HOMEWORK #9
WORKED SOLUTIONS

(1) Let I be an open interval, and suppose $p : I \to I$ is differentiable on I. Suppose $a \in I$ is such that $p(a) = a$. Let $p_n : I \to I$ be the n-fold composition $p \circ p \circ \ldots \circ p$. Show that $p'_n(a) = (p'(a))^n$.

Solution:

Proof by induction on n:

Base Case (n=1): In this case $p_1(x) = p(x)$, so $p'_1(a) = p'(a) = (p'(a))^1$, as required.

Inductive Step: Suppose that $p'_k(a) = (p'(a))^k$. Then

$$p_{k+1}(x) = p_k(p(x)),$$

so by the Chain Rule we have

$$\begin{aligned} p'_{k+1}(a) &= p'_k(p(a)).p'(a) \\ &= p'_k(a).p'(a) \\ &= (p'(a))^k.p'(a), \text{ by induction} \\ &= (p'(a))^{k+1}, \end{aligned}$$

as required.

(2) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function so that for all $x \in \mathbb{R}$ we have $f'(x) = f(x)$. (Think $f(x) = e^x$, if you like).

For $n \in \mathbb{N}$, find a formula for the n^{th} derivative of $g(x) = x^2 f(x)$.

Solution:

Let's do the first few cases:

$$\begin{aligned} g'(x) &= x^2 f'(x) + 2xf'(x) = (x^2 + 2x)f(x) \\ g''(x) &= (x^2 + 2x)f'(x) + (2x + 2)f(x) = (x^2 + 4x + 2)f(x) \\ g'''(x) &= (x^2 + 4x + 2)f'(x) + (2x + 4)f(x) = (x^2 + 6x + 6)f(x) \\ g^{(4)}(x) &= (x^2 + 6x + 6)f'(x) + 2x + 6)f(x) = (x^2 + 8x + 12)f(x). \end{aligned}$$

The general formula is:

$$g^{(n)}(x) = (x^2 + 2nx + n(n-1))f(x).$$

We can prove this by induction on n. We've already done the base case, so suppose that

$$g^{(k)}(x) = (x^2 + 2kx + k(k-1))f(x).$$

Then

$$\begin{aligned} g^{(k+1)}(x) &= (x^2 + 2kx + k(k-1))f'(x) + (2x + 2k)f(x) \\ &= (x^2 + 2kx + 2x + k^2 - k + 2k)f(x) \\ &= (x^2 + 2(k+1)x + (k+1)k)f(x), \end{aligned}$$

as required.

(3) Let $f : A \to \mathbb{R}$ be a function, and suppose that $f''(a)$ exists. Prove that

$$\lim_{h\to 0} \frac{f(a+h) + f(a-h) - 2f(a)}{h^2} = f''(a).$$

Solution:

We have

$$f''(a) = \lim_{x\to a} \frac{f'(x) - f'(a)}{x - a}.$$

In particular, there is $\delta_1 > 0$ so that $f'(x)$ exists for all $x \in (a-\delta_1, a+\delta_1)$. Therefore, f is continuous at c for all $c \in (a - \delta_1, a + \delta + 1)$. Let $\delta_2 = \frac{\delta_1}{2}$. Let $g_1 : (a - \delta_1, a + \delta_1) \to \mathbb{R}$ be defined by

$$g_1(t) = f'(a + t) + f(a - t) - 2f(a),$$

and $g_2 : (a - \delta_1, a + \delta_2) \to \mathbb{R}$ be defined by

$$g_2(t) = t^2.$$

Then $g_1(t) = g_2(t) = 0$, and both g_1 and g_2 are differentiable on $(a - \delta_1, a + \delta_1)$. We have $g_2'(t) = 2t$ and

$$g_1'(t) = f'(a + t) - f'(a - t).$$

We'll use l'Höpital's Rule.

[Note that we don't know that $f''(x)$ exists for any x other than a, so we can only apply l'Höpital's Rule once.]

$$\begin{aligned} \lim_{h\to 0} \frac{f'(a+h) - f'(a-h)}{2h} &= \lim_{h\to 0} \left(\frac{1}{2}.\frac{f'(a+h) - f'(a)}{h} + \frac{1}{2}.\frac{f'(a) - f'(a-h)}{h} \right) \\ &= \lim_{h\to 0} \frac{1}{2}.\frac{f'(a+h) - f'(a)}{h} + \lim_{k\to 0} \frac{1}{2}.\frac{f'(a+k) - f'(a)}{k} \quad \text{with k = -h} \\ &= \frac{1}{2}f''(a) + \frac{1}{2}f''(a) = f''(a). \end{aligned}$$

So, the limit

$$\lim_{h\to 0} \frac{g_1(h)}{g_2(h)}$$

exists and equals $f''(a)$. This is exactly what we were required to prove.

(4) A function $f : (a,b) \to \mathbb{R}$ is *convex* if for all $x, y \in (a,b)$ and any $0 < \lambda < 1$ we have

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y).$$

(a) Prove that if f is convex on (a,b) then it is continuous at every point of (a,b).

(b) Prove that if $w < x < y < z$ for $w, x, y, z \in (a,b)$ then

$$\frac{f(x) - f(w)}{x - w} \le \frac{f(z) - f(y)}{z - y}.$$

(c) Suppose that $g : (a,b) \to \mathbb{R}$ is differentiable. Prove that g is convex on (a,b) if and only if $g' : (a,b) \to \mathbb{R}$ is a nondecreasing function.

(d) Suppose that $h : (a,b) \to \mathbb{R}$ is twice differentiable (so $h''(x)$ exists for all $x \in (a,b)$). Prove that h is convex on (a,b) if and only if $h''(x) \ge 0$ for all $x \in (a,b)$.

Solution:

Part (a) Let $c \in (a,b)$ be arbitrary.

Let $d_1, d_2, d_3 \in (a,b)$ be arbitrary so that $d_1 < d_2 < d_3$. Let $\lambda = \frac{d_3 - d_2}{d_3 - d_1}$. Then, $0 < \lambda < 1$. Also, $\lambda d_1 + (1-\lambda)d_3 = d_2$ and the convexity condition says that

$$f(d_2) \le \frac{d_3 - d_2}{d_3 - d_1} f(d_1) + \left(1 - \frac{d_3 - d_2}{d_3 - d_1}\right) f(d_3).$$

This equation can be rewritten as

$$f(d_2) \le \frac{d_3 - d_2}{d_3 - d_1} f(d_1) + \frac{d_2 - d_1}{d_3 - d_1} f(d_3) \tag{1}$$

This equation holds for all $d_1, d_2, d_3 \in (a,b)$ with $d_1 < d_2 < d_3$.

Now let $c \in (a,b)$ be arbitrary, and choose any $d \in (c,b)$. If $x \in (a,b)$ we have

$$f(c) \le \frac{d - c}{d - x} f(x) + \frac{c - x}{d - x} f(d),$$

which can be rewritten as

$$f(x) \ge \frac{d - x}{d - c} f(c) - \frac{c - x}{d - c} f(d).$$

Now let $e \in (a,x)$ be arbitrary. Then $e < x < c$ and we have

$$f(x) \le \frac{c - x}{c - e} f(e) + \frac{x - e}{c - e} f(c).$$

Think of c, d and e as fixed and x as a variable in (e, c). Then we have

$$\frac{d-x}{d-c}f(c) - \frac{c-x}{d-c}f(d) \leq f(x) \leq \frac{c-x}{c-e}f(e) + \frac{x-e}{c-e}f(c).$$

If we let x go to c (from below) then the left hand side and the right hand side of the above equation both tend to $f(c)$, so by the Sandwich Principle, we get

$$\lim_{x \to c^-} f(x) = f(c).$$

By fixing $u \in (c, b)$, and $y \in (c, u)$ we can see (in an entirely similar way) that

$$\lim_{xt\o c^+} f(x) = f(c),$$

so that $\lim_{x \to c} f(x) = f(c)$, and f is continuous as c as required.

Part (b):

Equation (1) gives, for any $d_1, d_2, d_3 \in (a, b)$ with $d_1 < d_2 < d_3$,

$$\begin{aligned}
\frac{f(d_2) - f(d_1)}{d_2 - d_1} &\leq \frac{\left(\frac{d_3-d_2}{d_3-d_1}\right) f(d_1) + \left(\frac{d_2-d_1}{d_3-d_1}\right) f(d_3) - f(d_1)}{d_2 - d_1} \\
&= \frac{(d_3 - d_2)f(d_1) + (d_2 - d_1)f(d_3) - (d_3 - d_1)f(d_1)}{(d_3 - d_1)(d_2 - d_1)} \\
&= \frac{(d_2 - d_1)(f(d_3) - f(d_1))}{(d_3 - d_1)(d_2 - d_1)} \\
&= \frac{f(d_3) - f(d_1)}{d_3 - d_1}.
\end{aligned}$$

We can similarly prove that

$$\frac{f(d_3) - f(d_1)}{d_3 - d_1} \leq \frac{f(d_3) - f(d_2)}{d_3 - d_2}$$

(almost the same proof applies).

Now suppose that $w, x, y, z \in (a, b)$ and $w < x < y < z$. Then, applying the above twice, we get:

$$\begin{aligned}
\frac{f(x) - f(w)}{x - w} &\leq \frac{f(z) - f(w)}{z - w} \\
&\leq \frac{f(z) - f(y)}{z - y},
\end{aligned}$$

which is what we were required to prove.

Part (c): Suppose that g is convex.

Now, all of the above calculations for f apply to g, and g is differentiable.

Let $x, z \in (a,b)$ and let $x < y$. Then, since for any $w, z \in (a,b)$ so that $w < x$ and $x < y < z$ we have

$$\frac{f(x)-f(w)}{x-w} \leq \frac{f(z)-f(y)}{z-y},$$

we must have

$$\lim_{w \to x^-} \frac{f(x)-f(w)}{x-w} \leq \lim_{y \to z^-} \frac{f(z)-f(y)}{z-y}.$$

But, the lefthand limit in the above is equal to $f'(x)$ and the righthand limit is $f'(z)$, since $f'(x)$ and $f'(z)$ both exist. Therefore,

$$f'(x) \leq f'(z),$$

as required.

Now suppose that g' is a nondecreasing function on (a,b).

Let $x, y \in (a,b)$ and suppose that $\lambda \in (0,1)$. By replacing λ by $1-\lambda$ and reversing x and y, if necessary, we may assume that $x < y$. We have to prove:

$$g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda)g(y).$$

We use Taylor's Theorem. There are $r \in (x, \lambda x + (1-\lambda)y)$ and $s \in (\lambda x + (1-\lambda)y, y)$ so that

$$g(\lambda x+(1-\lambda)y) = g(x)+g'(r)(\lambda x+(1-\lambda)y-x) = g(x)+g'(r)(y-x)(1-\lambda),$$

and

$$g(\lambda x + (1-\lambda)y) = g(y) + g'(s)(\lambda x + (1-\lambda)y - y) = g(y) - g'(s)(y-x)\lambda.$$

Note that $r < s$ so $g'(r) \leq g'(s)$ by assumption. Also $\lambda(1-\lambda)(y-x) > 0$. Then we have:

$$\begin{aligned}
g(\lambda x + (1-\lambda)y) &= \lambda g(\lambda x + (1-\lambda)y) + (1-\lambda)g(\lambda x + (1-\lambda)y) \\
&= \lambda g(x) + \lambda g'(r)(y-x)(1-\lambda) + (1-\lambda)g(y) + (1-\lambda)g'(s)(y-x)\lambda \\
&= \lambda g(x) + (1-\lambda)g(y) + \lambda(1-\lambda)(y-x)(g'(r) - g'(s)) \\
&\leq \lambda g(x) + (1-\lambda)g(y),
\end{aligned}$$

as required. Therefore, g is convex.

Part (d):

Suppose that h is twice differentiable and convex. Then by Part (c) we know that h' is a nondecreasing function. Let $x \in (a,b)$. If $y < x$ then $h'(y) \leq h'(x)$ and

$$\frac{h'(y)-h'(x)}{y-x} \geq 0.$$

On the other hand, if $y > x$ then $h'(y) \geq h'(x)$ and

$$\frac{h'(y) - h'(x)}{y - x} \geq 0.$$

Therefore, in any case,

$$f''(x) = \lim_{y \to x} \frac{h'(y) - h'(x)}{y - x} \geq 0,$$

as required.

Conversely, suppose that h is twice differentiable and $h''(x) \geq 0$ for all $x \in (a, b)$.

Let $x, y \in (a, b)$ and $\lambda \in (0, 1)$.

Let $w = \lambda x + (1 - \lambda)y$, so $w \in (x, y)$. Note that $x - w = (1 - \lambda)(x - y)$ and $y - w = -\lambda(x - y)$.

Now, we use Taylor's Theorem again. There is $r \in (x, w)$ and $s \in (w, y)$ so that

$$h(x) = h(w) + (x - w)h'(w) + \frac{1}{2}(x - w)^2 h''(s),$$

and

$$h(y) = h(w) + (y - w)h'(w) + \frac{1}{2}(y - w)^2 h''(r).$$

Rearranging, and proceeding as before, we get:

$$\begin{aligned} h(w) &= \lambda h(w) + (1 - \lambda)h(w) \\ &= (\lambda h(x) - \lambda(1 - \lambda)(x - y)h'(w) + \frac{\lambda}{2}(x - w)^2 h''(s)) \\ &\quad +((1 - \lambda)h(y) + (1 - \lambda)\lambda(x - y)h'(w) + \frac{1 - \lambda}{2}(y - w)^2 h''(r) \\ &\leq \lambda h(x) + (1 - \lambda)h(y), \end{aligned}$$

Since $h''(r), h''(s), \lambda, (1 - \lambda), (x - w)^2, (y - w)^2$ are all at least 0.

This proves that h is convex, as required.