Math 313 - Analysis I **Spring 2009**

SPRING BREAK PRACTICE PROBLEMS - WORKED SOLUTIONS

(1) Suppose that $f : [a,b] \to \mathbb{R}$ is a continuous function so that for all $x \in [a,b]$ we have $f(x) > 0$. Prove that there is some $c > 0$ so that $f(x) \geq c$ for all $x \in [a,b]$. (We say that f is *bounded away from* 0.)

Solution

Note that f is continuous on $[a,b]$, and never 0, so $\frac{1}{f} : [a,b] \to \mathbb{R}$ exists and is continuous on $[a,b]$. Therefore, $\frac{1}{f}$ is a uniformly continuous function on $[a,b]$, and hence is bounded. Let M be an upper bound for the values of $\frac{1}{f(x)}$, for $x \in [a,b]$. Then since $\frac{1}{f(x)} \leq M < M+1$, we have $f(x) > \frac{1}{M+1}$ for all $x \in [a,b]$. We can take $c = \frac{1}{M+1}$.

(2) Give an example of a continuous function $f : A \to \mathbb{R}$ (you choose what A is) so that:

- (i) f is bounded above but does not attain its maximum;
- (ii) f is unbounded;
- (iii) f is bounded and attains its maximum but not its minimum.

Solutions:

(a): Let $A = (0,1)$ and let $f : (0,1) \to \mathbb{R}$ be given by $f(x) = x$.

(b): Let $A = (0,1)$ and let $f : (0,1) \to \mathbb{R}$ be given by $f(x) = \frac{1}{x}$.

(c): Let $A = (0,1]$ and let f be given by $f(x) = x$.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} t - t^3 & \text{if } t \in \mathbb{Q} \\ 0 & \text{if } t \notin \mathbb{Q} \end{cases}$$

Prove that f is continuous at $t = -1, 0, 1$, but not at any other point.

Solution:

Consider the case $t = 0$.

If $\epsilon > 0$ then let $\delta = \min\{\frac{\epsilon}{2}, \frac{\sqrt[3]{\epsilon}}{\sqrt[3]{2}}\}$. Suppose that $|x| < \delta$. If $x \notin \mathbb{Q}$ then $f(x) = 0$ and $|f(x) - f(0)| = 0$. If $x \in \mathbb{Q}$ then

$$\begin{aligned} |f(x) - f(0)| &= |x - x^3| \\ &\leq |x| + |x|^3 \\ &< \delta + \delta^3 \\ &\leq \frac{\epsilon}{2} + \left(\frac{\sqrt[3]{\epsilon}}{\sqrt[3]{2}}\right)^3 \\ &= \epsilon \end{aligned}$$

Therefore, f is continuous at 0.

Consider now the case $t = 1$.

Let $\epsilon > 0$ and let $\delta = \min\{\frac{1}{2}, \frac{4\epsilon}{3}\}$. If $|x-1| < \delta$ then $x \in (\frac{1}{2}, \frac{3}{2}$ and so $|x| < \frac{3}{2}$ and $|1-x| < \frac{1}{2}$. We have

$$\begin{aligned} |f(x) - f(1)| &= |x - x^3| \\ &= |x||1+x||1-x| \\ &< \frac{3}{4}|x-1| \\ &< \frac{3}{4}\delta \\ &\leq \epsilon. \end{aligned}$$

Therefore, f is continuous at 1.

The calculation that f is continuous at -1 is entirely similar.

Now suppose that $c \notin \{0, -1, 1\}$. If $c \in \mathbb{Q}$ then $f(c) \neq 0$, but there is certainly a sequence (x_n) of numbers not in $\mathbb{Q}$ so that $x_n \to c$. But $f(x_n) = 0$.

If $c \notin \mathbb{Q}$ then $f(c) = 0$. However, there is a sequence of rational numbers (y_n) so that $y_n \to c$. Now $f(y_n) = y_n - y_n^3$. Therefore,

$$\lim_{n \to \infty} f(y_n) = c - c^3 \neq 0.$$

Therefore, in both cases f is not continuous at c.

(4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function so that for all $x, y \in \mathbb{R}$ we have

$$|f(x) - f(y)| \leq (x-y)^2.$$

Prove that f is a constant function.

Solution:

We have, for any x, y,

$$-(x-y)^2 \leq f(x) - f(y) \leq (x-y)^2.$$

Therefore, if $x > y$ we have

$$-(x-y) \leq \frac{f(x) - f(y)}{x - y} \leq x - y,$$

whilst if $x < y$ we have

$$(x-y) \leq \frac{f(x) - f(y)}{x - y} \leq -(x-y).$$

Suppose that $y \in \mathbb{R}$. Then

$$0 = \lim_{x \to y^+} -(x-y) \le \lim_{x \to y^+} \frac{f(x) - f(y)}{x-y} \le \lim_{x \to y^+} x - y = 0,$$

and

$$0 = \lim_{x \to y^-} (x-y) \le \lim_{x \to y^-} \frac{f(x) - f(y)}{x-y} \le \lim_{x \to y^-} -(x - y) = 0.$$

Therefore

$$\lim_{x \to y^+} \frac{f(x) - f(y)}{x-y} = \lim_{x \to y^-} \frac{f(x) - f(y)}{x-y} = 0,$$

so $f'(y) = 0$ for all $y \in \mathbb{R}$.

We proved in lectures that this means that f is constant.

(5) Prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = (1 + \frac{1}{x})^x$$

is increasing when $x > 0$.

Solution: OK, so this question was slightly cheeky. But anyway...

So, we write:

$$f(x) = e^{x \log(1 + \frac{1}{x})}.$$

Differentiating, we get

$$f'(x) = \left(\log(x+1) - \log(x) - \frac{1}{x+1} \right) e^{x \log(1 + \frac{1}{x})}$$

Therefore, to show that $f'(x) > 0$ it is enough to show that $\log(x+1) - \log(x) - \frac{1}{x+1} > 0$.

Well, $\log(x+1) - \log(x)$ is the area under the curve $g(x) = \frac{1}{x}$ between x and $x+1$. If we take the disection $\mathcal{D} = \{x, x+1\}$, we see that $\inf\{g(y) \mid y \in (x, x+1)\} = \frac{1}{x+1}$ and

$$\mathcal{L}(g, \mathcal{D}) = \frac{1}{x+1},$$

is an estimate from below for this area (and clearly this is strict lower than the area).

This means that $\log(x+1) - \log(x) - \frac{1}{x+1} > 0$, and f is increasing, as required.

(6) Howie, Chapter 3.4, p.80: 3.16 (without using things from later sections, but directly from the definition). Also, 3.17. From 3.5, p.89: 3,21, 3.22, 3.24, 3.26, from 3.6, p. 94: 3.33, from 4.1, pp. 104-5: 4.2, 4.4, from Ch. 4.2, p.110: 4.7, 4.8. Finally, from Ch. 4.3, p.113: 4.13.

Solutions are in Howie, so I'm not going to give them...