lim_{x\to c} \frac{f(x)-f(c)}{(x-c)^2} = L$.

Since, $\lim_{x\to c} x = c$, we have $\lim_{x\to c}(x - c) = c - c = 0$, and so

$$\lim_{x\to c}(x - c)^2 = 0.0 = 0.$$

Therefore,

$$\begin{aligned}
\lim_{x\to c} f(x) - f(c) &= \lim_{x\to c} \frac{f(x) - f(c)}{(x - c)^2}(x - c)^2 \\
&= \lim_{x\to c} \frac{f(x) - f(c)}{(x - c)^2} \lim_{x\to c}(x - c)^2 \\
&= L.0 = 0
\end{aligned}$$

and so

$$\lim_{x\to c} f(x) = \lim_{x\to c} f(x)-f(c)+f(c) = \lim_{x\to c} f(x)-f(c)+\lim_{x\to c} f(c) = 0+f(c) = f(c),$$

which is to say that f is continuous at c, as required.

(2) Suppose that $f : [a, b] \to \mathbb{R}$ is a continuous function so that for all $p \in \mathbb{Q} \cap [a, b]$ $f(p) = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Solution:

Let $x \in [a, b]$. If $x \in \mathbb{Q}$ then we already know that $f(x) = 0$. So, suppose that $x \notin \mathbb{Q}$.

We know that $\lim_{y\to x} f(y) = f(x)$. Suppose that $f(x) \neq 0$ and let $\epsilon = |f(x)|$. There is some $\delta > 0$ so that for any $z \in [a, b]$ so that $0 < |x - z| < \delta$ we have $|f(z) - f(x)| < \epsilon = |f(x)|$. In particular, for such a z we have $f(z) \neq 0$. However, there is a $z \in \mathbb{Q}$ so that $0 < |x - z| < \delta$, and for such a z we have $f(z) = 0$. This is a contradiction, so we must have $f(x) = 0$, as required.

(3) (a) Suppose that $f : [a, b] \to [a, b]$ is a function so that for all $x, y \in [a, b]$ with $x \neq y$ we have

$$|f(x) - f(y)| < |x - y|$$

Prove that f is continuous on $[a, b]$.

(b) Let f be as in Part (a). Prove that there is a point $c \in [a, b]$ so that $f(c) = c$. (Such a point c is called a *fixed point* of f.)

(c) Let f be as in Part (a). Prove that f has exactly one fixed point.

(d) Give an example of a function $f : [0, 1] \to [0, 1]$ so that for all $x, y \in [0, 1]$ we have

$$|f(x) - f(y)| \leq |x - y|$$

but so that f has more than one fixed point.

Solution:

(a) Suppose that $c \in [a, b]$, and let $\epsilon > 0$. Let $\delta = \epsilon$. If $0 < |x - c| < \delta$ and $x \in [a, b]$ then $|f(x) - f(c)| < |x - c| < \delta = \epsilon$. Thus $\lim\limits_{x \to c} f(x) = f(c)$, and f is continuous at c.

(b) Consider the function $g : [a, b] \to \mathbb{R}$, defined by $g(x) = x - f(x)$. Since f is continuous on $[a, b]$, and so is the identity function, g is continuous on $[a, b]$. Since $f(a), f(b) \in [a, b]$ we have $g(a) \leq 0$ and $g(b) \geq 0$. If $g(a) = 0$ then $f(a) = a$ is the required fixed point. Similarly, if $g(b) = 0$ then b is the fixed point.

Therefore, suppose that $g(a) < 0$ and $g(b) > 0$. By the Intermediate Value Theorem, there is a $c \in [a, b]$ so that $g(c) = 0$. We have $f(c) = c$, and c is the fixed point.

(c) Suppose that $c \neq d$ and that $f(c) = c$ and $f(d) = d$. Then $|c - d| = |f(c) - f(d)| < |c - d|$, which is a contradiction. Therefore, there cannot be two fixed points.

(d) The function $f(x) = x$ will do.

(4) Let $f(x) = \frac{1}{x^2}$. Prove that

(a) f is continuous on (a, ∞) if $a \geq 0$.

(b) If $a > 0$ then f is uniformly continuous on (a, ∞).

(c) f is not uniformly continuous on $(0, \infty)$.

Solution:

(a) Let $a \geq 0$ and suppose that $c \in (a, \infty)$. We proved in lectures that the function $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2$ is continuous at every point in $\mathbb{R}$. Therefore, since $g(c) \neq 0$ we know that $f(x) = \frac{1}{x^2} = \frac{1}{g(x)}$ is also continuous at c.

(b) Let $a > 0$.

Then for any $x, c \in (a, \infty)$ we have

$$\begin{aligned} |f(x) - f(c)| &= |\frac{1}{x^2} - \frac{1}{c^2}| \\ &= |\frac{c^2 - x^2}{x^2c^2}| \\ &= |c - x|.|\frac{x + c}{x^2c^2}| \\ &= |x - c|.|\frac{1}{xc^2} + \frac{1}{x^2c}| \\ &< |x - c|.\frac{2}{a^3}, \end{aligned}$$

since $x, c > a > 0$.

Now, let $\epsilon > 0$ be arbitrary, and let $\delta = \frac{1}{2}\epsilon.a^3$. Then, if $x, c \in (a, \infty)$ and $0 < |x - c| < 0$ we have

$$|f(x) - f(c)| < |x - c|.\frac{2}{a^3} < \frac{1}{2}\epsilon.a^3\frac{2}{a^3} = \epsilon.$$

Thus f is uniformly continuous on (a, ∞), as required.

(c) By the above calculation, we have, for any $x, c \in (0, \infty)$, with $x \neq c$

$$|f(x) - f(c)| = |x - c|.|\frac{x + c}{x^2c^2}|$$

Let $\epsilon = 1$, and let $\delta > 0$ be arbitrary.

We need to find $x, c \in (0, \infty)$ so that $0 < |x - c| < \delta$, but $|f(x) - f(c)| \geq \epsilon = 1$.

Let $c = \min\{\delta, \sqrt{3}\}$, and let $x = \frac{c}{2}$. Since $c \leq \delta$, we have $|x - c| = \frac{c}{2} < c \leq \delta$. Since $c > 0$ we have $x \neq c$, Therefore, $0 < |x - c| < \delta$. We now estimate $|f(x) - f(c)|$.

Well, $x + c = \frac{3}{2}c$ and $x^2c^2 = \frac{1}{4}c^4$, since $x < c$. Therefore

$$\frac{x + c}{x^2c^2} = \frac{\frac{3}{2}c}{\frac{1}{4}c^4} = \frac{3}{\frac{1}{2}c^3}.$$

So, we have

$$\begin{aligned} |f(x) - f(c)| &= |x - c|.\frac{3}{\frac{1}{2}c^3} \\ &= \frac{c}{2}.\frac{3}{\frac{1}{2}c^3} \\ &= \frac{3}{c^2} \\ &\geq \frac{3}{(\sqrt{3})^2} \text{ since } c \leq \sqrt{3} \end{aligned}$$

$$
\begin{aligned}
&= \frac{3}{3} \\
&= 1.
\end{aligned}
$$

Thus, we have shown that f is not uniformly continuous on $(0, \infty)$.