HOMEWORK #8
WORKED SOLUTIONS

(1) Let A be a nonempty subset of $\mathbb{R}$. Define a function $f_A : \mathbb{R} \to \mathbb{R}$ by

$$f_A(x) = \inf \{|x - a| \mid a \in A\}.$$

(a) Prove that for any $x \in \mathbb{R}$ the infimum of the set $\{|x - a| \mid a \in A\}$ exists, so the function f_A is well-defined.

(b) Prove that f_A is uniformly continuous on $\mathbb{R}$.

Solution:

(a) Fix $x \in \mathbb{R}$. Since A is nonempty, the set $\{|x - a| \mid a \in A\}$ is nonempty. Also, for each $a \in A$ we have $|x-a| \geq 0$. Therefore the set $\{|x - a| \mid a \in A\}$ is bounded below, by 0.

Therefore, this set has a greatest lower bound, so $f_A(x)$ is well-defined.

(b) Fix $\epsilon > 0$ and let $\delta = 0$.

Let $x, y \in \mathbb{R}$ be so that $0 < |x - y| < \delta$. For any $a \in A$, we have

$$f_A(y) = \inf \{|y - a| \mid a \in A\} \leq |y - a| \leq |x - y| + |x - a|.$$

Therefore, $f_A(y) - |x - y| \leq |x - a|$ for all a. Therefore, since the infimum is the *greatest* lower bound, we have

$$f_A(y) - |x - y| \leq \inf \{|x - a| \mid a \in A\} = f_A(x).$$

Rewriting this, we get

$$f_A(y) - f_A(x) \leq |x - y|.$$

Reversing the roles of x and y we get

$$f_A(x) - f_A(y) \leq |y - x| = |x - y|.$$

Therefore,

$$|f_A(x) - f_A(y)| \leq |x - y| < \delta = \epsilon.$$

This proves that f_A is uniformly continuous at every point $x \in \mathbb{R}$.

(2) Let $A, B \subseteq \mathbb{R}$ be sets so that $A \subseteq B$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions. We say that g *extends* f if for all $x \in A$ we have $g(x) = f(x)$.

(a) Suppose that $f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b). Show that f can be extended to a continuous function $g : [a, b] \to \mathbb{R}$.

(b) Suppose that $f : (a, b) \to \mathbb{R}$ is a function that can be extended to a continuous function $g : [a, b] \to \mathbb{R}$. Show that f is uniformly continuous on (a, b).

Solutions:

(a) We want to see that

$$\lim_{x \to a^+} f(x)$$

exists, so that we can define $g(a)$ to be this limit. (And similarly with $g(b) = \lim_{x \to b^-} f(x)$.)

First, let (x_n) be a sequence of points in (a, b) converging to a. We claim that $\lim_{n\to\infty} f(x_n)$ exists. To see this, it is enough to see that $f(x_n)$ is Cauchy.

Well, let $\epsilon > 0$ be arbitrary. By uniform continuity, there is $\delta > 0$ so that if $x, y \in (a, b)$ and $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. Since (x_n) converges, it is Cauchy and there is $N \in \mathbb{N}$ so that for all $m, n > N$ we have $|x_n - x_m| < \delta$.

For such an m, n we have $|x_n - x_m| < \delta$, so $|f(x_n) - f(x_m)| < \epsilon$. Therefore $(f(x_n))$ is Cauchy, and converges.

Let L be the limit of the $f(x_n)$. If $\epsilon_0 > 0$ is arbitrary, and choose δ_0 so that if $x, y \in (a, b)$ and $|x - y| < \delta_0$ then $|f(x) - f(y)| < \frac{\epsilon_0}{2}$. There is also $M \in \mathbb{N}$ so that if $n > M$ we have $|f(x_n) - L| < \frac{\epsilon_0}{2}$.

Well, choose $y \in (a, b)$ so that $|y - a| < \delta_0$. Then there is some $m > M$ so that $x_m \in (a, y)$. We have $|y - x_m| < |y - a| < \delta_0$, so

$$|f(y) - L| \leq |f(y) - f(x_m)| + |f(x_m) - L| < \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = \epsilon_0.$$

This proves that $\lim_{y \to a^-} f(y) = L$, so this limit exists.

In an entirely similar manner, there is L' so that $\lim_{x \to b^-} f(x) = L'$.

Now define $g(a) = L$, $g(b) = L'$ and $g(x) = f(x)$ for $x \in (a, b)$. Then g is the required function.

(b) g is continuous on $[a, b]$, so it is uniformly continuous on $[a, b]$. Certainly then, restricted to any subinterval of $[a, b]$, g is still uniformly continuous. But the function f is exactly g restricted to the subinterval (a, b), so f must be uniformly continuous.

(3) Prove that the function $f : (1, 2) \to \mathbb{R}$ defined by $f(x) = \frac{1}{(x^2+1)^2}$ is continuous and strictly decreasing. Determine the image $f((1, 2))$ and find an explicit expression for $f^{-1}(x)$.

Solution:

$f(x)$ is continuous, since $g(x) = (x^2+1)^2$ is continuous (by a few applications of Theorems 3.11 and 3.16 from Howie). Now, $g(x) \neq 0$ for any x, so $f(x)$ is continuous by another application of Theorem 3.11.

Suppose that $x < y$ and $x, y \in (1,2)$. Then $0 < (x^2+1)^2 < (y^2+1)^2$. So

$$\frac{1}{(x^2+1)^2} > \frac{1}{(y^2+1)^2},$$

as required.

Since $f(1) = \frac{1}{4}$ and $f(2) = \frac{1}{25}$, the image $f((1,2)) = (\frac{1}{25}, \frac{1}{4})$.

Finally, we want the inverse function. The formula is

$$f^{-1}(y) = \sqrt{\frac{1}{\sqrt{y}} - 1}.$$

(It is straightforward to check that $f^{-1}(f(x)) = x$ and that $f(f^{-1}(y)) = y$.)

(4) Suppose that $f : (a,b) \to \mathbb{R}$ is continuous and differentiable at every point in (a,b) and that for all $x \in (a,b)$ we have $f'(x) > 0$. Prove that f is a strictly increasing function on (a,b).

Solution:

[The easiest way is to apply the Mean Value Theorem, but that would be cheating since we only did it in class on Friday after the HW was due. So here's a proof that's basically the proof of Rolle's Theorem]

Suppose, in order to obtain a contradiction, that there are $x, y \in (a,b)$ so that $x < y$ and $f(x) \geq f(y)$.

On the interval $[x,y]$ f is a continuous function, so that there is $d \in [x,y]$ so that

$$f(d) = \sup\{f(c) \mid c \in [x,y]\}.$$

First, if $d = y$ then $f(x)$ must be equal to $f(y)$ and the function is constant on $[x,y]$. Therefore for any point $c \in (x,y)$ we have $f'(c) = 0$, which is a contradiction.

Therefore, $d < y$. Now, if $z > d$ and $z \in [x,y]$ then $f(z) \leq f(d)$, by the choice of d. Hence we have

$$\frac{f(z) - f(d)}{z - d} \leq 0.$$

This proves that $\lim\limits_{z \to d^+} \frac{f(z)-f(d)}{z-d} \leq 0$. But since $f'(d)$ exists, this right-hand limit must be equal to $f'(d)$, so $f'(d) \leq 0$. This is a contradiction, so there are no such points x, y, and f must be strictly increasing.