Math 313 - Analysis I **Spring 2009**

MIDTERM #2 - PRACTICE
NOON, APRIL 3, 2009

(1) Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 2x(1 - x)$.

   (a) Using only the definition involving limits, prove that f is continuous on $[0, 1]$.

   (b) Using only the definition involving limits, prove that f is differentiable on $(0, 1)$.

   (c) What is the image $f([0, 1])$?

   (d) Prove that there is $c \in [0, 1]$ so that $f(c) = c$.

(2) Suppose that $f : (-1, 1)$ is a function so that if $\frac{p}{q} \in (-1, 1) \cap \mathbb{Q}$ is a rational number written in lowest terms, with $q \in \mathbb{N}$, then $f(\frac{p}{q}) = p$.

   (a) Prove that f is not continuous at 0.

   (b) Can f be continuous at any point in $(-1, 1)$?

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $x_0, a, b \in \mathbb{R}$, with $a \neq 0$ and $b \neq 0$. Suppose also that f is differentiable at x_0. Prove that

$$\lim_{n\to\infty} n \left(f(x_0 + \frac{a}{n}) - f(x_0 - \frac{b}{n}) \right) = (a + b)f'(x_0).$$