# HOMEWORK #8
# DUE NOON, MARCH 13, 2009

(1) Let $A$ be a nonempty subset of $\mathbb{R}$. Define a function $f_A : \mathbb{R} \to \mathbb{R}$ by

$$f_A(x) = \inf \{|x - a| \mid a \in A\}.$$

(a) Prove that for any $x \in \mathbb{R}$ the infimum of the set $\{|x - a| \mid a \in A\}$ exists, so the function $f_A$ is well-defined.

(b) Prove that $f_A$ is uniformly continuous on $\mathbb{R}$.

(2) Let $A, B \subseteq \mathbb{R}$ be sets so that $A \subseteq B$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions. We say that $g$ *extends* $f$ if for all $x \in A$ we have $g(x) = f(x)$.

(a) Suppose that $f : (a, b) \to \mathbb{R}$ is uniformly continuous on $(a, b)$. Show that $f$ can be extended to a continuous function $g : [a, b] \to \mathbb{R}$.

(b) Suppose that $f : (a, b) \to \mathbb{R}$ is a function that can be extended to a continuous function $g : [a, b] \to \mathbb{R}$. Show that $f$ is uniformly continuous on $(a, b)$.

(3) Prove that the function $f : (1, 2) \to \mathbb{R}$ defined by $f(x) = \frac{1}{(x^2+1)^2}$ is continuous and strictly decreasing. Determine the image $f((1, 2))$ and find an explicit expression for $f^{-1}(x)$.

(4) Suppose that $f : (a, b) \to \mathbb{R}$ is continuous and differentiable at every point in $(a, b)$ and that for all $x \in (a, b)$ we have $f'(x) > 0$. Prove that $f$ is a strictly increasing function on $(a, b)$.