HOMEWORK #9
DUE NOON, MARCH 20, 2009

(1) Let I be an open interval, and suppose $p : I \to I$ is differentiable on I. Suppose $a \in I$ is such that $p(a) = a$. Let $p_n : I \to I$ be the n-fold composition $p \circ p \circ \ldots \circ p$. Show that $p_n'(a) = (p'(a))^n$.

(2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function so that for all $x \in \mathbb{R}$ we have $f'(x) = f(x)$. (Think $f(x) = e^x$, if you like).

For $n \in \mathbb{N}$, find a formula for the n^{th} derivative of $g(x) = x^2 f(x)$.

(3) Let $f : A \to \mathbb{R}$ be a function, and suppose that $f''(a)$ exists. Prove that

$$\lim_{h \to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

(4) A function $f : (a, b) \to \mathbb{R}$ is *convex* if for all $x, y \in (a, b)$ and any $0 < \lambda < 1$ we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).$$

(a) Prove that if f is convex on (a, b) then it is continuous at every point of (a, b).

(b) Prove that if $w < x < y < z$ for $w, x, y, z \in (a, b)$ then

$$\frac{f(x) - f(w)}{x - w} \leq \frac{f(z) - f(y)}{z - y}.$$

(c) Suppose that $g : (a, b) \to \mathbb{R}$ is differentiable. Prove that g is convex on (a, b) if and only if $g' : (a, b) \to \mathbb{R}$ is a nondecreasing function.

(d) Suppose that $h : (a, b) \to \mathbb{R}$ is twice differentiable (so $h''(x)$ exists for all $x \in (a, b)$). Prove that h is convex on (a, b) if and only if $h''(x) \geq 0$ for all $x \in (a, b)$.