Math 313 - Analysis I **Spring 2009**

MIDTERM #2
SOLUTIONS

(1) Suppose that $f : (-1,1)$ is a continuous function. Suppose also that for each natural number $n \in \mathbb{N}$ we have $|f(\frac{1}{n})| < \frac{1}{n}$. Prove that $f(0) = 0$.

Solution:

We know that $\lim_{xt\o 0} f(x) = f(0)$, since f is continuous. Therefore, it suffices to prove that $\lim_{x\to 0} f(x) = 0$. (We know that this limit exists, by continuity)

There is a theorem in Howie (Theorem 3.5) which says that if f is continuous at c and if (a_n) is a sequence of numbers as that $a_n \to c$ as $n \to \infty$ then

$$\lim_{n\to\infty} f(a_n) = \lim_{x\to c} f(x).$$

So, let $a_n = \frac{1}{n}$. We know that $a_n \to 0$ as $n \to \infty$.

Also, since $|f(a_n)| < \frac{1}{n}$ we must have $\lim_{n\to\infty} f(a_n) = 0$. Therefore, we have proved that $\lim_{x\to 0} f(x) = 0$, which means that $f(0) = 0$, as required.

(2) The aim of this question is to prove that the function $f : [0,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous. You do not have to do the two parts below if you want to just prove this directly.

(a) Prove that $f : [1,\infty) \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is uniformly continuous.

(b) You may assume that $f : [0,1] \to \mathbb{R}$ given by $f(x) = \sqrt{x}$ is continuous. Use this and part (a) to prove that the square root function is uniformly continuous on $[0,\infty)$.

Solutions:

(a) Suppose that $\epsilon > 0$ is arbitrary, and let $\delta = 2\epsilon$. If $x, c \in [1,\infty)$ and $|x - c| < \delta$, then $\sqrt{x} + \sqrt{c} \geq 2$ and

$$\begin{aligned}
|f(x) - f(c)| &= |\sqrt{x} - \sqrt{c}| \\
&= (\sqrt{x} - \sqrt{c}).\frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \\
&= \frac{x - c}{\sqrt{x} - \sqrt{c}} \\
&\leq \frac{x - c}{2} \\
&< \frac{\delta}{2} = \epsilon.
\end{aligned}$$

Therefore, f is uniformly continuous no $[1,\infty)$ as required.

(b) Because we know f is continuous on $[0,1]$, it is continuous on all of $[0,\infty)$ and in particular it is continuous on $[0,2]$. Since $[0,2]$ is a closed, bounded interval, f is uniformly continuous on $[0,2]$.

Now, let $\epsilon > 0$ be arbitrary. Because f is uniformly continuous on $[1,\infty)$ there is $\delta_1 \in (0,1)$ so that if $x, c \in [1,\infty)$ and $|x-c| < \delta_1$ then $|\sqrt{x}-\sqrt{c}| < \epsilon$.

Also, since f is uniformly continuous on $[0,2]$, there is $\delta_2 \in (0,1)$ so that if $x, c \in [0,2]$ and $|x-c| < \delta_2$ then $|\sqrt{x}-\sqrt{c}| < \epsilon$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and suppose that $x, c \in [0,\infty)$ satisfy $|x-c| < \delta$. Since $\delta < 1$, we have either $x, c \in [0,2]$ or $x, c \in [1,\infty)$. In the first case, since $\delta \leq \delta_2$ we have $|\sqrt{x}-\sqrt{c}| < \epsilon$. In the second case, since $\delta \leq \delta_1$ we have $|\sqrt{x}-\sqrt{c}| < \epsilon$.

Thus f is uniformly continuous on $[0,\infty)$, as required.

(3)
- (a) Let $f : (a,b) \to \mathbb{R}$ be a function, and suppose that $\lim_{x\to c} f(x) = 0$. Prove that $\lim_{x\to c} \frac{1}{f(x)}$ does not exist.
- (b) Define the function $g : (-1,1) \to \mathbb{R}$ by $g(x) = \sqrt{|x|}$. Prove that $g'(0)$ does not exist.
- (c) Let g be as in Part (b). Using only the definition of derivative using limits, prove that if $c \in (-1,1) \setminus \{0\}$ then $g'(c)$ does exist. Find an expression for $g'(c)$, for such a c.

Solutions:

(a) [Thanks to Jonathan R. for pointing out this slick solution...]

Suppose, in order to obtain a contradiction, that there is $L \in \mathbb{R}$ so that $\lim\limits_{x\to c} \frac{1}{f(x)} = L$. Then there is some $\delta > 0$ so that for all $x \in (c-\delta, c+\delta)\setminus\{c\}$ we have $f(x) \neq 0$, so $\frac{1}{f(x)}$ is defined.

Now,

$$\lim_{x\to c} f(x).\frac{1}{f(x)} = \lim_{x\to c} f(x).\lim_{x\to c}\frac{1}{f(x)} = 0.L = 0.$$

But $f(x).\frac{1}{f(x)} = 1$ whenever $f(x) \neq 0$. Therefore, we must have

$$\lim_{x\to c} f(x).\frac{1}{f(x)} = 1.$$

Since $0 \neq 1$, we have a contradiction.

(b)

Consider the limit

$$\lim_{x\to 0}\frac{g(x)-g(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt{|x|}}{x}.$$

We have to prove that this limit doesn't exist. Well, if $x \neq 0$, we have

$$\frac{\sqrt{|x|}}{x} = \begin{cases} \frac{1}{\sqrt{|x|}}, & \text{if } x > 0 \\ \frac{-1}{\sqrt{-x}}, & \text{if } x < 0 \end{cases}$$

Define a function $h : (-1, 1)$ by

$$h(x) = \begin{cases} \sqrt{x}, & \text{if } x > 0 \\ -\sqrt{-x}, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then, for $x \neq 0$ we have $\frac{g(x)-g(0)}{x-0} = \frac{1}{h(x)}$. It is easy to see that $\lim\limits_{x \to 0} h(x) = 0$, so by Part (a) we must have that

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x - 0}$$

does not exist. Thus $g'(0)$ does not exist, as required.

(c)

Let $c \in (-1, 1) \setminus \{0\}$. Then we have

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c} = \lim_{x \to c} \frac{\sqrt{|x|} - \sqrt{|c|}}{x - c}.$$

If $x, c > 0$ we have

$$\frac{\sqrt{|x|} - \sqrt{|c|}}{x - c} = \frac{1}{\sqrt{x} + \sqrt{c}},$$

whilst if $x, c < 0$ we have

$$\frac{\sqrt{|x|} - \sqrt{|c|}}{x - c} = \frac{-1}{\sqrt{-x} + \sqrt{-c}}.$$

Therefore, if $c < 0$ we have

$$g'(c) = \frac{-1}{2\sqrt{-c}},$$

while if $c > 0$ we have

$$g'(c) = \frac{1}{2\sqrt{c}}.$$