Math 313 - Analysis I **Spring 2009**

MIDTERM #2 - PRACTICE WORKED SOLUTIONS

(1) Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = 2x(1-x)$.

(a) Using only the definition involving limits, prove that f is continuous on $[0,1]$.

(b) Using only the definition involving limits, prove that f is differentiable on $(0,1)$.

(c) What is the image $f([0,1])$?

(d) Prove that there is $c \in [0,1]$ so that $f(c) = c$.

Solution:

(a): Let $c \in [0,1]$, and let $\epsilon > 0$ be arbitrary. Let $\delta = \frac{\epsilon}{4}$. Suppose that $x \in [0,1]$ and that $|x - c| < \delta$.

Then $2x \leq 2$, and $2(1-c) \leq 2$. Now

$$\begin{aligned} |f(x) - f(c)| &= |2x(1-x) - 2c(1-c)| \\ &= 2x(1-x) - 2x(1-c) + 2x(1-c) - 2c(1-c) \\ &= 2x(1-x-(1-c)) + (1-c)(2x-2c) \\ &= 2x(c-x) + 2(1-c)(x-c) \\ &\leq 2|x-c| + 2|x-c| \\ &= 4|x-c| < 4\delta = \epsilon \end{aligned}$$

This is enough to prove that f is continuous at c. (We have, for $c \neq 0, 1$, $\lim_{x\to c} f(x) = f(c)$, and also $\lim_{x\to 0^+} f(x) = f(0)$ and $\lim_{x \to 1^-} f(x) = f(1)$.)

(b): Let $c \in (0,1)$. We have to prove that

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}.$$

exists.

Well, if $x \neq c$ then:

$$\begin{aligned} \frac{f(x) - f(c)}{x - c} &= \frac{2x(1-x) - 2c(1-c)}{x-c} \\ &= \frac{2x(c-x) + 2(1-c)(x-c)}{x-c} \\ &= \frac{(2-2c-2x)(x-c)}{x-c} \\ &= 2 - 2c - 2x \end{aligned}$$

Therefore,

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c}=2-4c.$$

(In particular, the limit exists.)

(c): If $c\in(0,\frac{1}{2})$ then $f'(c)>0$, so f is increasing. If $c\in(\frac{1}{2},1)$ then $f'(c)<0$ so f is decreasing. We have $f(0)=0$, $f(\frac{1}{2})=\frac{1}{2}$ and $f(1)=0$. Therefore,

$$f([0,1])=[0,\frac{1}{2}].$$

(d): Well, $f(0)=0$ and $f(\frac{1}{2})=\frac{1}{2}$, so they'll do.

(2) Suppose that $f:(-1,1)$ is a function so that if $\frac{p}{q}\in(-1,1)\cap\mathbb{Q}$ is a rational number written in lowest terms, with $q\in\mathbb{N}$, then $f(\frac{p}{q})=p$.

(a) Prove that f is not continuous at 0.

(b) Can f be continuous at any point in $(-1,1)$?

Solutions:

(a): Consider the following two sequences of points:

$$(\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots),$$

so $a_n=\frac{1}{n}$, and

$$(\frac{2}{3},\frac{2}{5},\frac{2}{7},\ldots),$$

so $b_n=\frac{2}{2n+1}$.

Well, $a_n\to 0$ and $b_n\to 0$ as $n\to\infty$. However, for each n we have $f(a_n)=1$ and $f(b_n)=2$. Therefore, it cannot be that

$$\lim_{x\to 0}f(x)$$

exists, so f is not continuous at 0, no matter what the definition of $f(0)$ is.

(b): No.

Let $c\in(-1,1)$. We know that f is not continuous at 0, so assume that $c\neq 0$. Then there is a sequence of distinct rational numbers $x_n=\frac{r_n}{s_n}$ with $x_n\neq x$. In order that a sequence of distinct rational numbers $(\frac{r_n}{s_n})$ converges, we must have $s_n\to\infty$ as $n\to\infty$. However, if $(\frac{r_n}{s_n})$ is going to converge to $c\neq 0$, and $s_n\to\infty$ then we must have $p_n\to\infty$ (if $c>0$) of $p_n\to-\infty$ (if $c<0$). In any case, near to x there are points x_n so that $|f(x_n)|$ is arbitrarily large as n gets large. This means that $\lim_{x\to c}f(x)$ does not exist, so f cannot be continuous as c.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and suppose that $x_0, a, b \in \mathbb{R}$, with $a \neq 0$ and $b \neq 0$. Suppose also that f is differentiable at x_0. Prove that

$$\lim_{n\to\infty} n\left(f(x_0 + \frac{a}{n}) - f(x_0 - \frac{b}{n})\right) = (a+b)f'(x_0).$$

Solution:

Well,

$$\begin{aligned} n\left(f(x_0 + \frac{a}{n}) - f(x_0 - \frac{b}{n})\right) &= n\left(f(x_0 + \frac{a}{n}) - f(x_0) + f(x_0) - f(x_0 - \frac{b}{n})\right) \\ &= n.\frac{a}{n}\left(\frac{f(x_0 + \frac{a}{n}) - f(x_0)}{\frac{a}{n}}\right) + n.\frac{b}{n}\left(\frac{f(x_0 + \frac{-b}{n}) - f(x_0)}{\frac{-b}{n}}\right) \\ &= a\left(\frac{f(x_0 + \frac{a}{n}) - f(x_0)}{\frac{a}{n}}\right) + b\left(\frac{f(x_0 + \frac{-b}{n}) - f(x_0)}{\frac{-b}{n}}\right). \end{aligned}$$

As $n \to \infty$ we have $\frac{a}{n} \to 0$ and $\frac{-b}{n} \to 0$ (and $\frac{a}{n} \neq 0$, $\frac{-b}{n} \neq 0$. Therefore:

$$\begin{aligned} \lim_{n\to\infty} n\left(f(x_0 + \frac{a}{n}) - f(x_0 - \frac{b}{n})\right) &= \lim_{n\to\infty} a\left(\frac{f(x_0 + \frac{a}{n}) - f(x_0)}{\frac{a}{n}}\right) + \lim_{n\to\infty} b\left(\frac{f(x_0 + \frac{-b}{n}) - f(x_0)}{\frac{-b}{n}}\right) \\ &= af'(x_0) + bf'(x_0) \\ &= (a+b)f'(x_0), \end{aligned}$$

as required.