Math 313 - Analysis I **Spring 2009**

PRACTICE QUESTIONS FOR FINAL - WORKED SOLUTIONS

These questions are intended to represent approximately how difficult and long the Final Exam will be, and also to indicate some of the types of questions that might arise. They should not be contrued as a complete list of the topics that are examinable.

(1) Show that the series
$$\sum_{n=0}^{\infty} \frac{2n+3}{(n+1)(n+2)}$$
does not converge.

Solution:

Note that
$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}.$$
Therefore, we have
$$\begin{aligned}\sum_{n=0}^{\infty} \frac{2n+3}{(n+1)(n+2)} &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} \right) \\ &= \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) + \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) - 1 \\ &= 2 \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) - 1\end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ is the harmonic series, which does not converge, we see that the sum
$$\sum_{n=0}^{\infty} \frac{2n+3}{(n+1)(n+2)}$$
does not converge either.

(2) Let $f : (-1, 1) \smallsetminus \{0\}$ be defined by:
$$f(x) = \begin{cases} \frac{e^x - 1}{x}, & \text{if } x \in (-1, 0) \\ \frac{3(x+1)}{x^2 - 4x + 3}, & \text{if } x \in (0, 1) \end{cases}$$

(a) Prove that $\lim_{x \to 0} f(x)$ exists.

(b) Give a value of $f(0)$ so that you get a function $f : (-1, 1)$ and prove that the function thus obtained is continuous at all points of $(-1, 1)$.

Solution:

(a): We consider first the limit from the left:

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^x - 1}{x}.$$

When we substitute $x = 0$ into this function, we get the 'indeterminate' $\frac{0}{0}$, so we try to apply L'Höpital's Rule. The derivative of the numerator is e^x, and that of the denominator is 1, so we get

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{e^x - 1}{x} \\ &= \lim_{x\to 0} \frac{e^x - 1}{x} \\ &= \lim_{x\to 0} \frac{e^x}{1} = 1.\end{aligned}$$

Consider now the limit from the right. We substitute $x = 0$ into the numerator and denominator and get $\frac{3}{3}$. Therefore,

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{3(x+1)}{x^2 - 4x + 3} = \frac{3}{3} = 1.$$

Therefore, we have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = 1.$$

Hence $\lim_{x\to 0} f(x)$ exists and equals 1.

(b): Since we've shown that $\lim_{x\to 0} f(x) = 1$, we should define $f(0) = 1$ in order that f be continuous at 0.

Now, consider points $y \in (-1, 0)$. In this case, we know that $e^x - 1$ is continuous, and x is also continuous and nonzero, so $\frac{e^x-1}{x}$ is continuous on this interval.

Furthermore, $x^2 - 4x + 3 = (x - 1)(x - 3)$, so is zero only when $x = 1, 3$, and in particular is nonzero on $(0, 1)$. Therefore,

$$\frac{3(x+1)}{x^2 - 4x + 3}$$

is continuous on $(0, 1)$.

This proves that $f(x)$ (with $f(0) = 1$) is continuous at all points in $(-1, 1)$.

(3) Question 4.7, page 110 of Howie.

Solution: See the back of Gallian.

(4) Assuming the Mean Value Theorem for integrals, and the Fundamental Theorem of Calculus, but not the IVT or the ordinary MVT, prove the First Mean Value Theorem:

Let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable. There is a $c \in (a, b)$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Solution: (Sorry, I meant to say that the derivative of f is continuous)

Consider the function $f'(x)$. We know that

$$\int_a^x f' = f(x) - f(a)$$

In particular,

$$\int_a^b f' = f(b) - f(a).$$

The Mean Value Theorem for Integrals now says that there is some $c \in (a, b)$ so that

$$f'(c)(b - a) = \int_a^b f'$$

If we put these two equations together, we get

$$f'(c)(b - a) = f(b) - f(a),$$

which we can rewrite as

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

the required expression.

(5) Question 5.9, page 130 of Howie.

Solution: See the back of Gallian.